Kangweon-Kyungki Math. Jour. **13** (2005), No. 2, pp. 123–126

ESTIMATION OF THE BIPLANAR CROSSING NUMBERS

Ki Sung Park

Abstract. This paper is a sequel to our earlier research on biplanar drawings [4] and biplanar crossing numbers [3]. The biplanar crossing number $cr_2(G)$ of a graph G is $min\{cr(G_1) + cr(G_2)\}$, where cr is the planar crossing number and $G = G_1 \cup G_2$. In this paper we show that $cr_2(G) \leq \frac{3}{8} cr(G)$.

1. Introduction

Recall that a graph G is *biplanar*, if one can write $G = G_1 \cup G_2$, where G_1 and G_2 are planar graphs. Owen [2] introduced the biplanar crossing number of a graph G, that we denote by $cr_2(G)$. One can define $cr_k(G) = min\{cr(G_1) + cr(G_2) + \cdots + cr(G_k)\}$, similarly for any $k \geq 2$, making G a union of k subgraphs, but perhaps $k = 2$ is more relevant for VLSI for the following reason: one always can realize $cr_2(G)$ by drawing the edges of G_1 and G_2 one two different sides of the same plane, while identical vertices of G_1 and G_2 are placed to identical locations on the plane.

This article deals with the problem of estimating on the biplanar crossing number.

2. Preliminaries

Let $cr(G)$ denote the standard crossing number of the graph G, i.e. the minimum number of crossings of its edges over all possible drawings of G in the plane, under the usual rules for drawings for crossing numbers [6]. For instance, a graph is planar if and only if its crossing number is zero.

Received January 31, 2005.
2000 Mathematics Subject Classification: 05C90.
Key words and phrases: crossing number, biplanar crossing number .

Definition. For the planar graphs G_1 and G_2, we denote the *biplanar crossing number* of a graph G by $cr_2(G)$. Here

$$cr_2(G) = min\{cr(G_1) + cr(G_2)\},$$

where the minimum is taken over all unions $G = G_1 \cup G_2$.

A biplanar drawing of a graph G means drawings of two subgraphs, G_1 and G_2, of G, on two disjoint planes under the usual rules for drawings for crossing numbers, such that $G = G_1 \cup G_2$.

Definition. The thickness of the planar graphs G,

$$\Theta(G) = min\{k : G = G_1 \cup \cdots \cup G_k\},$$

where $G_1, \cdots, G_k$ are planar.

By definition, $cr_2(G) = 0$ if and only if $\Theta(G) \leq 2$, i.e. G is biplanar.

Euler's polyhedral formula states that $v - e + f = 2$, where v is the number of vertices, e is the number of edges and f is the number of faces. Any face is bounded by at least three edges and every edge touches at most two faces. Then $e - v + 2 = f \leq \frac{2}{3}e$ and $e \leq 3v - 6$ if $v \geq 3$, which is true for all planar graphs with m edges and n vertices.

For any graph G with m edges and n vertices, we consider a diagram of G which has exactly $cr(G)$ crossings. Each of these crossings can be removed by removing an edge from G. Thus we can find a graph with at least $m - cr(G)$ edges and n vertices with no crossings, and is thus a planar graph. From the above results follows $m - cr(G) \leq 3n - 6$ and hence

$$cr_2(G) \geq m - 6n + 12$$

3. The estimates of the biplanar crossing number

Theorem 3.1. *Let G be a graph with order n and size m. For all $c > 6$, if $m \geq cn$, then*

$$cr_2(G) \geq \frac{c-6}{c^3}\frac{m^3}{n^2}.$$

Proof. Let D be a nice drawing that realize $cr_2(G)$, take $p = \frac{cn}{m} \leq 1$, pick each vertex in D with probability p, which will result in a random subdrawing D', then

$$cr_2(D') \geq m(D') - 6n(D') + 12,$$

take expectations, we have

$$p^4 cr_2(G) \geq mp^2 - 6np,$$

and hence

$$cr_2(G) \geq \frac{m}{p^2} - \frac{6n}{p^3} = \frac{m^3}{c^2n^2} - \frac{6nm^3}{c^3n^3} = \frac{c-6}{c^3}\frac{m^3}{n^2}.$$

□

THEOREM 3.2. *For all graph G, $cr_2(G) \leq \frac{3}{8}cr(G)$.*

Proof. Without loss of generality we may assume that the input drawing is *nice*, i.e. any two edges of G cross at most once, edges do not "touch", and edges sharing an endvertex do not cross; since all these assumptions do not change the crossing number.

Splitting Algorithm. INPUT any nice drawing D of G in the plane. Let $cr(D)$ denote the number of crossings in this drawing. Consider a random bipartition (U, W) of $V(G)$: for every vertex, independently toss a fair coin, and if Head is obtained, add it to U, otherwise to V. Now any crossing in D occurs in 6 possible forms, according to which classes the endpoints of the crossing edges belong to:

it is a crossings of UU, UU edges with probability $\frac{1}{16}$
it is a crossings of WW, WW edges with probability $\frac{1}{16}$
it is a crossings of UW, UW edges with probability $\frac{1}{4}$
it is a crossings of UU, WW edges with probability $\frac{1}{8}$
it is a crossings of UU, UW edges with probability $\frac{1}{4}$
it is a crossings of WW, UW edges with probability $\frac{1}{4}$

Draw in the first plane the subdrawings spanned by U and spanned by W, draw in the second plane the subdrawing of edges connecting U to W. In the second plane we have the UW, UW type crossings, in expectation $\frac{1}{4}cr(D)$. In the first plane we have the UU, UU and WW, WW type crossings, and also the UU, WW type crossings. However, we easily get rid of the latter type of crossing, by a translation of the W point set and its induced edges to sufficiently far away. Therefore, the first plane has in expectation $\frac{1}{8}cr(D)$ crossings. □

Unfortunately, not any kind of converse of Theorem 3.2 can be true, as the following theorem shows:

THEOREM 3.3. *([5]) There are graphs G with crossing number $cr(G) = \Theta(m^2)$ (i.e. as large as possible) and biplanar crossing number $cr_2(G) = \Theta(\frac{m^3}{n^2})$ (i.e. as small as possible), for any $m = m(n)$, where $\frac{m}{n}$ exceeds a certain absolute constant.*

OPEN PROBLEM. *What is the infimum c^* of those constants c, for which $cr_2(G) \leq c \cdot cr(G)$ holds for every graph G ?*

In [2], Owens came up with a conjectured cr_2-optimal drawing of the complete graph K_n which has about $\frac{7}{24}$ of the crossings of a conjectured cr-optimal drawing of K_n. This might give some basis to conjecture that $c^* \leq \frac{7}{24}$.

References

1. L. W. Beineke, *Biplanar graphs: a survey*, Computers. Math. Applic. 34 (1997), 1–8.
2. A. Owens, *On the biplanar crossing number*, IEEE Transactions on Circuit Theory CT–18 (1971), 277–280.
3. O. Sykora, L. A. Szekely, I. Vrto, *Crossing numbers and biplanar crossing numbers* I*: a survey of problems and results, submitted.*
4. O. Sykora, L. A. Szekely, I. Vrto, *Two counterexamples in graph drawing, submitted.*
5. O. Sykora, L. A. Szekely, I. Vrto, *Crossing numbers and biplanar crossing numbers* II*: using the probabilistic method, submitted.*
6. F. Shahrokhi, O. Sykora, L. A. Szekely, I. Vrto, *Crossing numbers: bounds and applications*, Bolyai Society Mathematical Studies 6 (1997), 179–206.

Department of Mathematics
Kangnam University
Yongin 449–702, Korea
E-mail: parkks@kangnam.ac.kr