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STABILITY OF POSITIVE PERIODIC NUMERICAL SOLUTION OF AN EPIDEMIC MODEL

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Abstract. We study an age–dependent *s*-*i*-*s* epidemic model with spatial diffusion. The model equations are described by a nonlinear and nonlocal system of integro–differential equations. Finite difference methods along the characteristics in age-time domain combined with finite elements in the spatial variable are applied to approximate the solution of the model. Stability of the discrete periodic solution is investigated.

1. Introduction.

In recent years there has been an interest in modelling the effects of spatial diffusion on age–dependent population models. A general account of the spatial dispersion of biological populations was given in the classic work of Skellam [9]. Later spatial diffusion was introduced into age–dependent population models by Gurtin [3] and Rotemberg [8] and then has been investigated by several authors [1,4,5,6,7]. In this paper we consider a numerical method to approximate the solution of an *s*-*i*-*s* epidemic model within an age–structured population dynamics with "random" diffusion. Namely, we consider a population density in

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steady state, which is governed by the following system of integro-differential equations:

$$
\begin{aligned}
&\frac{\partial v}{\partial a} - k\Delta v + \mu(a)v = 0, \quad x \in \Omega, \quad 0 \le a < a_\dagger, \quad t \ge 0, \\
&v(x,0) = \int_0^{a_\dagger} \beta(a)v(x,a)da, \quad x \in \Omega, \\
&v(x,a) = 0, \quad x \in \partial\Omega, \quad 0 \le a < a_\dagger.
\end{aligned} \tag{1.1}
$$

Here the maximum age $a_\dagger$ is assumed to be finite. The function $v(x,a)$ denotes the population density of age $a > 0$ at location x. The non-negative functions μ and β are the age-specific death rate and the age-specific birth rate, respectively. The death rate is assumed to satisfy the following:

$$\int_0^{a_\dagger} \mu(a)\, da = +\infty.$$

In this population we consider the spread of a mild disease. Let $i(x,t,a)$ and $s(x,t,a)$ denote the age specific density of infected and susceptible individuals, respectively. We assume that the disease does not impart immunity and that

$$v(x,a) = s(x,t,a) + i(x,t,a). \tag{1.2}$$

In this model the disease does not significantly affect the death rate and μ is assumed to be the same for all subpopulations. We shall also assume that infected and susceptible individuals interact freely and randomly. Thus the dynamics are governed by the following system of the equations:

$$
\begin{aligned}
&\frac{\partial i}{\partial t} + \frac{\partial i}{\partial a} - k\Delta i + \mu(a)i = \gamma(a)i\{v(x,a) - i\}, \\
&\qquad x \in \Omega, \quad 0 \le a < a_\dagger, \quad t \ge 0, \\
&i(x,t,0) = q\int_0^{a_\dagger} \beta(a)i(x,t,a)da, \quad x \in \Omega, \quad t \ge 0, \\
&i(x,0,a) = i_0(x,a), \quad x \in \Omega, \quad 0 \le a < a_\dagger, \\
&i(x,t,a) = 0, \quad x \in \partial\Omega, \quad 0 \le a < a_\dagger, \quad t \ge 0,
\end{aligned} \tag{1.3}
$$

where $\gamma(a)$ is the force of infection and q the vertical transmission rate, that is, the ratio of infective newborns produced by infectives. We assume that γ is nonnegative and belong to $L^\infty(0, a_\dagger)$, $0 \le q < 1$. For the model to make sense we also assume that $0 \le i_0(x, a) \le v_0(x, a)$, $0 \le a < a_\dagger$.

2. A Numerical Method.

We first define some notations to be used in the paper. For $1 \le q \le \infty$ and m any nonnegative integer, let

$$W^{m,q}(\Omega) = \{f \in L^q(\Omega) \mid D^\alpha f \in L^q(\Omega) \text{ if } |\alpha| \le m\}$$

denote the Sobolev space endowed with the norm

$$\|f\|_{m,q;\Omega} = \left(\sum_{|\alpha| \le m} \|D^\alpha f\|^q_{L^q(\Omega)} \right)^{\frac{1}{q}},$$

with the usual modification for $q = \infty$. Let $H^m(\Omega) = W^{m,2}(\Omega)$ with norm $\|\cdot\|_{m;\Omega} = \|\cdot\|_{m,2;\Omega}$. $H_0^m(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in the norm $\|\cdot\|_{m;\Omega}$, where $\mathcal{D}(\Omega)$ is the set of infinitely differentiable functions with compact support in Ω. To take into account the discretization of age and/or time we shall also find useful the following notations:

$$\|\chi^n\|_{\ell^p(H^r)} = \left(\sum_{j \ge 1} \|\chi_j^n\|^p_{H^r} \Delta t \right)^{1/p},$$

$$\|\chi\|_{\ell^{q,p}(H^r)} = \left(\sum_{n=0}^{N} \|\chi^n\|^q_{\ell^p(H^r)} \Delta t \right)^{1/q},$$

where if $q = \infty$, the sum is replaced by the maximum. We shall use the same notation to indicate the dualities between $H_0^r(\Omega)$ and $H^{-r}(\Omega)$. We now make some assumptions on the data and solutions of the problem (1.3). Observe that the initial age–space distribution i_0 given in (1.3) must be nonnegative for biological reasons. Now, let $T > 0$ be the final time and let $J = [0, a_\dagger) \times [0, T]$. Let Ω be a bounded domain with

C^2–boundary $\partial\Omega$. We shall assume that the initial–boundary value problem (1.3) has a unique solution $i \in C^2(J; H^{2+\varepsilon}(\Omega))$ for some ε, $0 < \varepsilon << 1$. Now using the divergence theorem we arrive at a weak form of $(1.3)_1$, $(1.3)_3$: Seek a map $i : J \to H_0^1(\Omega)$ satisfying

$$\left(\frac{\partial i}{\partial t}+\frac{\partial i}{\partial a}, w\right)+k\Big(\nabla i, \nabla w\Big)+(\mu(a)i, w) = (\gamma(a)i(v(x,a)-i), w), \tag{2.1}$$

where $w \in H_0^1(\Omega)$. We shall discretize (2.1) using a finite difference method of characteristics in the age–time direction and a finite element method for the spatial variable. Let N be a fixed integer, and let $\Delta t = T/N$, $A_\dagger = [a_\dagger/\Delta t]$, and

$$t^n = n\Delta t, \quad 0 \le n \le N, \quad a_j = j\Delta t, \quad 0 \le j < A_\dagger.$$

We define a directional derivative $D\chi$ of χ along the characteristic $t = a$ and a finite difference operator $\overline{D}$ as follows:

$$D\chi(a,t) = \lim_{\Delta t \to 0} \frac{\chi(a+\Delta t, t+\Delta t) - \chi(a,t)}{\Delta t},$$

and, for $j \ge 1$, $n \ge 1$,

$$\overline{D}\chi_j^n = \frac{\chi_j^n - \chi_{j-1}^{n-1}}{\Delta t}.$$

We partition Ω into triangles K and denote by $\mathcal{T}_h$ the resulting mesh and h the mesh size. We also assume that $\{\mathcal{T}_h\}$ is a quasi–uniform family and that the family $\{\mathcal{T}_h\}$ is weakly acute in the sense that

$$\int_\Omega \nabla\phi_i \cdot \nabla\phi_j dx \le 0, \qquad 1 \le i \ne j \le L,$$

where ϕ_i is the canonical piecewise linear function with value 1 at the node x_i and 0 at the remaining nodes, and L is the total number of nodes. Let $V_h \subset H_0^1(\Omega)$ be the set of piecewise linear polynomials. Namely,

$$V_h = \{\phi \in C(\bar{\Omega}) \mid \phi|_K \text{ is linear for each } K \in \mathcal{T}_h\}. \tag{2.2}$$

Let Π_h be the corresponding Lagrange interpolation operator. We consider the following discrete inner product $(\cdot,\cdot)$, which uses the vertex quadrature rule [2]: For $\psi,\ \varphi \in V_h^1$,

$$(\psi,\varphi) := \int_\Omega \Pi_h(\psi\varphi)dx = \sum_{K\in\mathcal{T}_h} \frac{1}{3}\text{meas }(K)\sum_{j=1}^{3}(\psi\varphi)(\mathbf{b}_{K,j}),$$

and for α continuous,

$$(\alpha\nabla\psi,\nabla\varphi) := \sum_{K\in\mathcal{T}_h} \frac{1}{3}\text{meas }(K)\sum_{j=1}^{3}\alpha(\mathbf{b}_{K,j})\nabla\psi|_K\cdot\nabla\varphi|_K,$$

where $\mathbf{b}_{K,j}$, $j=1,2,3$, are vertices of K. We note that $i(x,t,a_\dagger)\equiv 0$. Thus we may assume that $i^n_{A_\dagger,h}\equiv 0$, for $n\ge 1$. Then the approximation scheme we shall analyze is given by seeking $i^n_{j,h}\in V_h$ for $n\ge 1$, $1\le j<A_\dagger$, such that

(2.3)

$$\begin{aligned}
&i^0_{j,h} = P_h i_0(\cdot,a_j), 0\le j<A_\dagger,\\
&\big(\overline{D}i^n_{j,h},w\big)+k\big(\nabla i^n_{j,h},\nabla w\big)+(\mu_j i^n_{j,h},w)+(\gamma_j v_j(\cdot)i^n_{j,h},w)=\\
&\qquad(\gamma_j i^{n-1}_{j,h}(2v_j(\cdot)-i^n_{j,h}),w),\ w\in V_h,\quad 1\le j<A_\dagger,\quad n\ge 1,\\
&i^n_{0,h}=\sum_{l=0}^{A_\dagger}\beta_j i^n_{j,h}\Delta t, n\ge 1,
\end{aligned}$$

where P_h is an interpolant such that $i^0_{j,h}\ge 0$.

3 Discrete maximum principle and comparison results.

If we express $i^n_{j,h}$ in terms of the canonical basis functions $\{\phi_j\}_{1\le j\le L}$, then we notice that the discrete problem yields to compute $i^n_{j,h}$ using $i^{n-1}_{j,h}$, $0\le j<A_\dagger$, for given time level $n-1$. This in turn amounts to solve $(A_\dagger-2)$ elliptic problems for each time level n. It essentially

leads to solve the following linear system for $1 \le j < A_\dagger$, $n \ge 1$:

$$\begin{aligned} &\{M + \Delta t k A + \Delta t \mu_j M + \Delta t \gamma_j D_V^j M + \Delta t \gamma_j D_I^{n-1,j} M\} I^{n,j} = \\ &\qquad M I^{n-1,j-1} + 2\Delta t \gamma_j D_V^j M I^{n-1,j}, \quad 0 \le j < A_\dagger, \quad n \ge 1, \\ &I^{n,0} = \frac{1}{1-\Delta t \beta_0} \sum_{l=1}^{A_\dagger} \Delta t I^{n,l} \beta_l, \quad n \ge 1, \\ &I^{0,j} = P_h i_{j,h}^0, \quad 0 \le j < A_\dagger, \end{aligned} \tag{3.1}$$

where $i_{j,h}^n = \sum_{l=1}^L I_l^{n,j} \phi_l$, $I^{n,j} = (I_1^{n,j}, \cdots, I_L^{n,j})^T$, $M_{rs} = (\phi_r, \phi_s)_h$, $A_{rs} = (\nabla \phi_r, \nabla \phi_s)_h$, $(D_I^{n,j})_{rs} = I_r^{n,j} \delta_{rs}$, $(D_V^j)_{rs} = v_j(x_r)\delta_{rs}$.

We now have the following estimate:

Theorem 3.1. *$0 \le i_{j,h}^n \le v_{j,h}$, for $0 \le j < A_\dagger$, $n \ge 0$.*

We consider the following linear problem:

$$\begin{aligned} &\big(\overline{D} z_{j,h}^n, w\big) + k\big(\nabla z_{j,h}^n, \nabla w\big) + (\mu_j z_{j,h}^n, w) = 0, \\ &w \in V_h, \quad 1 \le j < A_\dagger, \quad n \ge 1, \\ &z_{0,h}^n = \sum_{l=0}^{A_\dagger} \beta_j z_{j,h}^n \Delta t, n \ge 1, \\ &z_{j,h}^0 = v_{j,h} - i_{j,h}^0, 0 \le j < A_\dagger. \end{aligned} \tag{3.2}$$

For problem (3.2), we have the following "discrete maximum principle":

Theorem 3.2.

$$\min\{0, \min_{x\in\Omega_h} \{z_{0,h}^{n-j}, z_{j-n,h}^0\}\} \le z_{j,h}^n \le \max\{0, \max_{x\in\Omega_h}\{z_{0,h}^{n-j}, z_{j-n,h}^0\}\}.$$

Theorem 3.3. *If $z_{0,h}^n \ge C$, for some constant $C > 0$, $x \in \mathcal{O} \subset \Omega_h$, $n \ge N$, then there exists a positive constant $C(A)$ for each $A < A_\dagger$, such that $z_{j,h}^n \ge C(A)$, for $x \in \Omega_h$, $0 \le j < A$, $n \ge N + A_\dagger$, and $C(A) \to 0$, as $A \to A_\dagger$.*

Following comparison results also hold:

THEOREM 3.4. *(1)* $z^0_{j,h} \leq \underline{z}^0_{j,h}$ *implies that* $z^n_{j,h} \leq \underline{z}^n_{j,h}$.
(2) If $\mu_j \geq \underline{\mu}_j$, $\beta_j \leq \underline{\beta}_j$, $f_j \leq \underline{f}_j$, *and* $\gamma_j \leq \underline{\gamma}_j$, *then* $z^n_{j,h} \leq \underline{z}^n_{j,h}$.

Using the discrete maximum principle and comparison results, we have the following maximum principle for the problem (2.3):

THEOREM 3.5. *If* $i^n_{0,h} \geq C$, *for some constant* $C > 0$, $x \in \mathcal{O} \subset \Omega_h$, $n \geq N$, *then, there exists a positive constant* $C(A)$ *for each* $A < A_\dagger$, *such that* $i^n_{j,h} \geq C(A)$, *for* $x \in \Omega_h$, $0 \leq j < A$, $n \geq N + A_\dagger$, *and* $C(A) \to 0$, *as* $A \to A_\dagger$.

We need the following Lemma for the main result (Theorem 4.7):

LEMMA 3.6. *Let* $\mathcal{O}_h$ *be a nonempty subset of* Ω_h. *If* (λ^1_h, w^1_h) *is the smallest eigen-pair of the following:*

$$\begin{cases} (\nabla w^1_h, \nabla \chi) + (\lambda w^1_h, \chi) = 0, & \text{in } \mathcal{O}_h, \\ w^1_h = 0, \qquad \text{on } \partial\mathcal{O}_h, \end{cases}$$

then $(\lambda^1_h, w^1_h) > 0$.

We are now ready to state the main result of this section.

THEOREM 3.7. *Assume that the initial data is fertile. Then* $i^n_{j,h} > 0$, *for* $x \in \Omega_h$, $1 \leq j < A_\dagger$, $n \geq N$, *for sufficiently large* N. *More precisely, there exist a positive constant* $C(A, N)$ *for each* $A < A_\dagger$, *such that* $i^n_{j,h} > C(A, N)$, *for* $x \in \Omega_h$, $1 \leq j \leq A$, $n \geq N$, *for sufficiently large* N, *and* $C(A, N) \to 0$ *as* $A \to A_\dagger$.

4. Steady state problem.

We now consider the following steady state problem:

$$\begin{aligned} &\frac{\partial i}{\partial a} - k\Delta i + \mu(a)i = \gamma(a)i\{v(x,a) - i\}, \quad x \in \Omega, \quad 0 \leq a < a_\dagger, \\ &i(x,0) = q\int_0^{a_\dagger} \beta(a)i(x,a)da, \qquad x \in \Omega, \\ &i(x,a) = 0, \quad x \in \partial\Omega, \qquad 0 \leq a < a_\dagger, \end{aligned} \tag{4.1}$$

and corresponding discrete problem of the following:

$$(4.2)\quad \begin{aligned} \big(\overline{D}i_{j,h},w\big)+k\big(\nabla i_{j,h},\nabla w\big)+(\mu_j i_{j,h},w) =&(\gamma_j i_{j,h}(v_j(\cdot)-i_{j,h}),w),\\ & w\in V_h,\quad 1\le j<A_\dagger,\\ i_{0,h}=\sum_{l=0}^{A_\dagger}\beta_j i_{j,h}\Delta t. \end{aligned}$$

We note that problem (4.2) is nonlinear and nonlocal, and an iteration is required to solve them. Now let $\underset{\sim}{i}_{j,h}\le i_{j,h}\le \widetilde{i}_{j,h}$ be the lower and upper solutions of (4.2). Consider the following iteration with an initial guess $\underset{\sim}{i}^{(0)}_{j,h}=\underset{\sim}{i}_{j,h}$, ($\widetilde{i}^{(0)}_{j,h}=\widetilde{i}_{j,h}$, respectively).

$$(4.3)\quad \begin{aligned} &\underset{\sim}{i}^{(0)}_{j,h}=\underset{\sim}{i}_{j,h},\quad 1\le j<A_\dagger,\\ &\big(\overline{D}\,\underset{\sim}{i}^{(k)}_{j,h},w\big)+k\big(\nabla \underset{\sim}{i}^{(k)}_{j,h},\nabla w\big)+(\mu_j \underset{\sim}{i}^{(k)}_{j,h},w)+(\gamma_j v_j(\cdot)\underset{\sim}{i}^{(k)}_{j,h},w)=\\ &\qquad (\gamma_j \underset{\sim}{i}^{(k-1)}_{j,h}(2v_j(\cdot)-\underset{\sim}{i}^{(k-1)}_{j,h}),w),\quad w\in V_h,\quad 1\le j<A_\dagger,\\ &\underset{\sim}{i}^{(k)}_{0,h}=\sum_{l=0}^{A_\dagger}\beta_j \underset{\sim}{i}^{(k)}_{j,h}\Delta t, \end{aligned}$$

and

$$(4.4)\quad \begin{aligned} &\widetilde{i}^{(0)}_{j,h}=\widetilde{i}_{j,h},\quad 1\le j<A_\dagger,\\ &\big(\overline{D}\widetilde{i}^{(k)}_{j,h},w\big)+k\big(\nabla \widetilde{i}^{(k)}_{j,h},\nabla w\big)+(\mu_j \widetilde{i}^{(k)}_{j,h},w)+(\gamma_j v_j(\cdot)\widetilde{i}^{(k)}_{j,h},w)=\\ &\qquad (\gamma_j \widetilde{i}^{(k-1)}_{j,h}(2v_j(\cdot)-\widetilde{i}^{(k-1)}_{j,h}),w),\quad w\in V_h,\quad 1\le j<A_\dagger,\\ &\widetilde{i}^{(k)}_{0,h}=\sum_{l=0}^{A_\dagger}\beta_j \widetilde{i}^{(k)}_{j,h}\Delta t. \end{aligned}$$

Noting that $G(i)=\gamma_j i(2v_j-i)$ is monotone increasing in i, we see that by comparison result,

$$0\le \widetilde{i}^{(1)}_{j,h}\le \widetilde{i}^{(0)}_{j,h}\le \widetilde{i}_{j,h}.$$

Thus $\widetilde{i}^{(k)}_{j,h}$ ($\underset{\sim}{i}^{(k)}_{j,h}$, respectively) is decreasing (increasing, respectively) in the iteration level k . Let $\bar{i}_{j,h} = \lim_{k\to\infty} \widetilde{i}^{(k)}_{j,h}$ and $\underline{i}_{j,h} = \lim_{k\to\infty} \underset{\sim}{i}^{(k)}_{j,h}$. Then $\underset{\sim}{i}^{(k)}_{j,h} \le i_{j,h} \le \widetilde{i}^{(k)}_{j,h}$.

Based on the monotone property of G, we have the following asymptotic behavior of the solution of the time dependent problem:

Theorem 4.1. *Let $\underset{\sim}{i}_{j,h}$ and $\widetilde{i}_{j,h}$ be a pair of lower and upper solutions of* (4.2) *with $\widetilde{i}_{j,h} \ge \underset{\sim}{i}_{j,h} \ge 0$ and let $\underset{\sim}{i}^{n}_{j,h}$ and $\widetilde{i}^{n}_{j,h}$ be the solutions of* (2.3) *with $\underset{\sim}{i}^{0}_{j,h} = \underset{\sim}{i}_{j,h}$ and $\widetilde{i}^{0}_{j,h} = \widetilde{i}_{j,h}$, respectively. Then, we have the following:*

(1) $\widetilde{i}^{n}_{j,h}$ ($\underset{\sim}{i}^{n}_{j,h}$, respectively) is monotone decreasing (increasing, respectively) in n and $\widetilde{i}^{n}_{j,h} \ge \underset{\sim}{i}^{n}_{j,h}$.

(2) $\bar{i}_{j,h} = \lim_{n\to\infty} \widetilde{i}^{n}_{j,h} \ge \underline{i}_{j,h} = \lim_{n\to\infty} \underset{\sim}{i}^{n}_{j,h}$ and $\bar{i}_{j,h}$ and $\underline{i}_{j,h}$ are the maximal and minimal solutions of (4.2)*.*

(3) $\underset{\sim}{i}_{j,h} \le i^{0}_{j,h} \le \widetilde{i}_{j,h}$ implies that $\underline{i}_{j,h} \le i^{n}_{j,h} \le \bar{i}_{j,h}$.

(4) If $i^{*}_{j,h}$ is the unique solution of (4.2) *such that $\underline{i}_{j,h} \le i^{*}_{j,h} \le \bar{i}_{j,h}$, then $i^{n}_{j,h} \to i^{*}_{j,h}$ as $n \to \infty$, whenever $\underset{\sim}{i}_{j,h} \le i^{0}_{j,h} \le \widetilde{i}_{j,h}$.*

5. Uniqueness and stability of a periodic solution.

In this section we use the results of §4 to show the unique existence and stability of a non-trivial periodic solution. We assume in this section, that the biological parameters μ, β, γ are T^*–periodic for some $T^* > 0$ and that $v(x,t,a)$ is a non–trivial T^*–periodic population density. We then consider the following problem:

$$(5.1)\quad \begin{aligned} &\frac{\partial i}{\partial t} + \frac{\partial i}{\partial a} - k\Delta i + \mu(t,a)i = \gamma(t,a)i\{v(x,t,a) - i\},\\ &\qquad x \in \Omega, \quad 0 \le a < a_\dagger, \quad t \ge 0,\\ &i(x,t,0) = q\int_0^{a_\dagger} \beta(t,a)i(x,t,a)da, \quad x \in \Omega, \quad t \ge 0,\\ &i(x,t,a) = 0, \quad x \in \partial\Omega, \quad 0 \le a < a_\dagger, \quad t \ge 0,\\ &i(x,t,a) = i(x,t+T^*,a), \quad x \in \Omega, \quad 0 \le a < a_\dagger, \quad t \ge 0, \end{aligned}$$

We are then interested in finding stable M-periodic endemic solutions. Namely, we want to find periodic solutions for $n \ge 1$, $1 \le j < A_\dagger$, to

$$(5.2)\quad \begin{aligned} &i^0_{j,h} = P_h i_0(\cdot, a_j), 0 \le j < A_\dagger,\\ &\big(\overline{D} i^n_{j,h}, w\big) + k\big(\nabla i^n_{j,h}, \nabla w\big) + (\mu^n_j i^n_{j,h}, w) + (\gamma^n_j v^n_j(\cdot) i^n_{j,h}, w) =\\ &\qquad (\gamma^n_j i^{n-1}_{j,h}(2v^n_j(\cdot) - i^n_{j,h}), w),\ w \in V_h,\ 1 \le j < A_\dagger,\ n \ge 1,\\ &i^n_{0,h} = \sum_{l=0}^{A_\dagger} \beta^n_j i^n_{j,h}\Delta t, n \ge 1,\\ &i^n_{j,h} = i^{n+M}_{j,h}, w \in V_h, \quad 1 \le j < A_\dagger, \quad n \ge 0. \end{aligned}$$

Following theorem concerns the existence of a maximal M-periodic solution in the desired range.

THEOREM 5.1. *There is an M-periodic solution ℓ^n_j of* (5.2) *such that $0 \le \ell^n_j \le v_j$. If there is an M-periodic solution i^n_j of* (5.2) *such that $0 \le i^n_j \le v_j$, then $0 \le i^n_j \le \ell^n_j$.*

THEOREM5.2. *Any non-trivial M-periodic solution is positive and there is at most one non-trivial M-periodic solution. Furthermore, it is asymptotically stable in that range when $0 \le i^0_j \le v^0_j$, if it exists.*

References

1. S. Busenberg and M. Iannelli, *A class of nonlinear diffusion problems in age–dependent population dynamics*, Nonlin. Anal. Th. Math. Appl. **7** (1983), 501–529.

2. P. G. Ciarlet, *The Finite Element Method for Elliptic Equations*, North–Holland, Amsterdam, 1978.
3. M. E. Gurtin, *A system of equations for age dependent population diffusion*, J. Theoret. Biol. **40** (1973), 389–392.
4. M. Gurtin and R. MacCamy, *Diffusion models for age–structured populations*, Math. Biosci. **54** (1981), 49–59.
5. M. Kubo and M. Langlais, *Periodic solutions for a population dynamics problem with age–dependence and spatial structure*, J. Math. Biol. **v.29** (1991), 363–378.
6. M. Langlais, *A nonlinear problem in age–dependent population diffusion*, SIAM J. Math. Anal. **16** (1985), 510–529.
7. M. Langlais, *Large time behavior in a nonlinear age–dependent population dynamics problem with spatial diffusion*, J. Math. Biol. **26** (1988), 319–346.
8. J. Rotemberg, *Theory of population transport*, J. Theoret. Biol. **37** (1972), 291–305.
9. J. G. Skellam, *Random dispersal in theoretical populations*, Biometrika **38** (1951), 196–218.

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