Kangweon-Kyungki Math. Jour. **13** (2005), No. 2, pp. 223–234

QUASI-SMOOTH α-STRUCTURE OF SMOOTH TOPOLOGICAL SPACES

Won Keun Min and Chun-Kee Park*

Abstract. We introduce the concepts of weak smooth α-closure and weak smooth α-interior of a fuzzy set and obtain some of their structural properties. We also introduce the concepts of several types of quasi-smooth α- compactness in terms of the concepts of weak smooth α-closure and weak smooth α-interior of a fuzzy set and investigate some of their properties.

1. Introduction

Badard [1] introduced the concept of a smooth topological space which is a generalization of Chang's fuzzy topological space [2]. Many mathematical structures in smooth topological spaces were introduced and studied. In particular, Gayyar, Kerre , Ramadan [5] and Demirci [3, 4] introduced the concepts of smooth closure and smooth interior of a fuzzy set and several types of compactness in smooth topological spaces and obtained some of their properties. In [6] we introduced the concepts of smooth α-closure and smooth α-interior of a fuzzy set which are generalizations of smooth closure and smooth interior of a fuzzy set defined in [3] and also introduced several types of α-compactness in smooth topological spaces and obtained some of their properties.

In this paper, we introduce the concepts of weak smooth α-closure and weak smooth α-interior of a fuzzy set and obtain some of their structural properties. We also introduce the concepts of several types of quasi-smooth α- compactness in terms of the concepts of weak smooth

Received July 26, 2005.
2000 Mathematics Subject Classification: 54A40, 03E72.
Key words and phrases: fuzzy sets, smooth topology, weak smooth α-closure, weak smooth α-interior, quasi-smooth α-compactness.
*Corresponding author

α-closure and weak smooth α-interior of a fuzzy set and investigate some of their properties.

2. Preliminaries

Let X be a set and $I = [0,1]$ be the unit interval of the real line. I^X will denote the set of all fuzzy sets of X. 0_X and 1_X will denote the characteristic functions of ϕ and X, respectively.

A smooth topological space (s.t.s.) [7] is an ordered pair (X, τ), where X is a non-empty set and $\tau : I^X \to I$ is a mapping satisfying the following conditions:

(O1) $\tau(0_X) = \tau(1_X) = 1$;

(O2) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(O3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$,

$$\tau(\cup_{i\in J}\ A_i) \geq \wedge_{i\in J}\ \tau(A_i).$$

Then the mapping $\tau : I^X \to I$ is called a smooth topology on X. The number $\tau(A)$ is called the degree of openness of A.

A mapping $\tau^* : I^X \to I$ is called a smooth cotopology [7] iff the following three conditions are satisfied:

(C1) $\tau^*(0_X) = \tau^*(1_X) = 1$;

(C2) $\forall A, B \in I^X,\ \tau^*(A \cup B) \geq \tau^*(A) \wedge \tau^*(B)$;

(C3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\tau^*(\cap_{i\in J}\ A_i) \geq \wedge_{i\in J}\ \tau^*(A_i)$.

If τ is a smooth topology on X, then the mapping $\tau^* : I^X \to I$, defined by $\tau^*(A) = \tau(A^c)$ where A^c denotes the complement of A, is a smooth cotopology on X. Conversely, if τ^* is a smooth cotopology on X, then the mapping $\tau : I^X \to I$, defined by $\tau(A) = \tau^*(A^c)$, is a smooth topology on X [7].

For the s.t.s. (X, τ) and $\alpha \in [0,1]$, the family $\tau_\alpha = \{A \in I^X : \tau(A) \geq \alpha\}$ defines a Chang's fuzzy topology (CFT) on X [2]. The family of all closed fuzzy sets with respect to τ_α is denoted by τ^*_α and we have $\tau^*_\alpha = \{A \in I^X : \tau^*(A) \geq \alpha\}$. For $A \in I^X$ and $\alpha \in [0,1]$, the τ_α-closure (resp., τ_α-interior) of A, denoted by $cl_\alpha(A)$ (resp., $int_\alpha(A)$), is defined by $cl_\alpha(A) = \cap\{K \in \tau^*_\alpha : A \subseteq K\}$ (resp., $int_\alpha(A) = \cup\{K \in \tau_\alpha : K \subseteq A\}$).

Demirci [3] introduced the concepts of smooth closure and smooth interior in smooth topological spaces as follows:

Let (X,τ) be a s.t.s. and $A \in I^X$. Then the τ-smooth closure (resp., τ-smooth interior) of A, denoted by $\overline{A}$ (resp., A^o), is defined by $\overline{A} = \cap\{K \in I^X : \tau^*(K) > 0, A \subseteq K\}$ (resp., $A^o = \cup\{K \in I^X : \tau(K) > 0, K \subseteq A\}$).

Let (X,τ) and (Y,σ) be two smooth topological spaces. A function $f : X \to Y$ is called smooth continuous with respect to τ and σ [7] iff $\tau(f^{-1}(A)) \geq \sigma(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is called weakly smooth continuous with respect to τ and σ [7] iff $\sigma(A) > 0 \Rightarrow \tau(f^{-1}(A)) > 0$ for every $A \in I^Y$. In this paper, a weakly smooth continuous function is called a quasi-smooth continuous function.

A function $f : X \to Y$ is smooth continuous with respect to τ and σ iff $\tau^*(f^{-1}(A)) \geq \sigma^*(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is weakly smooth continuous with respect to τ and σ iff $\sigma^*(A) > 0 \Rightarrow \tau^*(f^{-1}(A)) > 0$ for every $A \in I^Y$ [7].

A function $f : X \to Y$ is called smooth open (resp., smooth closed) with respect to τ and σ [7] if and only if $\tau(A) \leq \sigma(f(A))$ (resp., $\tau^*(A) \leq \sigma^*(f(A))$) for every $A \in I^X$.

A function $f : X \to Y$ is called smooth preserving (resp., strict smooth preserving) with respect to τ and σ [5] if and only if $\sigma(A) \geq \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) \geq \tau(f^{-1}(B))$ (resp., $\sigma(A) > \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) > \tau(f^{-1}(B))$) for every $A, B \in I^Y$.

If $f : X \to Y$ is a smooth preserving function (resp., a strict smooth preserving function) with respect to τ and σ, then $\sigma^*(A) \geq \sigma^*(B) \Leftrightarrow \tau^*(f^{-1}(A)) \geq \tau^*(f^{-1}(B))$ (resp., $\sigma^*(A) > \sigma^*(B) \Leftrightarrow \tau^*(f^{-1}(A)) > \tau^*(f^{-1}(B))$) for every $A, B \in I^Y$ [5].

A function $f : X \to Y$ is called smooth open preserving (resp., strict smooth open preserving) with respect to τ and σ [5] iff $\tau(A) \geq \tau(B) \Rightarrow \sigma(f(A)) \geq \sigma(f(B))$ (resp., $\tau(A) > \tau(B) \Rightarrow \sigma(f(A)) > \sigma(f(B))$) for every $A, B \in I^X$.

3. Weak smooth α-closure and weak smooth α-interior

In this section, we introduce the concepts of weak smooth α-closure and weak smooth α-interior of a fuzzy set in smooth topological spaces and investigate some of their properties.

Definition 3.1[6]. Let (X,τ) be a s.t.s., $\alpha \in [0,1)$ and $A \in I^X$. The τ-smooth α-closure (resp., τ-smooth α-interior) of A, denoted by $\overline{A}_\alpha$ (resp., A^o_α), is defined by $\overline{A}_\alpha = \cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A), A \subseteq K\}$ (resp., $A^o_\alpha = \cup\{K \in I^X : \tau(K) > \alpha\tau(A), K \subseteq A\}$).

Demirci [4] defined the families $W(\tau) = \{A \in I^X : A = A^o\}$ and $W^*(\tau) = \{A \in I^X : A = \overline{A}\}$, where (X,τ) is a s.t.s. Note that $A \in W(\tau) \Leftrightarrow A^c \in W^*(\tau)$.

We define the families $W_\alpha(\tau) = \{A \in I^X : A = A^o_\alpha\}$ and $W^*_\alpha(\tau) = \{A \in I^X : A = \overline{A}_\alpha\}$, where (X,τ) is a s.t.s. and $\alpha \in [0,1)$. Note that $A \in W_\alpha(\tau) \Leftrightarrow A^c \in W^*_\alpha(\tau)$.

Definition 3.2. Let (X,τ) be a s.t.s., $\alpha \in [0,1)$ and $A \in I^X$. The weak τ-smooth α-closure (resp., weak τ-smooth α-interior) of A, denoted by $wcl_\alpha(A)$ (resp., $wint_\alpha(A)$), is defined by $wcl_\alpha(A) = \cap\{K \in I^X : K \in W^*_\alpha(\tau), A \subseteq K\}$ (resp., $wint_\alpha(A) = \cup\{K \in I^X : K \in W_\alpha(\tau), K \subseteq A\}$).

Theorem 3.3. *Let (X,τ) be a s.t.s., $\alpha \in [0,1)$ and $A \in I^X$. Then*
(a) $A \subseteq wcl_\alpha(A) \subseteq \overline{A} \subseteq \overline{A}_\alpha$,
(b) $A^o_\alpha \subseteq A^o \subseteq wint_\alpha(A) \subseteq A$.

Proof. (a) Let $K \in I^X$ and $A \subseteq K$. Then $\tau^*(K) > \alpha\tau^*(A) \Rightarrow \tau^*(K) > 0$ and $\tau^*(K) > 0 \Rightarrow K = \overline{K}_\alpha$, i.e., $K \in W^*_\alpha(\tau)$ by Theorem 3.6[6]. From the definitions of $\overline{A}_\alpha$, $\overline{A}$ and $wcl_\alpha(A)$ we have $A \subseteq wcl_\alpha(A) \subseteq \overline{A} \subseteq \overline{A}_\alpha$.

(b) Let $K \in I^X$ and $K \subseteq A$. Then $\tau(K) > \alpha\tau(A) \Rightarrow \tau(K) > 0$ and $\tau(K) > 0 \Rightarrow K = K^o_\alpha$, i.e., $K \in W_\alpha(\tau)$ by Theorem 3.6[6]. From the definition of A^o_α, A^o and $wint_\alpha(A)$ we have $A^o_\alpha \subseteq A^o \subseteq wint_\alpha(A) \subseteq A$. □

Theorem 3.4. *Let (X,τ) be a s.t.s., $\alpha \in [0,1)$ and $A, B \in I^X$. Then*
(a) $A \subseteq B \Rightarrow wcl_\alpha(A) \subseteq wcl_\alpha(B)$,
(b) $A \subseteq B \Rightarrow wint_\alpha(A) \subseteq wint_\alpha(B)$,
(c) $(wcl_\alpha(A))^c = wint_\alpha(A^c)$,
(d) $wcl_\alpha(A) = (wint_\alpha(A^c))^c$,
(e) $(wint_\alpha(A))^c = wcl_\alpha(A^c)$,
(f) $wint_\alpha(A) = (wcl_\alpha(A^c))^c$.

Proof. (a) and (b) follow directly from Definition 3.2.

(c) From Definition 3.2 we have

$$\begin{aligned}(wcl_\alpha(A))^c &= (\cap\{K \in I^X : K \in W^*_\alpha(\tau), A \subseteq K\})^c \\ &= \cup\{K^c : K \in I^X, K^c \in W_\alpha(\tau), K^c \subseteq A^c\} \\ &= \cup\{U \in I^X : U \in W_\alpha(\tau), U \subseteq A^c\} \\ &= wint_\alpha(A^c).\end{aligned}$$

(d), (e) and (f) can be easily obtained from (c).

□

THEOREM 3.5. *Let* (X,τ) *be a s.t.s.,* $\alpha \in [0,1)$ *and* $A, B \in I^X$. *Then*

(a) $wcl_\alpha(0_X) = 0_X$,

(b) $A \subseteq wcl_\alpha(A)$,

(c) $wcl_\alpha(A) = wcl_\alpha(wcl_\alpha(A))$,

(d) $wcl_\alpha(A) \cup wcl_\alpha(B) \subseteq wcl_\alpha(A \cup B)$,

(e) $wcl_\alpha(A \cap B) \subseteq wcl_\alpha(A) \cap wcl_\alpha(B)$.

Proof. (a) By Theorem 3.4[6], $\overline{(0_X)}_\alpha = 0_X$, i.e., $0_X \in W^*_\alpha(\tau)$. From Definition 3.2 we have $wcl_\alpha(0_X) = 0_X$.

(b) follows directly from Definition 3.2.

(c) From (b) we have $wcl_\alpha(A) \subseteq wcl_\alpha(wcl_\alpha(A))$. From Definition 3.2 we have

$$\begin{aligned}wcl_\alpha(wcl_\alpha(A)) &= \cap\{K \in I^X : K \in W^*_\alpha(\tau), wcl_\alpha(A) \subseteq K\} \\ &= \cap\{K \in I^X : K \in W^*_\alpha(\tau), \cap\{U \in I^X : U \in W^*_\alpha(\tau), \\ &\qquad A \subseteq U\} \subseteq K\} \\ &\subseteq \cap\{K \in I^X : K \in W^*_\alpha(\tau), A \subseteq K\} \\ &= wcl_\alpha(A).\end{aligned}$$

Hence $wcl_\alpha(A) = wcl_\alpha(wcl_\alpha(A))$.

(d) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $wcl_\alpha(A) \subseteq wcl_\alpha(A \cup B)$ and $wcl_\alpha(B) \subseteq wcl_\alpha(A \cup B)$ by Theorem 3.4. Hence $wcl_\alpha(A) \cup wcl_\alpha(B) \subseteq wcl_\alpha(A \cup B)$.

(e) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $wcl_\alpha(A \cap B) \subseteq wcl_\alpha(A)$ and $wcl_\alpha(A \cap B) \subseteq wcl_\alpha(B)$ by Theorem 3.4. Hence $wcl_\alpha(A \cap B) \subseteq wcl_\alpha(A) \cap wcl_\alpha(B)$.

□

THEOREM 3.6. *Let (X, τ) be a s.t.s., $\alpha \in [0, 1)$ and $A, B \in I^X$. Then*
(a) $wint_\alpha(1_X) = 1_X$,
(b) $wint_\alpha(A) \subseteq A$,
(c) $wint_\alpha(A) = wint_\alpha(wint_\alpha(A))$,
(d) $wint_\alpha(A) \cup wint_\alpha(B) \subseteq wint_\alpha(A \cup B)$,
(e) $wint_\alpha(A \cap B) \subseteq wint_\alpha(A) \cap wint_\alpha(B)$.

Proof. The proof is similar to the proof of Theorem 3.5.

□

THEOREM 3.7. *Let (X, τ) be a s.t.s., $\alpha \in [0, 1)$ and $A \in I^X$. Then*
(a) $\tau^*(A) > 0 \Rightarrow wcl_\alpha(A) = A$,
(b) $\tau(A) > 0 \Rightarrow wint_\alpha(A) = A$.

Proof. Let $\tau^*(A) > 0$. Then $\overline{A}_\alpha = A$, i.e., $A \in W^*_\alpha(\tau)$ by Theorem 3.6[6]. Hence $A \in \{K \in I^X : K \in W^*_\alpha(\tau), A \subseteq K\}$. By Definition 3.2, $wcl_\alpha(A) \subseteq A$. By Theorem 3.3, $A \subseteq wcl_\alpha(A)$. Hence $wcl_\alpha(A) = A$.
(b) Let $\tau(A) > 0$. Then $A^o_\alpha = A$, i.e., $A \in W_\alpha(\tau)$ by Theorem 3.6[6]. Hence $A \in \{K \in I^X : K \in W_\alpha(\tau), K \subseteq A\}$. By Definition 3.2, $A \subseteq wint_\alpha(A)$. By Theorem 3.3, $wint_\alpha(A) \subseteq A$. Hence $wint_\alpha(A) = A$.

□

THEOREM 3.8. *Let (X, τ) be a s.t.s., $\alpha \in [0, 1)$ and $A \in I^X$. Then*
(a) if there exists a $\beta \in (\alpha\tau^*(A), 1]$ such that $A = cl_\beta(A)$, then $A = wcl_\alpha(A) = \overline{A} = \overline{A}_\alpha$,
(b) if there exists a $\beta \in (\alpha\tau(A), 1]$ such that $A = int_\beta(A)$, then $A = wint_\alpha(A) = A^o = A^o_\alpha$.

Proof. (a) If there exists a $\beta \in (\alpha\tau^*(A), 1]$ such that $A = cl_\beta(A)$, then $A \subseteq wcl_\alpha(A) \subseteq \overline{A} \subseteq \overline{A}_\alpha = \cap_{\beta > \alpha\tau^*(A)} cl_\beta(A) \subseteq cl_\beta(A) = A$ by Theorem 3.8[6] and 3.3. Hence $A = wcl_\alpha(A) = \overline{A} = \overline{A}_\alpha$.

(b) If there exists a $\beta \in (\alpha\tau(A), 1]$ such that $A = int_\beta(A)$, then $A = int_\beta(A) \subseteq \cup_{\beta > \alpha\tau(A)} int_\beta(A) = A^o_\alpha \subseteq A^o \subseteq wint_\alpha(A) \subseteq A$ by Theorem 3.8[6] and 3.3. Hence $A = wint_\alpha(A) = A^o = A^o_\alpha$.

□

Definition 3.9. Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. A function $f : X \to Y$ is called weak smooth α-continuous with respect to τ and σ iff $A \in W_\alpha(\sigma) \Rightarrow f^{-1}(A) \in W_\alpha(\tau)$ for every $A \in I^Y$.

Theorem 3.10. *Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. If a function $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ, then*

(a) $f(wcl_\alpha(A)) \subseteq wcl_\alpha(f(A))$ for every $A \in I^X$,
(b) $wcl_\alpha(f^{-1}(A)) \subseteq f^{-1}(wcl_\alpha(A))$ for every $A \in I^Y$,
(c) $f^{-1}(wint_\alpha(A)) \subseteq wint_\alpha(f^{-1}(A))$ for every $A \in I^Y$.

Proof. (a) For every $A \in I^X$, we have

$$\begin{aligned} & f^{-1}(wcl_\alpha(f(A))) = f^{-1}(\cap\{U \in I^Y : U \in W^*_\alpha(\sigma), f(A) \subseteq U\}) \\ & \supseteq f^{-1}(\cap\{U \in I^Y : f^{-1}(U) \in W^*_\alpha(\tau), A \subseteq f^{-1}(U)\}) \\ & = \cap\{f^{-1}(U) \in I^X : U \in I^Y, f^{-1}(U) \in W^*_\alpha(\tau), A \subseteq f^{-1}(U)\} \\ & \supseteq \cap\{K \in I^X : K \in W^*_\alpha(\tau), A \subseteq K\} \\ & = wcl_\alpha(A). \end{aligned}$$

Hence $f(wcl_\alpha(A)) \subseteq wcl_\alpha(f(A))$.

(b) For every $A \in I^Y$, we have

$$\begin{aligned} & f^{-1}(wcl_\alpha(A)) = f^{-1}(\cap\{U \in I^Y : U \in W^*_\alpha(\sigma), A \subseteq U\}) \\ & \supseteq f^{-1}(\cap\{U \in I^Y : f^{-1}(U) \in W^*_\alpha(\tau), f^{-1}(A) \subseteq f^{-1}(U)\}) \\ & = \cap\{f^{-1}(U) \in I^X : U \in I^Y, f^{-1}(U) \in W^*_\alpha(\tau), \\ & \qquad f^{-1}(A) \subseteq f^{-1}(U)\} \\ & \supseteq \cap\{K \in I^X : K \in W^*_\alpha(\tau), f^{-1}(A) \subseteq K\} \\ & = wcl_\alpha(f^{-1}(A)). \end{aligned}$$

(c) For every $A \in I^Y$, we have

$$\begin{aligned} &f^{-1}(wint_\alpha(A)) = f^{-1}(\cup\{U \in I^Y : U \in W_\alpha(\sigma), U \subseteq A\}) \\ &\subseteq f^{-1}(\cup\{U \in I^Y : f^{-1}(U) \in W_\alpha(\tau), f^{-1}(U) \subseteq f^{-1}(A)\}) \\ &= \cup\{f^{-1}(U) \in I^X : U \in I^Y, f^{-1}(U) \in W_\alpha(\tau), \\ &\quad f^{-1}(U) \subseteq f^{-1}(A)\} \\ &\subseteq \cup\{K \in I^X : K \in W_\alpha(\tau), K \subseteq f^{-1}(A)\} \\ &= wint_\alpha(f^{-1}(A)). \end{aligned}$$

□

Definition 3.11. Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. A function $f : X \to Y$ is called weak smooth α-open (resp., weak smooth α-closed) with respect to τ and σ iff $A \in W_\alpha(\tau) \Rightarrow f(A) \in W_\alpha(\sigma)$ (resp., $A \in W^*_\alpha(\tau) \Rightarrow f(A) \in W^*_\alpha(\sigma)$) for every $A \in I^X$.

Theorem 3.12. *Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. If a function $f : X \to Y$ is weak smooth α-open with respect to τ and σ, then $f(wint_\alpha(A)) \subseteq wint_\alpha(f(A))$ for every $A \in I^X$.*

Proof. For every $A \in I^X$, we have

$$\begin{aligned} f(wint_\alpha(A)) &= f(\cup\{U \in I^X : U \in W_\alpha(\tau), U \subseteq A\}) \\ &\subseteq f(\cup\{U \in I^X : f(U) \in W_\alpha(\sigma), f(U) \subseteq f(A)\}) \\ &= \cup\{f(U) \in I^Y : U \in I^X, f(U) \in W_\alpha(\sigma), f(U) \subseteq f(A)\} \\ &\subseteq \cup\{K \in I^Y : K \in W_\alpha(\sigma), K \subseteq f(A)\} \\ &= wint_\alpha(f(A)). \end{aligned}$$

□

4. Several types of quasi-smooth α-compactness

In this section, we introduce the concepts of several types of quasi-smooth α-compactness in smooth topological spaces and investigate some of their properties.

Definition 4.1. Let $\alpha \in [0,1)$. A s.t.s. (X,τ) is called quasi-smooth nearly α-compact iff for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_X$.

Definition 4.2. Let $\alpha \in [0,1)$. A s.t.s. (X,τ) is called quasi-smooth almost α-compact iff for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} wcl_\alpha(A_i) = 1_X$.

Definition 4.3[3]. A s.t.s. (X,τ) is called smooth compact iff for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} A_i = 1_X$.

Definition 4.4[3]. A s.t.s. (X,τ) is called smooth nearly compact (resp., smooth almost compact) iff for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}(\overline{A}_i)^o = 1_X$ (resp., $\cup_{i\in J_0}\overline{A}_i = 1_X$).

Definition 4.5[6]. Let $\alpha \in [0,1)$. A s.t.s. (X,τ) is called smooth nearly α-compact (resp., smooth almost α-compact) iff for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha = 1_X$ (resp., $\cup_{i\in J_0}\overline{(A_i)}_\alpha = 1_X$).

Theorem 4.6. *Let (X,τ) be a s.t.s. and let $\alpha \in [0,1)$. Then (X,τ) is quasi-smooth almost α-compact $\Rightarrow$ (X,τ) is smooth almost compact $\Rightarrow$ (X,τ) is smooth almost α-compact.*

Proof. The proof follows directly from Theorem 3.3.

□

Theorem 4.7. *Let (X,τ) be a s.t.s. and let $\alpha \in [0,1)$. If (X,τ) is smooth compact, then (X,τ) is quasi-smooth nearly α-compact.*

Proof. Let (X,τ) be a smooth compact s.t.s. Then for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} A_i = 1_X$. Since $\tau(A_i) > 0$ for each $i \in J$, $A_i = wint_\alpha(A_i)$ for each $i \in J$ by Theorem 3.7. From Theorem 3.3 and 3.4 we have $A_i = wint_\alpha(A_i) \subseteq wint_\alpha(wcl_\alpha(A_i))$ for each $i \in J$. Thus $1_X = \cup_{i\in J_0} A_i \subseteq \cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i))$, i.e., $\cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_X$. Hence (X,τ) is quasi-smooth nearly α-compact.

□

THEOREM 4.8. *Let $\alpha \in [0,1)$. Then a quasi-smooth nearly α-compact s.t.s. (X,τ) is quasi-smooth almost α-compact.*

Proof. Let (X,τ) be a quasi-smooth nearly α-compact s.t.s. Then for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_X$. Since $wint_\alpha(wcl_\alpha(A_i)) \subseteq wcl_\alpha(A_i)$ for each $i \in J$ by Theorem 3.3, $1_X = \cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)) \subseteq \cup_{i\in J_0} wcl_\alpha(A_i)$. Thus $\cup_{i\in J_0} wcl_\alpha(A_i) = 1_X$. Hence (X,τ) is quasi-smooth almost α-compact.

□

THEOREM 4.9. *Let (X,τ) and (Y,σ) be two smooth topological spaces, $\alpha \in [0,1)$ and $f : X \to Y$ a surjective, quasi-smooth continuous and weak smooth α-continuous function with respect to τ and σ. If (X,τ) is quasi-smooth almost α-compact, then so is (Y,σ).*

Proof. Let $\{A_i : i \in J\}$ be a family in $\{A \in I^Y : \sigma(A) > 0\}$ covering Y, i.e., $\cup_{i\in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i\in J} f^{-1}(A_i)$. Since f is quasi-smooth continuous with respect to τ and σ, $\tau(f^{-1}(A_i)) > 0$ for each $i \in J$. Since (X,τ) is quasi-smooth almost α-compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} wcl_\alpha(f^{-1}(A_i)) = 1_X$. From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0} wcl_\alpha(f^{-1}(A_i))) = \cup_{i\in J_0} f(wcl_\alpha(f^{-1}(A_i)))$. Since $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ, from Theorem 3.10 we have $wcl_\alpha(f^{-1}(A)) \subseteq f^{-1}(wcl_\alpha(A))$ for every $A \in I^Y$. Hence

$$\begin{aligned} 1_Y &= \cup_{i\in J_0} f(wcl_\alpha(f^{-1}(A_i))) \subseteq \cup_{i\in J_0} f(f^{-1}(wcl_\alpha(A_i))) \\ &= \cup_{i\in J_0} wcl_\alpha(A_i), \end{aligned}$$

i.e., $\cup_{i\in J_0} wcl_\alpha(A_i) = 1_Y$. Thus (Y,σ) is quasi-smooth almost α-compact.

□

THEOREM 4.10. *Let (X,τ) and (Y,σ) be two smooth topological spaces, $\alpha \in [0,1)$ and $f : X \to Y$ a surjective, quasi-smooth continuous, weak smooth α-continuous and weak smooth α-open function with respect to τ and σ. If (X,τ) is quasi-smooth nearly α-compact, then so is (Y,σ).*

Proof. Let $\{A_i : i \in J\}$ be a family in $\{A \in I^Y : \sigma(A) > 0\}$ covering Y, i.e., $\cup_{i\in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i\in J} f^{-1}(A_i)$. Since f is quasi-smooth continuous, $\tau(f^{-1}(A_i)) > 0$ for each $i \in J$. Since (X,τ) is quasi-smooth nearly α-compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} wint_\alpha(wcl_\alpha(f^{-1}(A_i))) = 1_X$. From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0} wint_\alpha(wcl_\alpha(f^{-1}(A_i)))) = \cup_{i\in J_0} f(wint_\alpha(wcl_\alpha(f^{-1}(A_i))))$.

Since $f : X \to Y$ is weak smooth α-open with respect to τ and σ, from Theorem 3.12 we have

$$f(wint_\alpha(wcl_\alpha(f^{-1}(A_i)))) \subseteq wint_\alpha(f(wcl_\alpha(f^{-1}(A_i))))$$

for each $i \in J$. Since $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ, from Theorem 3.10 we have $wcl_\alpha(f^{-1}(A_i)) \subseteq f^{-1}(wcl_\alpha(A_i))$ for each $i \in J$. Hence we have

$$\begin{aligned} 1_Y &= \cup_{i\in J_0} f(wint_\alpha(wcl_\alpha(f^{-1}(A_i)))) \\ &\subseteq \cup_{i\in J_0} wint_\alpha(f(wcl_\alpha(f^{-1}(A_i)))) \\ &\subseteq \cup_{i\in J_0} wint_\alpha(f(f^{-1}(wcl_\alpha(A_i)))) \\ &= \cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)). \end{aligned}$$

Thus $\cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_Y$. Hence (Y,σ) is quasi-smooth nearly α-compact.

□

DEFINITION 4.11. Let $\alpha \in [0,1)$. A s.t.s. (X,τ) is called quasi-smooth α-regular iff each fuzzy set $A \in I^X$ satisfying $\tau(A) > 0$ can be written as $A = \cup\{K \in I^X : \tau(K) \geq \tau(A), wcl_\alpha(K) \subseteq A\}$.

THEOREM 4.12. *Let $\alpha \in [0,1)$. Then a quasi-smooth almost α-compact quasi-smooth α-regular s.t.s. (X,τ) is smooth compact.*

Proof. Let $\{A_i : i \in J\}$ be a family in $\{A \in I^X : \sigma(A) > 0\}$ covering X, i.e., $\cup_{i\in J} A_i = 1_X$. Since (X,τ) is quasi-smooth α-regular, $A_i = \cup_{j_i\in J_i}\{K_{j_i} \in I^X : \tau(K_{j_i}) \geq \tau(A_i), wcl_\alpha(K_{j_i}) \subseteq A_i\}$ for each $i \in J$. Since $\cup_{i\in J} A_i = \cup_{i\in J}[\cup_{j_i\in J_i} K_{j_i}] = 1_X$ and (X,τ) is quasi-smooth almost α-compact, there exists a finite subfamily $\{K_l \in I^X : \tau(K_l) > 0,\ l \in L\}$ such that $\cup_{l\in L} wcl_\alpha(K_l) = 1_X$. Since for each $l \in L$ there exists $i \in J$ such that $wcl_\alpha(K_l) \subseteq A_i$, $\cup_{i\in J_0} A_i = 1_X$, where J_0 is a finite subset of J. Hence (X,τ) is smooth compact.

□

We obtain the following corollary from Theorem 4.8 and 4.12.

COROLLARY 4.13. *Let $\alpha \in [0,1)$. Then a quasi-smooth nearly α-compact quasi-smooth α-regular s.t.s. (X,τ) is smooth compact.*

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Department of Mathematics
Kangwon National University
Chuncheon 200–701, Korea
E-mail: wkmin@kangwon.ac.kr, ckpark@kangwon.ac.kr