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THE EXACT BERGMAN KERNEL AND THE EXTREMAL PROBLEM

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Abstract. In this paper we find the Laurent series expansions representing the reproducing kernels. Also we find the number of zeroes of the exact Bergman kernel via parallel slit domain in order to relate the exact Bergman kernel to an extremal problem.

1. Introduction

The reproducing kernels such as the Bergman kernel, exact Bergman kernel, and the Szegő kernel associated to a bounded planar domain carry plenty of information about the domain. For example, conformal mappings from bounded planar domains onto canonical domains can be expressed simply in terms of the above reproducing kernels(see [2], [4], and [6]). Using the transformation formula of the Bergman kernel we get the property of the proper holomorphic map between two smoothly bounded domains(see [1]). Also if the kernel functions are algebraic, then any proper holomorphic map from the given domain to the unit disc is algebraic and vise versa(see [3]). So the issue is to find a domain whose reproducing kernels are algebraic. We proved in [5] that every non-degenerate n-connected planar domain with $n \geq 2$ is mapped biholomorphically onto a domain $W_{\mathbf{a},\mathbf{b}}$ defined by

$$\left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

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with suitable complex vectors $\mathbf{a} = (a_1, \cdots, a_{n-1})$ and $\mathbf{b} = (b_1, \cdots, b_{n-1})$. Note that the Bergman kernel and the Szegő kernel kernel asociated with $W_{\mathbf{a},\mathbf{b}}$ are algebraic.

Hence reproducing kernels have been useful tools in the study of conformal mappings and also of proper holomorphic mappings. These kernels are closely related via the holomorphic functions obtained by harmonic mesures. So we can expect that they have similar properties in some respect.

Suita and Yamada [8] proved that the Bergman kernel $B(z, w)$ associated with an n-connected planar domain Ω has $n-1$ zeroes in Ω as a function of z whenever $w \in \Omega$ is sufficiently close to $b\Omega$. If $w \in \Omega$ is not close to $b\Omega$, then the Bergman kernel may have fewer than $n-1$ zeroes.

The Szegő kernel $S(z, w)$ has $n-1$ zeroes in Ω as a function of z for fixed $w \in \Omega$. Bell [2] proved that if w is close to one of the boundary curve, the zeroes $w_1, w_2, \cdots, w_{n-1}$ become distinct simple zeroes. If w is a point in $b\Omega$, then $S(z, w)$ is nonvanishing on Ω as a function of z and has exactly $n-1$ zeroes on $b\Omega$.

In this paper, we find the Laurent series expansions representing the above mentioned reproducing kernels. Also we find the number of zeroes of the exact Bergman kernel via parallel slit domain in order to consider an extremal problem.

2. Preliminaries

Let Ω be an n-connected, bounded, planar domain with C^∞ boundary components and let $b\Omega$ denote the boundary of Ω.

Let $L^2(\Omega)$ denote the space of square integrable complex-valued functions on Ω with the inner product given by $< u, v >= \int_\Omega u\bar{v}dz$ and let $H^2(\Omega)$ denote the closed subspace of $L^2(\Omega)$ consisting of holomorphic functions on Ω.

The orthogonal projection $B : L^2(\Omega) \to H^2(\Omega)$ called the Bergman projection is well-defined and represented by the Bergman kernel $B(z, w)$ on $\Omega \times \overline{\Omega}$ via

$$B\varphi(z) = \int_\Omega B(z, w)\varphi(w)\, dw$$

for φ in $L^2(\Omega)$ and z in Ω. The Bergman kernel is holomorphic in z, antiholomorphic in w, and $B(z, w) = \overline{B(w, z)}$.

Let $E^2(\Omega)$ denote the closed subspace of $L^2(\Omega)$ consisting of holomorphic functions on Ω which are derivatives of single-valued functions.

The orthogonal projection $E : L^2(\Omega) \to E^2(\Omega)$ called the exact Bergman projection is well-defined and represented by exact Bergman kernel $E(z,w)$ on $\Omega \times \overline{\Omega}$ via

$$E\varphi(z) = \int_\Omega E(z,w)\varphi(w)\,dw$$

for φ in $L^2(\Omega)$ and z in Ω.

Let $L^2(b\Omega)$ denote the space of square integrable complex-valued functions on $b\Omega$ with the inner product given by $< u, v >_b = \int_{b\Omega} u\bar{v}ds$ where ds denotes the arc length measure. Let $H^2(b\Omega)$ denote the closed subspace of $L^2(b\Omega)$ consisting of boundary values of holomorphic functions on Ω.

The orthogonal projection $S : L^2(b\Omega) \to H^2(b\Omega)$ called the Szegő projection is well-defined and represented by the Szegő kernel $S(z,w)$ on $\Omega \times \overline{\Omega}$ via

$$S\varphi(z) = \int_{b\Omega} S(z,w)\varphi(w)\,ds_w$$

for φ in $L^2(b\Omega)$ and z in Ω. Here we have identified $S\varphi \in H^2(b\Omega)$ with its unique holomorphic extension to Ω. The Szegő kernel is holomorphic in z, antiholomorphic in w, and $S(z,w) = \overline{S(w,z)}$.

To see the relation among the reproducing kernels, first let $\{\gamma_j\}_{j=1}^n$ denote the n boundary curves of Ω. Without loss of generality, assume that γ_n is the outer boundary curve which bounds the unbounded component of the complement of Ω in $\mathbb{C}$. Let $\{\omega_j\}_{j=1}^n$ denote the harmonic measure functions associated to Ω. They are harmonic functions on Ω which extend C^∞ smoothly to $\overline{\Omega}$ and $\omega_j(\gamma_i) = \delta_{ij}$. We can get a multi-valued holomorphic function W_j by analytically continuing around Ω a germ of $\omega_j + i\omega_j^*$ where ω_j^* is a local harmonic conjugate for ω_j. Then $W_j' = 2\partial\omega_j/\partial z$ is also a holomorphic function. It is known that (see [2] p.119)

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{j=1}^{n-1} \lambda_j W_j'(z)$$

where λ_j are constants in z which depend on w.

Given a point $a \in \Omega$, the Ahlfors map g_a associated to the pair (Ω, a) is a proper holomorphic mapping of Ω onto the unit disc. It is an n-to-one mapping (counting multiplicity), it extends to be in $C^\infty(\overline{\Omega})$, and it maps each boundary curve γ_j one-to-one onto the unit circle. Furthermore, $g_a(a) = 0$, and g_a is the unique function mapping Ω into the unit disc maximizing $|g_a'(a)|$ with $g_a'(a) > 0$. The Ahlfors map is related to the Szegő kernel $S(z, a)$ and the Garabedian kernel $L(z, a)$ via

$$g_a(z) = \frac{S(z, a)}{L(z, a)}.$$

Since g_a is n-to-one, g_a has n zeroes. The simple pole of $L(z, a)$ at a accounts for the simple zero of g_a at a. The other $n - 1$ zeroes of g_a are given by $n - 1$ zeroes of $S(z, a)$ in $\Omega - \{a\}$.

3. The series expansions

Holomorphic functions in a multiply connected planar domain can be expressed by the Laurent series. We will find the Laurent series expansions of the reproducing kernels.

EXAMPLE 3.1. Let $\Omega = \{z \in \mathbb{C} : \rho < |z| < 1\}$.
(1) We note that

$$< z^n, z^n >= \iint_\Omega z^n \bar{z}^n dxdy = \frac{\pi}{n+1}(1 - \rho^{2n+2}).$$

Hence the set $\{u_n = \frac{\sqrt{n+1}z^n}{\sqrt{\pi(1-\rho^{2n+2})}}\}, (n = \cdots, -1, 0, 1, \cdots)$ is the basis for $H^2(\Omega)$ orthonormalized by $< u_n, u_m >= \delta_{nm}$. Therefore,

$$B(z, w) = \sum_{n=-\infty}^{n=\infty} u_n(z)\overline{u_n}(w) = \frac{1}{\pi}\sum_{n=-\infty}^{n=\infty} \frac{(n+1)(z\bar{w})^n}{1 - \rho^{2n+2}}.$$

(2) We note that

$$< (z^n)', (z^n)' >= \iint_\Omega (z^n)'(\bar{z}^n)' dxdy = \pi n(1 - \rho^{2n}).$$

Hence the set $\{u_n = \frac{z^n}{\sqrt{\pi n(1-\rho^{2n})}}\}, (n = \cdots, -1, 1, \cdots)$ is the basis for $E^2(\Omega)$ orthonormalized by $< u_n', u_m' >= \delta_{nm}$. Therefore,

$$E(z, w) = \sum_{n=-\infty}^{n=\infty} u_n'(z)\overline{u_n}'(w) = \frac{1}{\pi}\sum_{n=-\infty}^{n=\infty} \frac{n(z\bar{w})^{n-1}}{1 - \rho^{2n}}.$$

(3) We note that

$$< z^n, z^n >_b = \iint_{b\Omega} z^n \bar{z}^n dxdy = 2\pi(1+\rho^{2n+1}).$$

Hence the set $\{u_n = \frac{z^n}{\sqrt{2\pi(1+\rho^{2n+1})}}\}, (n = \cdots, -1, 0, 1, \cdots)$ is the basis for $H^2(b\Omega)$ rthonormalized by $< u_n, u_m >_b = \delta_{nm}$. Therefore,

$$S(z,w) = \sum_{n=-\infty}^{n=\infty} u_n(z)\overline{u_n}(w) = \frac{1}{2\pi} \sum_{n=-\infty}^{n=\infty} \frac{(z\bar{w})^n}{1+\rho^{2n+1}}.$$

4. The exact Bergman kernel and the extremal problem

Since holomorphic functions are derivatives of single-valued functions in simply connected planar domains, $H^2(\Omega)$ and $E^2(\Omega)$ are the same and hence the Bergman kernel and the exact Bergman kernel are the same in simply connected planar domains. But in multiply connected planar domain, they are not the same.

The parallel slit domain is a canonical domain for planar domains. For any planar domain Ω there is a conformal map φ from Ω onto a parallel slit domain mapping $w \in \Omega$ into the point at ∞. When parallel slits form an angle θ with positive real axis, its representation is

$$\varphi(z) = \varphi_\theta(z,w) = \frac{1}{z-w} + a_{1,\theta}(z-w) + a_{2,\theta}(z-w)^2 + \cdots$$

near $z = w$. By using two basic mappings φ_0 and $\varphi_{\pi/2}$, φ_θ is given by

$$\varphi_\theta(z,w) = e^{i\theta}[cos\theta \ \varphi_0(z,w) - i \ sin\theta \ \varphi_{\pi/2}(z,w)]$$

(see [7] p.339).

For w in the unit disc $\{z \in \mathbb{C} : |z| < 1\}$, the function $f_1(z) = \frac{1-\bar{w}z}{z-w}$ maps the unit disc onto the outside of the unit disc mapping the point $z = w$ into the point at ∞. The mapping $f_2(z) = \frac{1}{1-|w|^2}(z + 1/z)$ maps the outside of the unit disc onto the extended complex plane except the line segment $[-(\frac{1}{1-|w|^2})2, (\frac{1}{1-|w|^2})2]$. Hence the composition function $f_2 \circ f_1(z)$ maps the unit disc onto the extended complex plane except the line segment $[-(\frac{1}{1-|w|^2})2, (\frac{1}{1-|w|^2})2]$ and has a pole of order 1 at $z = w$ with residue 1.

Similarly the mapping $f_3(z) = \frac{1}{1-|w|^2}(z-1/z)$ maps the outside of the unit disc onto the complex plane except the line segment

$$[-(\frac{1}{1-|w|^2})2i, (\frac{1}{1-|w|^2})2i].$$

Hence, the composition function $f_3 \circ f_1(z)$ maps the unit disc onto the the extended complex plane except the line segment $[-(\frac{1}{1-|w|^2})2i, (\frac{1}{1-|w|^2})2i]$.

Thus we can represent a conformal map from the unit disc onto a parallel slit domain.

EXAMPLE 4.1. Let $\Omega = \{z \in \mathbb{C} : |z| < 1\}$. A conformal map of Ω onto a parallel slit domain mapping $w \in \Omega$ into the point at ∞ has a representation

$$\varphi_0(z, w) = \frac{1}{1-|w|^2}(\frac{1-\bar{w}z}{z-w} + \frac{z-w}{1-\bar{w}z})$$

and

$$\varphi_{\pi/2}(z, w) = \frac{1}{1-|w|^2}(\frac{1-\bar{w}z}{z-w} - \frac{z-w}{1-\bar{w}z}).$$

Hence

$$\varphi_\theta(z, w) = \frac{1}{1-|w|^2}(\frac{1-\bar{w}z}{z-w} + e^{2i\theta}\frac{z-w}{1-\bar{w}z}).$$

Let Ω be a planar domain with piecewise smooth boundary $b\Omega$, and both of the functions p and q have continuous first derivatives in Ω and on $b\Omega$. Then,

$$\frac{\partial}{\partial \bar{z}}(pq) = p\frac{\partial q}{\partial \bar{z}} + q\frac{\partial p}{\partial \bar{z}}$$

and it implies the following generalized Green's formula.

PROPOSITION 4.2. *For a planar domain Ω with piecewise smooth boundary $b\Omega$, and for the functions p and q with continuous first derivatives in Ω and on $b\Omega$, we have the following formula*

$$\iint_\Omega p\frac{\partial q}{\partial \bar{z}}dx\,dy = \frac{1}{2i}\int_{b\Omega} pq dz - \iint q\frac{\partial p}{\partial \bar{z}}dx\,dy.$$

By using the above proposition, we have the following theorem for the zeroes of the exact Bergman kernel.

THEOREM 4.3. *Let Ω be a bounded n-connected smooth planar domain. Then $E(z, w)$ has $2(n-1)$ zeroes in Ω for $w \in \Omega$.*

Proof. Let

$$M(z,w) = \frac{1}{2}(\varphi_0(z,w) - \varphi_{\pi/2}(z,w))$$

and

$$N(z,w) = \frac{1}{2}(\varphi_0(z,w) + \varphi_{\pi/2}(z,w)).$$

Then $M(z,w)$ is holomorphic in Ω and $N(z,w)$ has a pole of order 1 with residue 1 at $z = w$.

Let $f(z)$ be a single-valued holomorphic function in a domain Ω and on its boundary $b\Omega$. Then the generalized Green's formula and $\overline{M'(z,w)\ dz} = N'(z,w)dz$ in [7] p.362 imply that

$$\begin{aligned}
\iint_\Omega f'(z)\overline{M'(z,w)}dxdy &= \overline{\iint_\Omega \overline{f'(z)}M'(z,w)dxdy} \\
&= \overline{\int_{b\Omega} \overline{f(z)}M'(z,w)\frac{1}{2i}dz} \\
&= \overline{\int_{b\Omega} \overline{f(z,w)}\ \overline{N'(z,w)}\frac{1}{2i}\overline{dz}} \\
&= \int_{b\Omega} f(z)(-\frac{1}{2i}N'(z,w))dz.
\end{aligned}$$

Since the residue of $N(z,w)$ at $z = w$ is 1, the principal part of $N'(z,w)$ is $\frac{-1}{(z-w)^2}$. By using Cauchy's Integral Formula, it holds that

$$\int_{b\Omega} f(z)(-\frac{1}{2i}N'(z,w))dz = -\frac{1}{2i}(-2\pi i f'(w)).$$

Therefore,

$$\iint_\Omega f'(z)\overline{M'(z,w)}dxdy = \pi f'(w).$$

and hence $E(z,w) = \frac{1}{\pi}M'(z,w)$. By using the tangential derivatives of φ_0 and $\varphi_{\pi/2}$ as well as the argument principle, we get $M'(z,w)$ has $2(n-1)$ zeroes in Ω for $w \in \Omega$ and so does $E(z,w)$. □

By using the above theorem, we get

Theorem 4.4. *Let Ω be an n-connected planar domain and let*

$$F_w(z) = \frac{M'(z,w)}{N'(z,w)}.$$

Then F_w is a proper $2n$-to-one map from Ω onto the unit disc with $F_w(w) = F'_w(w) = 0$ and $F''_w(w) = 2E(z,w)$.

Proof. Let

$$F_w(z) = \frac{M'(z,w)}{N'(z,w)} = \frac{\frac{\varphi_0'(z,w)}{\varphi_{\pi/2}'} - 1}{\frac{\varphi_0'(z,w)}{\varphi_{\pi/2}'} + 1}.$$

It is the composition of

$$f_1(z) = \frac{\varphi_0'(z,w)}{\varphi_{\pi/2}'(z,w)}$$

and

$$f_2(z) = \frac{z-1}{z+1}.$$

We know that the univalent holomorphic function $\varphi_\theta(z,w)$ has the expression

$$\varphi_\theta(z,w) = e^{i\theta}[cos\theta\ \varphi_0(z,w) - i\, sin\theta\ \varphi_{\pi/2}(z,w)]$$

and $\varphi_\theta'(z,w)$ can't vanish in Ω. Hence, $f_1(z)$ can't be $i\,tan\theta$ and $f_1(z)$ maps Ω onto the right half plane with the $Ref_1(z) = 0$ on the boundary of Ω. By the property of f_1 and f_2, F_w is a proper holomorphic map from Ω onto the unit disc. Moreover F_w has $2n$ zeroes in Ω since $N'(z,w)$ has no zero in Ω with a pole of order 2 at $z = w$ and $M'(z,w)$ has $2(n-1)$ zeroes in Ω with no pole. Therefore F_w is a proper $2n$-to-one map from Ω onto the unit disc.

Since $N'(z,w) = -\frac{1}{(z-w)^2} + h_w(z)$ where h_w is a holomorphic function in Ω,

$$F_w(z) = \frac{M'(z,w)}{-\frac{1}{(z-w)^2} + h_w(z)}.$$

Therefore $F_w(w) = F_w'(w) = 0$ and $F''(z) = 2M'(w,w) = 2\pi E(w,w)$ □

We conjecture that the above theorem can be used to solve an extremal problem. In this sense, the above approach to find the zeroes of the exact Bergman kernel is useful.

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