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# GENERALIZED BOUNDED ANALYTIC FUNCTIONS IN THE SPACE $H_{\omega,p}$

Jun-Rak Lee

Abstract. We define a general space $H_{\omega,p}$ of the Hardy space and improve that Aleman's results to the space $H_{\omega,p}$. It follows that the multiplication operator on this space is cellular indecomposable and that each invariant subspace contains nontrivial bounded functions.

## 1. Introduction

For a positive integrable function $\omega \in C^2[0,1)$, we define the space $H_{\omega,p}$ of analytic functions $f$ in the unit disc $U$ that satisfies

$$\|f\|^p_{\omega,p} = |f(0)|^p + p^2 \int_U |f(z)|^{p-2}|f'(z)|^2\omega(|z|)dm < \infty, \tag{1.1}$$

where $m$ is the area measure on $C$. Some simple computations with power series show that if $f(z) = \sum_{n\geq 0} a_n z^n$ is analytic in $U$, then

$$\|f\|^p_{\omega,p} = \sum_{n\geq 0} |a_n|^p \omega_n, \tag{1.2}$$

where $\omega_0 = 1$ and for $n \geq 1$,

$$\omega_n = 2\pi n^p \int_0^1 r^{pn-p+1}\omega(r)dr. \tag{1.3}$$

If $p = 2$, Aleman([1]) proved that every function in $H_{\omega,2}$ is the quotient of two bounded analytic functions in $H_{\omega,2}$. In this paper, by the similar proofs, we shall prove that every function in $H_{\omega,p}$ is the quotient of two bounded analytic functions in $H_{\omega,p}$ for $p \geq 2$. This result which is proved

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in section 2 has some applications concerning the invariant subspaces of the multiplication operator defined on $H_{\omega,P}$ by

$$(M_z f)(\zeta) = \zeta f(\zeta), \qquad \zeta \in U, \quad f \in H_{\omega,p}. \tag{1.4}$$

Using (1.1) and (1.2), it follows easily that $M_z$ is a bounded weighted shift on $H_{\omega,p}$. From the result mentioned above it turns out that every nontrivial invariant subspace of $M_z$ contains a nontrivial bounded function, that each two nontrivial invariant subspaces have a nontrivial intersection and that each nontrivial invariant subspace has the codimension one property. For the usual Dirichlet space $D$, this was proved by S.Richter and A.Shields in [4]. These results provide positive answers to the corresponding questions for the space $D_\alpha$ [4, Conjectures 1 and 2] and are proved in section 3. The method used for the proofs implies the following cyclicity theorem for the spaces $H_{\omega,p}$, related to Question 3 in [2]. A function whose modulus is greater or equal to the modulus of a cyclic vector for $M_z$ must also be a cyclic vector.

## 2. Bounded Functions in $H_{\omega,p}$

We begin with a general change of variable formula that is used in order to obtain an equivalent form of the norm on $H_{\omega,p}$.

LEMMA 2.1. *([1])Let $\phi$ be a nonconstant analytic function in $U$ and $u, v$ be nonnegative measurable functions on $C$ with respect to area measure. Then*

$$\int_U (u \circ \phi) v |\phi'|^2 dm = \int_{\phi(U)} u(\zeta) \left( \sum_{\phi(z)=\zeta} v(z) \right) dm(\zeta). \tag{2.1}$$

*This result is actually known and was proved for $v(z) = -log|z|$ in [5].*

For a nonconstant analytic function $f$ in $U$, $\zeta \in f(U)$ and $u$ a nonnegative measurable function on $[0, 1)$, we denote

$$N_{u,f}(\zeta) = \sum_{f(z)=\zeta} u(|z|). \tag{2.2}$$

In the special case when $u(r) = u_0(r) = log1/r, r \in [0, 1)$, (2.2) gives the usual Nevanlinna counting function of $f$ and we denote $N_{u_0,f} = N_f$. Substituting that $u(\zeta) = |\zeta|^{p-2}$,$v(z) = \omega(|z|)$, and $\phi(z) = f(z)$,we obtain the following Corollary.

COROLLARY 2.2. *If $f \in H_{\omega,p}$ is nonconstant, then*

$$\|f\|_{\omega,p}^p = |f(0)|^p + p^2 \int_{f(U)} |\zeta|^{p-2} \left( \sum_{f(z)=\zeta} \omega(|z|) \right) dm(\zeta).$$

$$= |f(0)|^p + p^2 \int_{f(U)} |\zeta|^{p-2} N_{\omega,f}(\zeta) dm(\zeta). \tag{2.3}$$

LEMMA 2.3. *([1])Let $f$ be nonconstant analytic function in $U$ and for $z, \lambda \in U$ and let $\varphi_z(\lambda) = (z+\lambda)/(1+\bar{z}\lambda)$. Then for every $\zeta \in f(U)$*

$$N_{\omega,f}(\zeta) = -\frac{1}{2\pi} \int_U \triangle\bar{\omega}(z) N_{f\circ\varphi_z}(\zeta) dm(z), \tag{2.4}$$

*where $\bar{\omega}$ is defined on $U$ by $\bar{\omega}(z) = \omega(|z|)$ and $\triangle$ denotes the Laplace operator.*

From the fact that $H_{\omega,p}$ is contained in $H^2$, it follows that each function $f \in H_{\omega,p}$ has nontangential limits $f(e^{i\theta})a.e.$ on $[0, 2\pi]$ and that its boundary function is in $L^2[0, 2\pi]$. For $z \in U$, let $P_z(\theta) = Re(e^{i\theta} + z)/(e^{i\theta} - z)$, be the Poisson kernel.

PROPOSITION 2.4. *Let $f \in H_{\omega,p}$, then*

$$\|f\|_{\omega,p}^p - |f(0)|^p = -\int_U \triangle\bar{\omega}(z) \left( \frac{1}{2\pi} \int_0^{2\pi} P_z(\theta) |f(e^{i\theta})|^p d\theta - |f(z)|^p \right) dm(z). \tag{2.5}$$

*Proof.* For a nonconstant $f \in H_{\omega,p}$, we have

$$\|f\|_{\omega,p}^p - |f(0)|^p = p^2 \int_{f(U)} |\zeta|^{p-2} \left( -\frac{1}{2\pi} \int_U \triangle\bar{\omega}(z) N_{f\circ\varphi_z}(\zeta) dm(z) \right) dm(\zeta)$$

$$= -\int_U \triangle\bar{\omega}(z) \left( \frac{p^2}{2\pi} \int_{f(U)} |\zeta|^{p-2} N_{f\circ\varphi_z}(\zeta) dm(\zeta) \right) dm(z), \tag{2.6}$$

by Lemma 2.3 and Fubini's theorem. Letting $\nu(\lambda) = log1/|\lambda|$,$u(\zeta) = |\zeta|^{p-2}$, $\phi = f \circ \varphi_z$ in (2.1), the Littlewood-Paley formula gives

$$\frac{p^2}{2\pi} \int_{f(U)} |\zeta|^{p-2} N_{f\circ\varphi_z}(\zeta) dm(\zeta) = \frac{p^2}{2\pi} \int_{f(U)} |\zeta|^{p-2} \left( \sum_{f\circ\varphi_z(\lambda)=\zeta} log\frac{1}{|\lambda|} \right) dm(\zeta)$$

$$= \frac{p^2}{2\pi}\int_U |(f\circ\varphi_z)|^{p-2}|(f\circ\varphi_z)'|^2 log\frac{1}{|\lambda|}dm$$

$$= \|f\circ\varphi_z\|_{H^p}^p - |f\circ\varphi_z(0)|^p. \tag{2.7}$$

Obviously $f\circ\varphi_z(0) = f(z)$ and by elementary computations with harmonic measures for the unit disk, we obtain

$$\|f\circ\varphi_z\|_{H^p}^p - |f\circ\varphi_z(0)|^p = \frac{1}{2\pi}\int_0^{2\pi} |f\circ\varphi_z(e^{i\theta})|^p d\theta - |f(z)|^p$$

$$= \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f(e^{i\theta})|^p d\theta - |f(z)|^p, \tag{2.8}$$

and the proof is complete.

□

By the similar proofs of [1, Corollary 2.5], we obtain the following.

COROLLARY 2.5. *If $f\in H_{\omega,p}$ and $F$ is its outer factor, then $F\in H_{\omega,p}$ and*

$$\|F\|_{\omega,p}^p - |F(0)|^p \le \|f\|_{\omega,p}^p - |f(0)|^p. \tag{2.9}$$

For a function $f$ in the Nevanlinna class $N, f\neq 0$, we denote by $\phi_f$ the outer function satisfying $|\phi_f(e^{i\theta})| = min\{1, 1/|f(e^{i\theta})|\}$ a.e. on $[0, 2\pi]$; that is

$$\phi_f(z) = exp\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z} log\, min\{1, 1/|f(e^{i\theta})|\}d\theta. \tag{2.10}$$

Our main result is

THEOREM 2.6. *Let $f\in H_{\omega,p}, f\neq 0$. Then $\phi_f, f\phi_f$ are in $H_{\omega,p}$ and satisfy*

$$\|\phi_f\|_{\omega,p}^p - |\phi_f(0)|^p \le \|f\|_{\omega,p}^p - |f(0)|^p \tag{2.11}$$

*and*

$$\|f\phi_f\|_{\omega,p} \le \|f\|_{\omega,p}. \tag{2.12}$$

The proof uses the following inequalities:

LEMMA 2.7. *([1])Let $(X, \mu)$ be a probability space and $f \in L^1(\mu)$ such that $f > 0\, \mu -$ a.e. on $X$ and $log\, f \in L^1(\mu)$. Let*

$$E(f) = \int_X f d\mu - exp \int_X log\, f d\mu. \tag{2.13}$$

*Then*

$$E(min\{1, f\}) \leq E(f), \tag{2.14}$$

*and*

$$E(max\{1, f\}) \leq E(f), \tag{2.15}$$

*Proof of Theorem* 2.6. Let $f \in H_{\omega,p},\ f = IF$ with $I$ inner and $F$ outer. An application of (2.14) with $X = [0, 2\pi],\ d\mu = (1/2\pi)P_z d\theta$, yields

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f\phi_f(e^{i\theta}|^p d\theta - |F\phi_f(z)|^p \\
&= \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|F\phi_f(e^{i\theta})|^p d\theta - |F\phi_f(z)|^p \\
&= \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\, min\{1, |F(e^{i\theta})|^p\} d\theta \\
&\qquad - exp\left(\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\, log\, min\{1, |F(e^{i\theta})|^p\} d\theta\right) \\
&= E(min\{1, |F|^p\}) \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|F(e^{i\theta})|^p d\theta - exp\left(\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\, log\, |F(e^{i\theta})|^p d\theta\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|F(e^{i\theta})|^p d\theta - |F(z)|^p.
\end{aligned}$$

Hence

$$\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f\phi_f(e^{i\theta})|^p d\theta - \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\, |F(e^{i\theta})|^p d\theta \\
&\leq |F\phi_f(z)|^p - |F(z)|^p = -|F(z)|^p(1 - |\phi_f(z)|^p) \\
&\leq -|I(z)F(z)|^p(1 - |\phi_f(z)|^p) = |f\phi_f(z)|^p - |f(z)|^p. \qquad (2.16)
\end{aligned}$$

Therefore

(2.17)

$$\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f\phi_f(e^{i\theta})|^p d\theta - |f\phi_f(z)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f(e^{i\theta})|^p d\theta - |f(z)|^p,$$

and the inequality in (2.12) follows by Proposition 2.4. Furthermore, we have

$$\left|\frac{1}{\phi_f(e^{i\theta})}\right| = max\{1, |f(e^{i\theta})|\}\, a.e.\, on\, [0, 2\pi]$$

and for $z \in U$,

$$\left|\frac{1}{\phi_f(e^{i\theta})}\right|^p = exp\left(\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\, log\, max\{1, |f(e^{i\theta})|^p\} d\theta\right). \tag{2.18}$$

We apply (2.15) to obtain

$$\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\left|\frac{1}{\phi_f(e^{i\theta})}\right|^p d\theta - \left|\frac{1}{\phi_f(z)}\right|^p = E(max\{1, |f|^p\}) \leq E(|f|^p)$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f(e^{i\theta})|^p d\theta - |f(z)|^p, \tag{2.19}$$

for all $z \in U$. Thus by Proposition 2.4, $1/\phi_f \in H_{\omega,p}$ and

$$\|\frac{1}{\phi_f}\|^p_{\omega,p} - |\frac{1}{\phi_f(0)}|^p \leq \|f\|^p_{\omega,p} - |f(0)|^p. \tag{2.20}$$

Finally, since $\phi'_f = -\phi_f^2(1/\phi_f)'$ and $|\phi_f| \leq 1$ in $U$, the definition (1.1) implies that

$$\left\|\frac{1}{\phi_f}\right\|^p_{\omega,p} - \left|\frac{1}{\phi_f(0)}\right|^p = p^2\int_U \left|\frac{1}{\phi_f(z)}\right|^{p-2}\left|\left(\frac{1}{\phi_f(z)}\right)'\right|^2 \omega(|z|)dm$$

$$\geq p^2\int_U |\phi_f(z)|^{p-2}|\phi'_f|^2\omega(|z|)dm$$

$$= \|\phi_f\|^p_{\omega,p} - |\phi_f(0)|^p \tag{2.21}$$

for $p \geq 2$. Hence we obtain

$$\|\phi_f\|^p_{\omega,p} - |\phi_f(0)|^p \leq \|f\|^p_{\omega,p} - |f(0)|^p,$$

and the proof is complete.

COROLLARY 2.8. *If $p \geq 2$, then every function in $H_{\omega,p}$ is the quotient of two bounded functions in $H_{\omega,p}$.*

*Proof.* Since $|\phi_f|$, $|f\phi_f| \leq 1$ in $U$ and $f = f\phi_f/\phi_f$, we have the result. □

## 3. Invariant subspaces

The present section contains some applications of Theorem 2.6 concerning the invariant subspaces of the multiplication operator on $H_{\omega,p}$ defined by (1.4). A closed subspace $M$ of $H_{\omega,p}$ is called invariant if $M_z M \subset M$. For a function $f \in H_{\omega,p}$ we denote by $[f]$ the smallest invariant subspace containing $f$ ; that is the closure of the polynomial multiplies of $f$ in $H_{\omega,p}$. In order to prove the main result of this section we use the same method as in [4]. Let $H^\infty$ be the algebra of bounded analytic functions in $U$ with the norm $\|g\|_\infty = sup_{z\in U}|g(z)|$, $g \in H^\infty$. We have

LEMMA 3.1. *If $f \in H_{\omega,p}$ and $g \in H^\infty$ such that $gf \in H_{\omega,p}$, then $gf \in [f]$.*

The proof of the lemma is based on some simple properties of the linear operators $T_i$, $0 \leq t < 1$, defined on the set $H(U)$ of analytic functions in $U$ by

$$(3.1) \qquad (T_i h)(z) = \frac{1}{1-t}\int_t^1 h(sz)ds \;\; z \in U, \;\; h \in H(U).$$

By the theorem of L'hospital, we have

$$lim_{t->1}(T_i h)(z) = lim_{t->1}\frac{\int_1^t h(sz)ds}{t-1} = lim_{t->1}h(tz) = h(z)$$

for all $z \in U$ and $h \in H(U)$. Some other properties are summarized below. For $h \in H(U)$, and $t \in [0,1)$, we denote by $h_t$ the function given by $h_t(z) = h(tz)$, $z \in U$.

LEMMA 3.2. *For every $t \in [0,1)$, we have*

$$(i)(M_z T_t h)' = \frac{h - th_t}{1-t}, \;\; h \in H(U).$$

$$(ii) If \;\; h \in H_{\omega,p}\,,\; then \; T_t h \in H_{\omega,p} \; and \; \|T_t h\|_{\omega,p} \leq \|h\|_{\omega,p}\,.$$

$(iii) If \;\; h \in H^\infty\,,\; then \; T_t h \in H_{\omega,p} \; and \; fT_t h \in H_{\omega,p} \; whenever \; f \in H_{\omega,p}\,.$
*Furthermore in this case, $fT_t h \in [f]$.*

*Proof.* (i)For every $\zeta \in U$ and $h \in H(U)$,
(3.2)

$$(M_z T_t h)(\zeta) = \zeta (T_t h)(\zeta) = \zeta \frac{1}{1-t} \int_t^1 h(s\zeta) ds = \frac{1}{1-t} \int_{t\zeta}^{\zeta} h(\lambda) d\lambda.$$

Hence

$$(M_z T_t h)'(\zeta) = \left[ \frac{1}{1-t} \int_0^{\zeta} h(\lambda) d\lambda - \frac{1}{1-t} \int_0^{t\zeta} h(\lambda) d\lambda \right]'$$

$$= \frac{1}{1-t} [h(\zeta) - t h_t(\zeta)]$$

(ii)If $h(z) = \sum_{n \geq 0} a_n z^n$, $z \in U$, then

$$(T_t h)(z) = \sum_{n \geq 0} \left( \frac{1 - t^{n+1}}{1-t} \right) \frac{a_n}{n+1} z^n, \quad z \in U, \tag{3.3}$$

which shows that $T_t h \in H_{\omega,p}$ whenever $h \in H_{\omega,p}$ and $\|T_t h\|_{\omega,p} \leq \|h\|_{\omega,p}$.
(iii)From (i) we obtain that if $h \in H^\infty$ then $T_t h$ and $(T_t h)'$ are both in $H^\infty$,hence $T_t h \in H_{\omega,p}$ and also a multiplier. Moreover, there exists a sequence of polynomials $\{p_n\}$ with $sup_n \|p_n'\|_\infty < \infty$, conversing pointwise to $T_t h$ on $U$. It follows that $sup_n \|p_n f\|_\omega < \infty$, and that at least a subsequence of $\{p_n f\}$ converges weakly in $H_{\omega,p}$. Also, its limit must be $f T_t h$ because the point evaluations are bounded linear functionals on $H_{\omega,p}$. Thus, $f T_t h \in [f]$. □

*Proof of Lemma* 3.1. Let $f, g$ be as in the statement. For every $t \in [0,1)$ we have $f T_t g \in [f]$ by Lemma 3.2 and $lim_{t->1} (f T_t g)(z) = f(z) g(z)$ for all $z \in U$. We are going to show that the norms $\|f T_t g\|_{\omega,p}$ remain bounded when $t$ tends to 1. Note first that by the definition of $H_{\omega,p}$, it follows easily that the operator $M_z$ is injective and has closed range on $H_{\omega,p}$, hence there exists a positive constant $c$ such that $\|f T_t g\| \leq c \|M_z f T_t g\|_{\omega,p}$, $t \in [0,1)$. Further,

$$\|M_z f T_t g\|_{\omega,p}^p \leq 2 \int_U |f (M_z T_t g)'|^p \omega \, dm + 2 \int_U |f' M_z T_t g|' \omega \, dm \tag{3.4}$$

$$\leq 2 \int_U |f (M_z T_t g)'|^p \omega \, dm + 2 \|g\|_\infty^p \|f\|_\omega^p,$$

and by Lemma 3.2(i),
(3.5)

$$|f(M_zT_tg)'| = \left|f\frac{g-tg_t}{1-t}\right| = \left|\frac{fg-tf_tg-t-g_tf+g_ttf_t+fg_t(1-t)}{1-t}\right|$$

$$\leq \left|\frac{fg-tf_tg_t}{1-t}\right| + \left|g_t\frac{f-tf_t}{1-t}\right| + |fg_t|$$

$$= |(M_zT_tfg)'| + |g_t(M_zT_tf)'| + |fg_t|.$$

We obtain

$$\int_U |f(M_zT_tg)'|^p\omega\, dm \leq 3\|M_zT_tfg\|^p_{\omega,p} + 3\|g\|^p_{\omega,p}\|M_zT_tf\|^p_{\omega,p} + 3\|g\|^p_{\omega,p}\int_U |f|^p\omega\, dm. \tag{3.6}$$

This leads to an estimation of the form

$$\|fT_tg\|^p_{\omega,p} \leq c_1\|fg\|^p_{\omega,p} + c_2\|g\|^p_{\omega,p}\|f\|^p_{\omega,p}, \tag{3.7}$$

where $c_1$, $c_2$ are positive constants independent of $t$. As in the proof of Lemma 3.2(iii), there exists a sequence $\{t_n\}$ in $[0,1)$, tending to 1 such that $fT_{t_n}g$ tends weakly to $fg$ in $H_{\omega,p}$, hence $gf \in [f]$.

A function $f \in H_{\omega,p}$ is called a cyclic vector for $M_z$ if $[f] = H_{\omega,p}$. From Lemma 3.1, we obtain

Corollary 3.3. *If $f$, $g \in H_{\omega,p}$, $g$ is a cyclic vector for $M_z$ and $|f(z)| \geq |g(z)|$ for all $z \in U$, then $f$ is also cyclic.*

*Proof.* If $h = g/f$, then $h \in H^\infty$ and $hf = g$,i.e. $hf \in H_{\omega,p}$. Then by Lemma 3.1, $g \in [f]$, which shows that $f$ is cyclic. □

*Remark.* The above result remains true if $H_{\omega,p}$ is replaced by the Dirichlet space $D$, because Lemma 3.1 holds for this space as well, with the same proof. This problem was raised for general Banach spaces of analytic functions by L. Brown and A. Shields [2, Question 3].

The main result of this section is

Theorem 3.4. *Let $M \neq \{0\}$, $M \neq \{0\}$ be invariant subspaces for the operator $M_z$ on $H_{\omega,p}$. Then (i) $M \cap H^\infty \neq \{0\}$ and (ii) $M \cap N \neq \{0\}$.*

*Proof.* (i)Let $f \in M$, $f =\neq 0$. By Theorem 2.6 there exist functions $g, h \in H^\infty \cap H_{\omega,p}$ such that $f = g/h$ and, by Lemma 3.1, $g = hf \in [f] \subset M$. (ii) If $g \in M$, $h \in N$, $g, h \neq 0$ are bounded then $gh \in [g] \cap [h] \subset M \cap N$. □

Theorem 3.4(ii) states that the operator $M_z$ on $H_{\omega,p}$ is cellular indecomposable and answers affirmatively Conjecture 1 of [4]. As it was pointed out in [4] this implies the fact that each nontrivial invariant subspace $M$ of $M_z$ has the codimension one property that is, $(z - \lambda)M$ is a closed subspace of $M$ having codimension 1 in $M$, for every $\lambda \in U$. This follows from results obtained by S. Richter in [3] and Theorem 3.4. Indeed, if $M$ is such a subspace and $f, g \in M \backslash \{0\}$, then $[f] \cap [g] \neq \{0\}$ by Theorem 3.4 and by [3, Corollaries 3.12 and 3.15] $M$ has the codimension one property.

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Samcheok National University
Samchok, Kangwondo 245-711,Korea
*E-mail*: jrlee@samcheok.ac.kr