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ON THE SEMI CONTINUOUS FUNCTIONS WITH THE OPEN PROPERTY

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Abstract. Some of the generalized continuous functions and their basic properties are introduced in concern with the cover theory. The open property of a function is a crucial tool for the survey of this area.

1. Introduction

As a generalization of a continuous function, L. Levin gave the weakly continuous function ([2]). In the second section of this present note we will introduce several generalized continuous functions like quasi, weakly quasi, almost, semi, relatively continuous functions and will show through counterexamples that they are independent concepts to each other. In next two sections the properties of semi continuous functions which have additional open property are studied in concern with the cover theory.

For this purpose we define in following basic concepts. Throughout this paper alphabet $X, Y, ..$ denote the topological spaces. The closure of S and interior of S are denoted by Cl(S) and Int(S), respectively.

$S \subseteq X$ is said to be *quasi open* if to each $x \in S$ and to each open set $Q \subseteq X$ containing x there is a non empty open $G \subseteq Q$ such that $G \subseteq S$. $S \subseteq X$ is said to be *semi open* if there is an open set Q such that $Q \subseteq S \subseteq Cl(Q)$.

Remark 1.1. *The quasi openness and the semi openness are equivalent.*

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Recall that a function $f : X \to Y$ is said to be *weakly continuous at* $x \in X$ if to each open set V in Y containing $f(x)$ there is an open set U containing x such that $U \subseteq f^{-1}(Cl(V))$. $f : X \to Y$ is said to be *weakly continuous* if f is weakly continuous at every point of X. A function $f : X \to Y$ is said to be *quasi continuous (weakly quasi-continuous) at* $x \in X$ if to each open set V in Y containing $f(x)$ and to each open set U in X containing x there is a non empty open set G contained in U such that $G \subseteq f^{-1}(V)$ $(G \subseteq f^{-1}(Cl(V)))$, respectively. $f : X \to Y$ is said to be *quasi continuous* (*weakly quasi continuous*) if it is quasi continuous (weakly quasi continuous) at each point $x \in X$, respectively. $f : X \to Y$ is said to be *semi continuous at* $x \in X$ if for each open set V in Y containing $f(x)$ $f^{-1}(V)$ is semi open. $f : X \to Y$ is said to be *semi continuous* if it is semi continuous at each point $x \in X$. From the remark 1.1 the following statement is straightforward.

Remark 1.2. *The quasi continuity and the semi continuity are equivalent.*

$f : X \to Y$ is said to be *almost continuous at* $x \in X$ if for each open subset V of Y containing $f(x)$ $Cl(f^{-1}(V))$ is a neighborhood of x. $f : X \to Y$ is said to be *almost continuous* if it is almost continuous at each point $x \in X$ ([3]). J. Chew ([1]) defined a function $f : X \to Y$ *relatively continuous* at $x \in X$ if given an open set V in Y containing $f(x)$, the set $f^{-1}(V)$ is an open set in the subspace $f^{-1}(Cl(V))$. f is said to be *relative continuous* if this condition is satisfied for each $x \in X$. He showed that a continuous function is relatively continuous but the inverse is not true.

2. Examples

Here we will show that the almost, quasi, almost and relatively continuity are pairwise independent concepts.

Example 1 (a) There is a function which is quasi continuous but not relatively continuous. A function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} x \text{ if } x \in (-\infty, 0) \\ x+1 \text{ if } x \in [0, \infty) \end{cases}$ where on both $\mathbb{R}$ the topology $\tau_\infty = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) | a \in \mathbb{R}\}$ are given is quasi continuous. But it is not relatively continuous at $x = 0$. To see this choose $0 < \epsilon < 1$ and $V := (-\epsilon + 1, \infty)$ as a neighborhood of the point $f(0) = 1$. Then $f^{-1}(V) = f^{-1}((-\epsilon + 1, \infty)) = [0, \infty)$ and $f^{-1}(Cl(V)) = f^{-1}(Cl(-\epsilon + 1, \infty)) = f^{-1}(\mathbb{R}) = \mathbb{R}$. It is obvious that $f^{-1}(V)$ can not be open in $f^{-1}(Cl(V)$.

(b) There is a quasi continuous function which is not almost continuous. Consider the same function of example (a). Given the usual topologies on domain and range $\mathbb{R}$, the function is quasi continuous but not almost continuous at $x = 0$.

Example 2 **(a)** There is an almost continuous function which is not relatively continuous. Consider an identity function id: $\mathbb{R} \to \mathbb{R}$ where on domain $\mathbb{R}$ the indiscrete topology and on range $\mathbb{R}$ the usual topology are given. Then id is obviously almost continuous, but for any $\epsilon > 0$ and $x \in \mathbb{R}$ $id^{-1}((-\epsilon + x, x + \epsilon)) = (-\epsilon + x, x + \epsilon)$ is not open in $id^{-1}(Cl(-\epsilon + x, x + \epsilon)) = id^{-1}([-\epsilon + x, x + \epsilon]) = [-\epsilon + x, x + \epsilon]$ relative to the indiscrete topology on range $\mathbb{R}$.

(b) There is an almost continuous function which is not quasi continuous. Consider an identity function id: $\mathbb{R} \to \mathbb{R}$ where on domain $\mathbb{R}$ the topology $\{\emptyset, \mathbb{R}, (-\infty, a), [a, \infty)\}$ for one $a \in \mathbb{R}$ and on range $\mathbb{R}$ the discrete topology are given. Then id is an almost continuous function but not quasi continuous.

Example 3 **(a)** There is a relatively continuous function which is not quasi continuous. Consider a Dirichlet function $D : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ otherwise} \end{cases}$ with the usual topology on both $\mathbb{R}$. Then D is relatively continuous but not quasi continuous.

(b) There is a relatively continuous function which is not almost continuous. Consider a function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 1 \text{ if } x \in [0, \infty) \\ 0 \text{ if } x \in (-\infty, 0) \end{cases}$ with the usual topology on both $\mathbb{R}$. Then f is relatively continuous but not almost continuous at $x = 0$.

3. The inverse images of (semi) open sets

REMARK 3.1. *$f : X \to Y$ is semi continuous if and only if $f^{-1}(V)$ is semi open for each open $V \subseteq Y$*

THEOREM 3.1. *Let $f : X \to Y$ be open. Then f is semi continuous if and only if $f^{-1}(S)$ is semi open for each semi open $S \subseteq Y$*

Proof. Necessity. Let $S \subseteq Y$ be semi open. There exists an open $V \subseteq Y$ such that $V \subseteq S \subseteq Cl(V)$. Hence $f^{-1}(V) \subseteq f^{-1}(S) \subseteq f^{-1}(Cl(V))$. By remark above there is an open $T \subseteq X$ such that $T \subseteq f^{-1}(V) \subseteq Cl(T)$. We claim $f^{-1}(Cl(V)) \subseteq Cl(T)$. Let $x \in f^{-1}(Cl(V))$ and $x \in W, W$ open in X. Then $f(x) \in Cl(V)$ and $f(W) \cap V \neq \emptyset$. Hence there is $y \in W$ such that $f(y) \in V$. Thus $y \in f^{-1}(V) \subseteq Cl(T)$ that means $W \cap T \neq \emptyset$. Therefore $T \subseteq f^{-1}(V) \subseteq f^{-1}(S) \subseteq f^{-1}(Cl(V)) \subseteq Cl(T)$. Sufficiency is obvious by the remark above. q.e.d.

THEOREM 3.2. *Let $f : X \to Y$ be open. Then f is weakly quasi continuous if and only if $f^{-1}(Cl(V))$ is semi open for each open $V \subseteq Y$*

Proof. Necessity. Let $V \subseteq Y$ be open. We show $f^{-1}(Cl(V)) \subseteq Cl(Int(f^{-1}$
$(Cl(V))))$. Let $y \in f^{-1}(Cl(V))$ and W be an open subset of X containing y. Then $f(y) \in Cl(V)$, thus $f(W) \cap V \neq \emptyset$. There exists $z \in W$ such tht $f(z) \in V$. By the quasi continuity of f there exists non empty open $G \subseteq W$ such that $G \subseteq f^{-1}(Cl(V))$. Therefore $G \subseteq Int(f^{-1}(Cl(V)))$ that means $W \cap Int(f^{-1}(Cl(V))) \neq \emptyset$. Sufficiency is obvious. q.e.d.

THEOREM 3.3. *If $f : X \to Y$ is almost continuous, then $Cl(f^{-1}(V))$ is semi open for each open $V \subseteq Y$*

Proof. Let $V \subseteq Y$ be open. We show $Cl(f^{-1}(V)) \subseteq Cl(Int(Cl$ $(f^{-1}(V))))$. Let $x \in Cl(f^{-1}(V))$ and $x \in W, W \subseteq X$ be open . Then $W \cap f^{-1}(V) \neq \emptyset$, thus there exists $y \in W$ such that $f(y) \in V$. Hence there exists an open set T containing y such that $T \subseteq Cl(f^{-1}(V))$ and also $W \cap T \subseteq Cl(f^{-1}(V))$. Therefore $W \cap Int(Cl(f^{-1}(V)) \neq \emptyset$ that means $x \in Cl(Int(Cl(f^{-1}(V))))$. q.e.d.

Recall that space X is said to be *extremally disconnected* if the closure of an open subset is open. If the condition of extremally disconnectedness of X is added in the theorem 3.3, it is obvious that for each open subset V of Y $Cl(f^{-1}(V))$ is open. Hence we have

Theorem 3.4. *Let X be extremally disconnected. Then $f : X \to Y$ is almost continuous if and only if $Cl(f^{-1}(V))$ is semi open for each open $V \subseteq Y$.*

J. Chew proved that a function $f : X \to Y$ is continuous if and only if it is weakly continuous and relatively continuous ([1]). Observe that, if a function $f : X \to Y$ is open, then for each $A \subseteq X$, $f^{-1}(Cl(A)) \subseteq Cl(f^{-1}(A))$. It is easy to see that from the weak continuity with this open property follows the almost continuity.

Theorem 3.5. *Weak continuity plus openness implies an almost continuity.*

Theorem 3.6. *Quasi continuity plus almost continuity implies the weak continuity.*

Proof. Let $f : X \to Y$ be quasi continuous, almost continuous and open function. Let $x \in X$ and $V \subseteq Y$ be an open set containing $f(x)$. By the almost continuity of f to each $y \in f^{-1}(V)$ there is an open $U_y \subseteq X$ such that $y \in U_y \subseteq Cl(f^{-1}(V))$. Put $U = \bigcup\{U_y | y \in f^{-1}(V)\}$. We claim $f(U) \subseteq Cl(V)$. Let $z \in U$. There is then $y \in f^{-1}(V)$ such that $z \in U_y$. Let $W \subseteq Y$ be an open set containing $f(z)$. By the quasi continuity of f, $f^{-1}(W)$ is semi open(remark 3.1). Hence there is an open $T \subseteq X$ such that $T \subseteq f^{-1}(W) \subseteq Cl(T)$. From $z \in Cl(T)$ follows $T \cap U_y \neq \emptyset$. By $U_y \subseteq Cl(f^{-1}(V))$, $T \cap Cl(f^{-1}(V)) \neq \emptyset$. Thus $T \cap f^{-1}(V) \neq \emptyset$. Therefore $f^{-1}(W) \cap f^{-1}(V) = f^{-1}(V \cap W) \neq \emptyset$ that means $V \cap W \neq \emptyset$. q.e.d.

4. Semi Connected Spaces

Definition 4.1. *X is said to be semi disconnected if there exist non empty disjoint semi open sets S_1 and S_2 such that $X = S_1 \cup S_2$.*
X is said to be semi connected if it is not semi disconnected.

It is obvious that a disconnected space is semi disconnected. Notice that a semi continuous functions does not preserve the semi connected property, but if we add the open property then it works.

Lemma 4.1. *Let A, B, C be subsets of topological space X. Then $A \setminus Cl(B) \subseteq Cl(A) \setminus Cl(B) \subseteq Cl(A \setminus Cl(B))$.*

We next show the open sets of the T_3-space with the $\sigma-$locally finite basis are expressible as a union of countably many closed sets and such spaces are under certain hypotheses semi disconnected.

THEOREM 4.2. *If X is a T_3-space with $\sigma-$locally finite basis, every open set is a union of countably many closed sets.*

Proof. Let $\mathcal{B} = \bigcup\{\mathcal{B}_n | n \in \mathbb{N}\}$ be a $\sigma-$locally finite basis of X and Q be an open subset of X. Putting for each $n \in \mathbb{N}$ $\mathcal{U}_n = \{V | V \in \mathcal{B}_n, Cl(V) \subseteq Q\}$ and $Cl(\bigcup \mathcal{U}_n) = A_n$, then it holds $Q = \bigcup\{A_n | n \in \mathbb{N}\}$.

" $\supseteq$:" Let $n \in \mathbb{N}$. It is enough to show $A_n \subseteq Q$. Let $x \in A_n = Cl(\bigcup \mathcal{U}_n)$. Let W be a neighborhood of x which intersects only finitely many sets of $\mathcal{B}_n$. Then there are just finitely many set of $\mathcal{U}_n$ which are not disjoint to W. Let $\mathcal{V}$ be a union of such sets and $Y = \bigcup \mathcal{V}$. Then Y is open and $W \setminus Cl(Y)$ also. Since $(W \setminus Y) \cap \bigcup \mathcal{U}_n = \emptyset$, hence also $(W \setminus Cl(Y)) \cap \bigcup \mathcal{U}_n = \emptyset$. Since x is an adherent point of $\bigcup \mathcal{U}_n$, $x \notin W \setminus Cl(Y)$. By $x \in W$, $x \in Cl(Y) = Cl(\bigcup \mathcal{V}) = \bigcup\{Cl(V) | V \in \mathcal{V}\} \subseteq Q$.
" $\subseteq$:" Let $x \in Q$. Then $x \notin X \setminus Q$, thus by T_3 property there are open sets V and W such that $x \in V, X \setminus Q \subseteq W$ and $V \cap W = \emptyset$. Hence $V \subseteq X \setminus W \subseteq Q$ and from this follows $Cl(V) \subseteq X \setminus W$. Since $\mathcal{B}$ is a basis, there exists $V' \in \mathcal{B}$ with $x \in V' \subseteq V$. It holds $Cl(V') \subseteq Cl(V) \subseteq Q$. Thus there exists $n \in \mathbb{N}$ such that $V' \in \mathcal{B}_n$, so that $V' \in \mathcal{U}_n$. Therefore $x \in \bigcup \mathcal{U}_n \subseteq A_n$. q.e.d

From the proof for the theorem 4.2 we see that

THEOREM 4.3. *If X is a T_3- space with a $\sigma-$locally finite basis, then X is expressible as a union of countably many semi open sets. More precisely, $X = \bigcup_{n \in \mathbb{N}} Cl(Q_n), Q_n$ open subset of X.*

Let X be a T_3-space with a $\sigma-$locally finite basis. According to theorem 4.3 X is expressible as $X = \bigcup_{n \in \mathbb{N}} Cl(Q_n), Q_n$ open subset of X. Let now $S_1 = Cl(Q_1)$ and $S_n = Cl(Q_1) \cup \cdots \cup Cl(Q_n) \setminus Cl(Q_1) \cup \cdots \cup Cl(Q_{n-1}), n \geq 2$. Then it holds $X = \bigcup_{n \in \mathbb{N}} S_n$. By the lemma above for each $n \in \mathbb{N}$ S_n is semi open. If there are two non empty and distinct $S_n, S_m, n, m \in \mathbb{N}$, then they are disjoint to each other. In this case, with the fact that the union of semi open sets is semi open, we can obtain the following result.

REMARK 4.2. *A T_3- space with a $\sigma-$locally finite basis is semi disconnected.*

5. The Quasi Continuity and S-Closed Property

Recall that a space X is said to be $S-$*closed* if for every semi open cover of X there exists a finite subfamily such that the union of their closures covers X.

LEMMA 5.1. *If $f: X \to Y$ be quasi continuous and X be extremally disconnected, then $f(Cl(T)) \subseteq Cl(f(T))$ for each open subset T of X.*

Proof. Let $T \subseteq X$ be open and assume there exists $x \in Cl(T)$ such that $f(x) \notin Cl(f(T))$. Then there exists an open subset W of Y such that $f(x) \in W$ and $W \cap f(T) = \emptyset$. Thus $f^{-1}(W) \cap T = \emptyset$ and also $Cl(f^{-1}(W)) \cap T = \emptyset$. By remark 3.1 there exists open subset Q of X such that $Cl(f^{-1}(W)) = Cl(Q)$. Since $x \in Cl(T)$ and $Cl(f^{-1}(W))$ is open, $Cl(f^{-1}(W)) \cap T \neq \emptyset$ that is impossible. q.e.d.

THEOREM 5.2. *Let $f : X \to Y$ be a quasi continuous,open and surjective function. Let X be extremally disconnected. If X is $S-$closed then Y also.*

Proof. Let $(S_\lambda)_\lambda$ be a semi open cover of Y. Then by theorem 3.1 $(f^{-1}(S_\lambda))_\lambda$ is a semi open cover of X. Since X is $S-$ closed, there exists $k \in \mathbb{N}$ such that $(Cl(f^{-1}(S_{\lambda_i})))_{i=1,\dots,k}$ is a cover of X. Besides there are opens subsets T_i of X such that $T_i \subseteq f^{-1}(S_{\lambda_i}) \subseteq Cl(T_i)$, thus $Cl(T_i) = Cl(f^{-1}(S_{\lambda_i})), i = 1, \dots, k$. Applying the lemma above to the last subset implications, we have $f(T_i) \subseteq f(f^{-1}(S_{\lambda_i})) \subseteq f(Cl(T_i)) \subseteq Cl(f(T_i))$, so that

$$\begin{aligned} Cl(f(T_i)) &= Cl(f(f^{-1}(S_{\lambda_i}))) = Cl(f(Cl(T_i)) \\ &= Cl(f(Cl(f^{-1}(S_{\lambda_i})))) \subseteq Cl(S_{\lambda_i}), i = 1, \dots, k. \end{aligned}$$

Therefore, it holds $f(Cl(f^{-1}(S_{\lambda_i}))) \subseteq Cl(S_{\lambda_i})$ that means $(Cl(S_{\lambda_i}))_{i=1,\dots,k}$ is a cover of Y. q.e.d.

References

1. J. Chew, *Some Remarks on Weak Continuity 2*, Mathematical Monthly, [December] (1991), pp. 931–934.
2. N. Levin, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly, vol 68 (1961), pp. 44–46.

3. Paul E.Long and Earl E.Mcgehee, Jr., *Properties of almost continuous functions*, Proc. Amer. Math. Soc. vol 24, (1970), MR 40# 4931, pp. 175–180

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