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ON PRIME LEFT(RIGHT) IDEALS OF GROUPOIDS-ORDERED GROUPOIDS

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Abstract. Recently, Kehayopulu and Tsingelis studied for prime ideals of groupoids-ordered groupoids. In this paper, we give some results on prime left(right) ideals of groupoid-ordered groupoid. These results are generalizations of their results.

If $(G, \cdot, \leq)$ is an ordered groupoid, a non-empty subset A of G is called a *left* (resp. *right*) *ideal* of G if 1) $GA \subseteq A$ (resp. $AG \subseteq A$) and 2) $a \in A, b \leq a$ for $b \in G$ implies $b \in A$ ([2-4]). If G is a groupoid, a *left* (resp. *right*) *ideal* of G is a non-empty subset A of G such that $GA \subseteq A$ (resp. $AG \subseteq A$). A subset P of a groupoid is said to be *prime* if $ab \in P$ implies $a \in P$ or $b \in P$ (see [5]). A *prime left* (resp. *right*) *ideal* of a groupoid (ordered groupoid) is prime as left (resp. right) ideals ([2, 3]). If $(G, \cdot, \leq)$ is an ordered groupoid, a *zero* of G is an element 0 of G such that $0x = x0 = 0$ and $0 \leq x$ for every $x \in G$ ([1]). If G is a groupoid, a *zero* of G is an element 0 of G such that $0x = x0 = 0$ for every $x \in G$.

Recently, Kehayopulu and Tsingelis gave some results for prime ideals of groupoids.

In this paper, we give analogous results for prime left(right) ideals of groupoids. These results are generalizations of the results of Kehayopulu and Tsingelis.

Proposition 1. *Let G be a groupoid (resp. ordered groupoid) and $\mathcal{C}$ a chain (under set inclusion) of prime left ideals of G. If $\bigcap_{L \in \mathcal{C}} L$ is non-empty, then it is a prime left ideal of G.*

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Proof. Since $\bigcap_{L\in\mathcal{C}} L$ is non-empty and

$$G\left(\bigcap_{L\in\mathcal{C}} L\right) = \bigcap_{L\in\mathcal{C}} (GL) \subseteq \bigcap_{L\in\mathcal{C}} L,$$

the set $\bigcap_{L\in\mathcal{C}} L$ is a left ideal of G.

Let $ab \in \bigcap_{L\in\mathcal{C}} L$ for $a, b \in G$. Suppose that $a \notin \bigcap_{L\in\mathcal{C}} L$ and $b \notin \bigcap_{L\in\mathcal{C}} L$. Then there exist $L_1 \in \mathcal{C}$ and $L_2 \in \mathcal{C}$ such that $a \notin L_1$ and $b \notin L_2$. Since $ab \in L_1$, $a \notin L_1$ and L_1 is prime, we get $b \in L_1$. Since $\mathcal{C}$ is a chain and $b \in L_1 \setminus L_2$, we get $L_2 \subseteq L_1$. Since $ab \in L_2$ and $b \notin L_2$, we get $a \in L_2 \subseteq L_1$. Thus $a \in L_1$, which is impossible. Hence if $ab \in \bigcap_{L\in\mathcal{C}} L$ for $a, b \in G$, then $a \in \bigcap_{L\in\mathcal{C}} L$ or $b \in \bigcap_{L\in\mathcal{C}} L$. Therefore $\bigcap_{L\in\mathcal{C}} L$ is a prime left ideal of G. □

By the similar method we have the following proposition.

PROPOSITION 2. *Let G be a groupoid (resp. ordered groupoid) and $\mathcal{C}$ a chain (under set inclusion) of prime right ideals of G. If $\bigcap_{R\in\mathcal{C}} R$ is non-empty, then it is a prime right ideal of G.*

From proposition 1 and 2, we have the following corollary.

COROLLARY 1. *[3] Let G be a groupoid (resp. ordered groupoid) and $\mathcal{B}$ a chain of prime ideals of G. If $\bigcap_{B\in\mathcal{B}} B$ is non-empty, then it is a prime ideal of G.*

PROPOSITION 3. *Let G be a groupoid (resp. ordered groupoid) and K a non-empty subset of G. If L is a prime left ideal containing K, then there exists a prime left ideal L^* of G having the properties:*
1) $L^* \subseteq L$.
2) For each prime left ideal T of G such that $K \subseteq T \subseteq L^*$, we have $T = L^*$.

Proof. Let $\mathcal{A} := \{A \mid A$ is a prime left ideal of G such that $K \subseteq A \subseteq L\}$. Since $L \in \mathcal{A}$, we have $\mathcal{A} \neq \emptyset$. Now we define the relation on $\mathcal{A}$ as follows:

$$\preccurlyeq := \{(A, B) \in \mathcal{A} \times \mathcal{A} \mid B \subseteq A\}$$

Then the family $\mathcal{A}$ is an ordered set with the relation "$\preccurlyeq$." Let $(\mathcal{B}, \subseteq)$ be a chain in $\mathcal{A}$. Since K is non-empty and $K \subseteq B$ for any $B \in \mathcal{B}$, we have $\bigcap_{B\in\mathcal{B}} B$ is non-empty. By Proposition 1, the set $\bigcap_{B\in\mathcal{B}} B$ is a prime left ideal of G. Moreover, $K \subseteq \bigcap_{B\in\mathcal{B}} B \subseteq L$. Thus $\bigcap_{B\in\mathcal{B}} B$ is an upper bound of $\mathcal{B}$. By Zorn's Lemma, the set $\mathcal{A}$ has a maximal element, say L^*. Since $L^* \in \mathcal{A}$, we have $L^* \subseteq L$.

Let T be a prime left ideal of G such that $K \subseteq T \subseteq L^*$. Then $K \subseteq T \subseteq L$, and so $T \in \mathcal{A}$. Since $L^* \in \mathcal{A}$ and $T \subseteq L^*$, we have $L^* \preccurlyeq T$. And since $L^* \preccurlyeq T$ and L^* is a maximal in $\mathcal{A}$, we have $L^* = T$. □

By the similar method we have the following proposition.

Proposition 4. *Let G be a groupoid (resp. ordered groupoid) and K non-empty subset of G. If R is a prime right ideal containing K, then there exists a prime right ideal R^* of G having the properties:*
1) $R^* \subseteq R$.
2) For each prime right ideal T of G such that $K \subseteq T \subseteq R^*$, we have $T = R^*$.

From proposition 3 and 4, we have the following corollary.

Corollary 2. *[3] Let G be a groupoid (resp. ordered groupoid) and K non-empty subset of G. If P is a prime ideal containing K, then there exists a prime ideal I^* of G having the properties:*
1) $P^* \subseteq P$.
2) For each prime ideal T of G such that $K \subseteq T \subseteq P^*$, we have $T = P^*$.

Proposition 5. *Let G be a groupoid (resp. ordered groupoid) with a zero and L a prime left ideals of G. Then there exists a minimal prime left ideal L^* of G such that $L^* \subseteq L$.*

Proof. Since $0 = 0l \in GL \subseteq L$ for a zero 0 of G and $l \in L$, we get $\{0\} \subseteq L$. By Proposition 3, there exists a prime left ideal L^* of G having the properties:

1) $L^* \subseteq L$.

2) For each prime left ideal T of G such that $\{0\} \subseteq T \subseteq L^*, T = L^*$.

Thus the set L^* is a minimal prime left ideal of G. □

By the similar method we have the following proposition.

Proposition 6. *Let G be a groupoid (resp. ordered groupoid) with a zero and R a prime right ideals of G. Then there exists a minimal prime right ideal R^* of G such that $R^* \subseteq R$.*

From proposition 5 and 6, we have the following corollary.

Corollary 3. *[3] Let G be a groupoid (resp. ordered groupoid) with a zero and P a prime ideals of G. Then there exists a minimal prime ideal P^* of G such that $P^* \subseteq P$.*

Proposition 7. *Let G be a groupoid (resp. ordered groupoid) and $\{L_i \mid i \in \Lambda\}$ a non-empty family of prime left ideals of G. Then the set $\bigcup_{i\in\Lambda} L_i$ is a prime left ideal of G.*

Proof. Since every L_i is non-empty, $\bigcup_{i\in\Lambda} L_i$ is non-empty. And

$$G\left(\bigcup_{i\in\Lambda} L_i\right) = \bigcup_{i\in\Lambda}(GL_i) \subseteq \bigcup_{i\in\Lambda} L_i.$$

Thus $\bigcup_{i\in\Lambda} L_i$ is a left ideal of G.

Let $ab \in \bigcup_{i\in\Lambda} L_i$ for $a, b \in G$. Then $ab \in L_j$ for some $j \in \Lambda$. Since L_j is prime, $a \in L_j$ or $b \in L_j$. Thus $a \in \bigcup_{i\in\Lambda} L_i$ or $b \in \bigcup_{i\in\Lambda} L_i$. Therefore $\bigcup_{i\in\Lambda} L_i$ is a prime left ideal of G. □

By the similar method we have the following proposition.

Proposition 8. *Let G be a groupoid (resp. ordered groupoid) and $\{R_i \mid i \in \Lambda\}$ a non-empty family of prime right ideals of G. Then the set $\bigcup_{i\in\Lambda} R_i$ is a prime right ideal of G.*

From proposition 7 and 8, we have the following corollary.

Corollary 4. *[3] Let G be a groupoid (resp. ordered groupoid) and $\{P_i \mid i \in \Lambda\}$ a non-empty family of prime ideals of G. Then the set $\bigcup_{i\in\Lambda} P_i$ is a prime ideal of G.*

Proposition 9. *Let G be a groupoid (resp. ordered groupoid), L a left ideal of G and R a right ideal of G such that $L \bigcap R$ be prime in G. Then $L \subseteq R$ or $R \subseteq L$.*

Proof. Assume that $L \nsubseteq R$. Then there exists $a \in L \setminus R$. Let $b \in R$. Since L is a left ideal and R is a right ideal of G,

$$ba \in RL \subseteq GL \subseteq L \ \text{ and } \ ba \in RL \subseteq RG \subseteq R.$$

Thus $ba \in L \bigcap R$. Since $L \bigcap R$ is prime, $b \in L \bigcap R$ or $a \in L \bigcap R$. Since $a \notin R$, $b \in L \bigcap R$. Hence $R \subseteq L \bigcap R$. Therefore $R \subseteq L$. □

Remark. In proposition 9, $R \bigcap L$ need not be ideal, left(right) ideal.

From proposition 9, we have the following corollary.

Corollary 5. *[3] Let G be a groupoid (resp. ordered groupoid), P_1, P_2 ideals of G such that $P_1 \bigcap P_2$ be a prime ideal of G. Then $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$.*

Proposition 10. *Let G be a groupoid (resp. ordered groupoid) and L a prime left ideal of G and R a prime right ideal of G. Then the following are equivalent:*
1) $L \subseteq R$ or $R \subseteq L$.
2) $L \bigcap R$ is prime in G.

Proof. 1) $\Rightarrow$ 2). If $L \subseteq R$, then $L \bigcap R = L$. Thus $L \bigcap R$ is prime in G. If $R \subseteq L$, $L \bigcap R = R$. Hence $L \bigcap R$ is prime in G.

2) $\Rightarrow$ 1). It is obvious by Proposition 9. □

From proposition 10, we have the following corollary.

Corollary 6. *[3] Let G be a groupoid (resp. ordered groupoid) and P_1, P_2 prime ideals of G. The following are equivalent:*
1) $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$.
2) $P_1 \bigcap P_2$ is a prime ideal of G.

PROPOSITION 11. *Let S be a groupoid (resp. ordered groupoid), I an ideal of S and P a prime left ideal of I. Then P is a left (resp. right) ideal of S.*

Proof. We note that $\emptyset \neq P \subseteq I$. Let $a \in S$, and $b \in P$. Then $b \in I$ and $ab \in SI \subseteq I$. Since $aba \in SIS \subseteq I$, we have $(ab)^2 = (aba)b \in IP \subseteq P$. Since P is a prime left ideal of I, we get $ab \in P$. Thus $SP \subseteq P$, and so P is a left ideal of S. $\square$

By the similar method we have the following proposition.

PROPOSITION 12. *Let S be a groupoid (resp. ordered groupoid), I an ideal of S and P a prime right ideal of I. Then P a left (resp. right) ideal of S.*

From proposition 11 and 12, we have the following corollary.

COROLLARY 7. *[3] Let S be a groupoid (resp. ordered groupoid), I an ideal of S and P a prime of I. Then P is a ideal of S.*

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