Bull. Korean Math. Soc. **42** (2005), No. 1, pp. 203–211

ON FACTORIZATION OF SOLUTIONS TO SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. If a meromorphic solution of second order homogeneous linear differential equation is factorizable, then the right factor of the factorization of the solution has order not more than the coefficient's. And some asympotic properties of solutions are studied.

1. Introduction

A transcendental meromorphic function $f(z)$, in the complex plane $\mathbf{C}$, is factorizable, if $f(z)$ is of the form

$$f(z) = g(h(z)),$$

where $g(z)$ is transcendental, meromorphic and $h(z)$ is transcendental, entire. $g(z)$ is called the left factor of the factorization of $f(z), h(z)$ the right factor of the factorization of $f(z)$. In this paper, we always study the form of $f(z) = g(h(z))$. $f(z)$ is prime, if for every factorization of $f(z)$, either $g(z)$ is bilinear or $h(z)$ is linear. If $g(z)$ is always a rational function, or $h(z)$ is always polynomial, $f(z)$ is said to be pseudo prime. Steinmetz [7] pointed out that any nontrivial solution of linear differential equations with rational coefficients is pseudo prime. Zheng and He [12] studied some second order homogeneous linear differential equations, and prove some solutions are prime, and some are factorizable. In the following, we continue to research the factorization of solutions of second order linear differential equations with meromorphic coefficients. And some asymptotical properties of solutions are studied under the consideration of the coefficients with order less than $\frac{1}{2}$.

Received November 3, 2003.
2000 Mathematics Subject Classification: 30D05, 34M05, 34M45.
Key words and phrases: factorization, left factor, linear differential equation, meromorphic solution, right factor.

Consider the following equation

$$f'' + A(z)f = 0 \tag{1.1}$$

where $A(z)$ is meromorphic, transcendental of finite order. It is known that every nontrivial solution $f(z)$ of (1.1) is transcendental, see [2]. If $A(z)$ is entire, any nontrivial solution of (1.1) is of infinite order, see [2] and if $f(z) \neq 0$, then $f(z)$ is factorizable, its right factor of the factorization is of the same order as $A(z)$, for the solution must be written as $f(z) = e^{g(z)}$, where $g(z)$ is entire and its order is equal to the order of $A(z)$, see [2]; if $A(z)$ is meromorphic, the (1.1) may have a nontrivial meromorphic solution of finite order, see Remarks in §5. It is easy to prove that for any nontrivial meromorphic solution $f(z)$, every pole of $f(z)$ must be a pole of $A(z)$.

Let $f(z)$ be a meromorphic function, $\sigma(f)$, $\mu(f)$ and $\lambda(f)$ denote the order of $f(z)$, the lower order of $f(z)$ and the exponent of convergence of zero-sequence of $f(z)$, respectively, i.e.,

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r},$$

$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r},$$

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log^+ N(r,\frac{1}{f})}{\log r}.$$

Nevanlinna theory and some standard notations come from [4].

Our main results may be stated as follows.

THEOREM 1.1. Let $f(z)$ be a nontrivial meromorphic solution of (1.1). If $f(z) = g(h(z))$ is factorizable, then $\sigma(h) \leq \sigma(A)$.

From Theorem 1.1, it follows the corollary below:

COROLLARY. Let $f(z)$ be a nontrivial meromorphic solution of (1.1). If $f(z) = g(h(z))$ and $\sigma(h) > \sigma(A)$, then $g(z)$ is rational.

THEOREM 1.2. Suppose $A(z)$ is a transcendental meromorphic function of order less than $\frac{1}{2}$, $f(z)$ is a nontrivial meromorphic solution of (1.1) and $f(z) = g(h(z))$ is factorizable. If $g(z)$ has a finite deficient value a, then there exists a line L approaching to ∞ such that

$$\liminf_{|z|\to\infty} \frac{\log\log|f(z)|}{\log|z|} \geq \frac{1}{2}, \quad z \in L.$$

Let $f(z)$ be meromorphic in $\mathbf{C}$. If $n(r,f) = O(r^k), k < \frac{1}{2} < \mu(f)$, W. Hayman asks whether the corresponding Boas' theorem holds or not. See [5,10]. Our Theorem 1.2 describes this property.

2. Lemmas

The following lemma is from Theorem 4.4.5 in [11], or see [3, 7].

LEMMA 2.1. *Let $F_j(z)$ and $h_j(z)(j = 0, 1, 2, \cdots, m)$ be not identically vanishing meromorphic functions, and $g(z)$ be a nonconstant entire function. There exists an unbounded positive sequence $\{r_n\}_{n=1}^{\infty}$ satisfying*

$$\sum_{j=0}^{m} T(r_n, h_j) \leq KT(r_n, g),$$

where K is a positive constant. If $F_j(z)$ and $h_j(z)(j = 0, 1, 2, \cdots, m)$ satisfy

$$F_0(g)h_0 + \cdots + F_m(g)h_m \equiv 0,$$

then there exist polynomials $P_0, P_1, \cdots, P_m$ not all identically zero such that

$$P_0(g)h_0 + P_1(g)h_1 + \cdots + P_m(g)h_m \equiv 0.$$

Furthermore, if h_j's are not all identically zero, then there exist not all identically zero polynomials $Q_0, Q_1, \cdots, Q_m$ such that

$$F_0(z)Q_0(z) + F_1(z)Q_1(z) + \cdots + F_m(z)Q_m(z) \equiv 0.$$

From a result of Valiron [8], one has

LEMMA 2.2 *Let $f(z)$ be a transcendental entire solution of (2.1)*

$$a_2(z)f'' + a_1(z)f' + a_0(z)f = 0, \tag{2.1}$$

where $a_2(z) \not\equiv 0, a_1(z), a_0(z)$ are polynomials. Then $\mu(f) > 0$.

LEMMA 2.3. *Let $H(z)$ and $h(z)$ be transcendental entire functions of finite order. If $\lambda(H) > 0$, then $\lambda(H(h)) = \infty$.*

Proof. Let $\{z_k\}_{k=1}^{\infty}$ be a sequence zeros of $H(z)$. By Hadamard-Borel's theorem, $H(z)$ can be expressed as

$$H(z) = z^m e^{P(z)} \Pi(z),$$

where $P(z)$ is a polynomial with degree $\deg P(z) \leq \sigma(H)$, m is a non-negative integer, $\Pi(z)$ is of the form as follows

$$\Pi(z) = \sum_{k=1}^{\infty}(1 - \frac{z}{z_k})e^{P_k(\frac{z}{z_k})},$$

and

$$P_k(\frac{z}{z_k}) = \sum_{t=1}^{q}\frac{1}{t}(\frac{z}{z_k})^t,$$

q is the minimum nonnegative integer satisfying

$$\sum_{k=1}^{\infty}\frac{1}{|z_k|^{q+1}} < \infty.$$

If $q = 0$, then $\sigma(\Pi) = 0$, this is a contradiction to $\lambda(\Pi) > 0$. So $q \geq 1$, and then $\lambda(H(h)) = \infty$. The proof is complete. □

From Theorem 3.5 in [10], we state the following.

LEMMA 2.4. *Let $f(z)$ be an entire function of a finite deficient value. Then $\mu(f) > \frac{1}{2}$.*

The following lemma from Theorem 4.16 [10].

LEMMA 2.5. *Let $f(z)$ be meromorphic in the plane and satisfy*

$$\limsup_{r\to\infty}\frac{\log^+ n(r,f)}{\log r} = k < \rho = \min\{\mu(f), \frac{1}{2}\}.$$

Then there exists a line L approaching to infinity such that

$$\liminf_{|z|\to\infty}\frac{\log\log|f(z)|}{\log|z|} \geq \rho$$

on L.

3. The proof of Theorem 1.1

By Theorem 5.1 in [2], we have

$$\limsup_{r\to\infty}\frac{\log^+\log^+ T(r,f)}{\log r} \leq \sigma(A). \tag{3.1}$$

Assume that $\sigma(h) > \sigma(A)$. Then there is a small number $\epsilon > 0$, such that

$$\sigma(h) > \sigma(A) + \epsilon.$$

There exists an unbounded sequence $\{r_n\}, r_n \to \infty$ as $n \to \infty$, such that

$$T(r_n, h) > r_n^{\sigma(A)+\epsilon}.$$

Take $r \in [r_n, 2r_n]$, and then

$$\begin{aligned} T(r,h) &\geq T(r_n,h) \\ &> r_n^{\sigma(A)+\epsilon} \\ &= 2^{-\sigma(A)-\epsilon}(2r_n)^{\sigma(A)+\epsilon} \\ &\geq 2^{-\sigma(A)-\epsilon}\ r^{\sigma(A)+\epsilon}. \end{aligned}$$

When r is sufficiently large, we have

$$T(r, A) < r^{\sigma(A)+\frac{\epsilon}{2}}.$$

And then

$$T(r,h) > T(r,A), \qquad r \in \bigcup_{n=N}^{\infty} [r_n, 2r_n]$$

where N is some positive integer. Since h is entire, by Nevanlinna theory, we obtain

$$T(r,h'') < 2T(r,h), \qquad r \notin E$$

and

$$T(r,h'^2) < 3T(r,h), \qquad r \notin E$$

where E is a possible set of values r of finite linear measure. Because $\bigcup_{n=N}^{\infty}[r_n, 2r_n]$ has infinite linear measure, we deduce that there is a sequence

$$\{\bar{r}_k\} \subset \bigcup_{n=N}^{\infty} [r_n, 2r_n] \backslash E,$$

$\bar{r}_k \to \infty$ as $k \to \infty$, such that

$$T(\bar{r}_k, h'^2) + T(\bar{r}_k, h'') + T(\bar{r}_k, A) < 6T(\bar{r}_k, h).$$

Substituting $f(z) = g(h(z))$ into (1.1), we have

$$g''(h)h'^2 + g'(h)h'' + Ag(h) \equiv 0.$$

By Lemma 2.1, we obtain that there exist not all identically zero polynomials Q_0, Q_1, Q_2 such that

$$g(z)Q_0(z) + g'(z)Q_1(z) + g''(z)Q_2(z) \equiv 0.$$

By Lemma 2.2, we get $\mu(g) > 0$. $g(z)$ has at most finitely many poles. Let

$$g(z) = \frac{H(z)}{Q(z)},$$

where $H(z)$ is entire, and $Q(z)$ is a nonzero polynomial. Obviously $\sigma(H) = \sigma(g)$, $\mu(H) = \mu(g)$ and $T(r, Q(h)) = o\{T(r, H(h))\}$. Since $H(z)$ has positive lower growth order, there is a positive number $\zeta > 0$, such that for all $r > 0$,

$$\log M(r, H) > r^{\zeta}.$$

By a result of Pòlya [6], there is a number $c (0 < c < 1)$, such that for all $r > 0$,

$$\begin{aligned} \log\log\log M(r, H(h)) &\geq \log\log\log M(cM(\tfrac{r}{2}, h), H) \\ &> \log\log M(\tfrac{r}{2}, h) - O(1). \end{aligned}$$

By (3.1), we have

$$\sigma(h) = \limsup_{r\to\infty} \frac{\log\log M(\frac{r}{2}, h)}{\log r} \leq \limsup_{n\to\infty} \frac{\log\log\log M(r, g(h))}{\log r} \leq \sigma(A).$$

This is a contradiction to our assumption. So $\sigma(h) \leq \sigma(A)$. The proof is complete.

4. Proof of Theorem 1.2

Proof. If $g(z)$ has at least one pole, then set

$$g(z) = \frac{G(z)}{H(z)},$$

where $G(z), H(z)$ are nonconstant entire and $H(z) \not\equiv 0$ is the canonical product of poles of $g(z)$ if $g(z)$ has infinitely many poles, if $g(z)$ has at most finitely many poles, $H(z)$ is a nonzero polynomial. Since every pole of $f(z)$ is a pole of $A(z)$, noting that $\sigma(A) < \infty$, by Lemma 2.3, $\lambda(H) = \sigma(H) = 0$. As a is a deficient value of $g(z)$, 0 is the deficient value of $G(z) - aH(z)$, by Lemma 2.4, $\mu(G) = \mu(G - aH) > \frac{1}{2}$, so does $\mu(g)$.

If $g(z)$ is entire, by Lemma 2.4, $\mu(g) > \frac{1}{2}$.

From $\mu(g) > \frac{1}{2}$, we have $\mu(g(h)) > \frac{1}{2}$. Note that

$$\limsup_{r\to\infty} \frac{\log^{+} n(r, f)}{\log r} \leq \sigma(A) < \frac{1}{2}.$$

By Lemma 2.5, there exists a path L tending to ∞, such that

$$\liminf_{|z|\to\infty} \frac{\log\log|f(z)|}{\log|z|} \geq \frac{1}{2}, \quad z \in L.$$

Theorem 1.2 follows. □

5. Remarks

The factorization $f(z) = g(h(z))$, satisfying (1.1), may have kinds of forms of the factorization, in the sense of the growth order of the left or right factor of the factorization.

(1) $\sigma(f) < \infty$ may occure. For example, $f(z) = e^{2z} + 1$ solves

$$f'' - \frac{4e^{2z}}{e^{2z}+1} f = 0,$$

and $f(z) = (z^2+1) \circ e^z$ is pseudo prime.

(2) The left (right) factor, of zero order, may occure. Take

$$g(z) = h(z) = \sum_{n=1}^{\infty} e^{-n(\log n)^2} z^n,$$

and

$$A(z) = -\left(\frac{d^2 g(h)}{dh^2}\left(\frac{dh}{dz}\right)^2 + \frac{dg(h)}{dh}\frac{d^2 h}{dz^2}\right)(g(h(z)))^{-1},$$

then $f(z) = g(h(z))$ solves

$$f'' + A(z) f = 0.$$

But Bake [1] showed that $\sigma(f)$ is finite, nonzero. So is $\sigma(A)$.

(3) If $\sigma(A) = 0$, we can find examples to show that $f(z)$, of zero order, is factorizable. For example, take

$$g(z) = h(z) = \sum_{n=0}^{\infty} e^{-n^2} z^n,$$

then Bake [1] showed that $\sigma(f) = \sigma(g(h)) = 0$. $f(z)$ is a solution of

$$f'' - \left(\frac{d^2 g(h)}{dh^2}\left(\frac{dh}{dz}\right)^2 + \frac{dg(h)}{dh}\frac{d^2 h}{dz^2}\right)(g(h(z)))^{-1} f = 0.$$

(4) For the case $\lambda(f) < \infty$, Zheng and He [12] studied that $f(z) = \psi(e^z)e^{\Phi(z)+dz}$ solves

$$w'' + A(e^z)w = 0,$$

where $A(x) = \sum_{j=0}^{p} b_j x^j, p_p \neq 0, p \leq 2$, d is not a rational number, $\psi(\xi)$ is a polynomial having at least a nonzero simple zero, and $\sigma(\Phi(z)) < \infty$, and obtained that $f(z)$ is prime.

If d is a rational number, or $\psi(\xi)$ is a rational function, we may have a factorizable solution $f(z)$, with $0 < \lambda(f) < \infty$. For example,

$$f(z) = (e^{-z} + 1)e^{\frac{1}{2}e^z + z} = (z+1)e^{\frac{1}{2}z} \circ e^z$$

solves

$$f'' - \frac{1}{4}e^z(e^z + 6)f = 0.$$

Obviously, $\lambda(f) = 1$.

(5) $\sigma(g) = \infty$ may occure. For example, let

$$h(z) = \sum_{n=1}^{\infty} e^{-n(\log n)^2} z^n,$$

and $g(z) = e^{h(z)}$. $f(z) = g(h(z))$ solves

$$f'' - (h''(z) \cdot h'(h(z)) + h'^2(z) \cdot h''(h(z)) + (h'(z) \cdot h''(h(z)))^2)f = 0.$$

By (2), $h''(z) \cdot h'(h(z)) + h'^2(z) \cdot h''(h(z)) + (h'(z) \cdot h''(h(z)))^2$ has finite order. But $\sigma(g) = \sigma(e^h) = \infty$. This example shows that the left factor is more different than the right factor shown in Theorem 1.1 or Corollary.

Theorem 1.1 was proved in [9].

ACKNOWLEGEMENT. In final, the author would like to thank the referee's help.

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