Evaluation of Effective Stiffness for 3D Beam with Repeated Structure

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Analysis of structures which are composed of numerous repeated unit structures can be simplified by using homogenized properties. If the unit structure is repeated in one direction, the whole structure may be regarded as a beam. Once the effective stiffness is obtained from the analysis of the unit structure in a proper way, the effort for the detail modeling of the global structure is not required, and the real structure can be replaced simply with a beam. This study proposes a kinematical periodicity constraint to be imposed on the FE model of the unit structure, which improves the accuracy of the effective stiffness. The method is employed to a one dimensionally arrayed 3D structure containing periodically repeated unsymmetrical holes. It is demonstrated that the deformation behavior of the homogenized beam agrees well with that of the real structure.

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1. Introduction

The periodically repeated structures are widely used in a variety of engineering applications, whether their representative volume elements are in microscopic or macroscopic levels. The detailed modeling of such structures is usually impractical due to the innumerable constituent elements. Instead, the domains are often simplified into homogeneous solids with homogenized effective properties.

The homogenized properties are evaluated through the analysis of the unit cell, but the mathematical difficulties caused by the geometric complexity and nonlinear behavior of material make the numerical approach more preferable. A number of studies concerning the homogenization of solids have been reported. Among many others, Anthoine[1] and Chung[2-5] noticed that the distribution of strain is periodic under the uniform stress field, and proposed to impose a certain constraint on the boundary faces of the unit cell model to realize the periodicity. In their studies, only a single cell was analyzed to yield accurate effective properties.

The macroscopic beams composed of repeated unit cell structures also need the proper homogenization technique. Pala and Ozmen[6] used the effective stiffness concept in the analysis for the design of an earthquake-resistant structure. Anthoine et al.[7] evaluated the effective properties of steel reinforced three dimensional concrete beam with progressive damage. Burgardt and Catraud[8] defined a unit cell for a truss structure, and derived its effective stiffness. Chung[9] derived the periodicity constraint equations for a beam-in-plane and showed that the resulting effective properties were good enough to predict the global behavior of in-homogeneous beams.

Unlike two or three dimensional solids, in which all the faces are surrounded by adjacent cells, the unit cells for beam problems are connected to the global structure through only two faces along the longitudinal direction of the beam, while the others are free. In addition, the uniform stress field needs to be replaced with the uniform section load field for the characterization of beams. These are the main reasons for the need of the distinct homogenization strategy from the two or three dimensional solid case. In this study, periodicity constraint equations for beams-in-space are proposed. These can be imposed on the finite element models for beams constructed of three dimensional unit structures arrayed in one direction.

For the validation purpose of the proposed method, a beam-in-space containing periodically spaced unsymmetrical voids is analyzed. The evaluated effective stiffness properties are employed to predict the deflection of a cantilever beam. The results are compared with the deformation behavior of the one modeled in detail.

2. Periodicity Constraints for Beams-in-Space

2.1 Behavior of Beams in Space

The structural properties of beams-in-space can be described by the stiffness or flexibility matrices against axial and flexural loads or deformations as Eq. (1).

$$\begin{Bmatrix} F_x \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \kappa_y \\ \kappa_z \end{Bmatrix} \tag{1a}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \kappa_y \\ \kappa_z \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ -\dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 v}{\partial x^2} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{Bmatrix} F_x \\ M_y \\ M_z \end{Bmatrix} \tag{1b}$$

Here. ε_{xx} and κ are the extensional strain and curvature, and F and M are the axial load and the bending moment, respectively. The reference coordinate frame and the directions of the section loads and the displacements are shown in Fig. 1. In case the internal structure of the beam is unsymmetrical with respect to xy and yz planes, the coupling effects between extension and bending or bending in y and z directions result in nonzero off-diagonal terms in the stiffness and flexibility matrices.

2.2 Periodicity Constraints

The effective stiffness and flexibility of beam, C_{ij} and S_{ij}, can be evaluated from the behavior of the unit cell under the uniform section loads, which is equivalent to the uniform stress field for in-plane or three dimensional solids. If the cells are under the uniform section loads, the strain fields are repeated with the same interval as the axial length of the cell, and therefore, Eq. (2) holds.

$$\left(\frac{\partial u}{\partial x}\right)_A - \left(\frac{\partial u}{\partial x}\right)_B = 0 \tag{2a}$$

$$\left(\frac{\partial v}{\partial y}\right)_A - \left(\frac{\partial v}{\partial y}\right)_B = 0 \tag{2b}$$

$$\left(\frac{\partial w}{\partial z}\right)_A - \left(\frac{\partial w}{\partial z}\right)_B = 0 \tag{2c}$$

$$\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)_A - \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)_B = 0 \tag{2d}$$

$$\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)_A - \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)_B = 0 \tag{2e}$$

$$\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)_A - \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)_B = 0 \tag{2f}$$

The subscripts A and B in the equation imply the corresponding material points with the same location within the adjacent unit cells as shown in Fig. 2. They have identical x and z coordinates, but different x coordinate by the length of the cell.

Subtraction of the x derivative of Eq. (2b) from the y derivative of Eq. (2d) gives Eq. (3a), and in a similar manner, Eq. (3b) can be derived from Eq. (2c) and (2f).

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_A - \left(\frac{\partial^2 u}{\partial y^2}\right)_B = 0 \tag{3a}$$

$$\left(\frac{\partial^2 u}{\partial z^2}\right)_A - \left(\frac{\partial^2 u}{\partial z^2}\right)_B = 0 \tag{3b}$$

It is easily noted that the difference in u's of the corresponding node

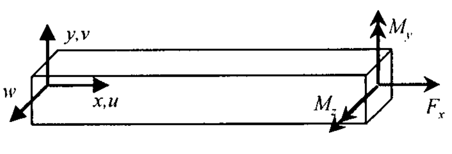

Fig. 1 Sign conventions for displacements and section loads

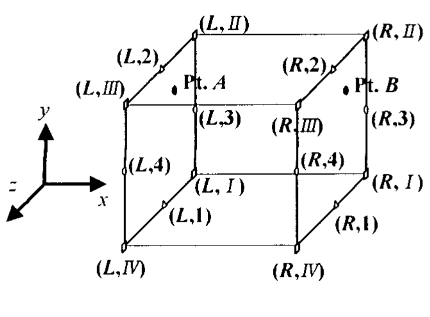

Fig.2 A unit cell with corresponding points and reference points

pair needs to be in the form of Eq. (4) in order to satisfy Eq. (3).

$$u_A - u_B = c_1 yz + c_2 y + c_3 z + c_4 \tag{4}$$

Anthoine et al.[7] assumed the linear distribution in y and z, i.e. c_1=0. However, it is true only when the internal structure of the beam is symmetric with respect to xy and xz planes. For arbitrarily chosen four reference point pairs in Fig. 2, Eq. (4) could be rewritten as Eq. (5). The reason for selecting four pairs is that Eq. (4) contains four unknown coefficients.

$$u_A - u_B = \lfloor yz \quad y \quad z \quad 1 \rfloor [A]^{-1} \{\Delta U\} \tag{5}$$

$$[A] = \begin{bmatrix} y_I z_I & y_I & z_I & 1 \\ y_{II} z_{II} & y_{II} & z_{II} & 1 \\ y_{III} z_{III} & y_{III} & z_{III} & 1 \\ y_{IV} z_{IV} & y_{IV} & z_{IV} & 1 \end{bmatrix}, \quad \{\Delta U\} = \begin{Bmatrix} u_I^L - u_I^R \\ u_{II}^L - u_{II}^R \\ u_{III}^L - u_{III}^R \\ u_{IV}^L - u_{IV}^R \end{Bmatrix}$$

Similarly, the differentiation and combination of Eq. (2a) and Eq. (2d) yield Eq. (6a), and also Eq. (6b) can be obtained from Eq. (2a) and Eq. (2f). The resulting equations imply that the corresponding point pairs have identical curvatures under the uniform section load condition.

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_A - \left(\frac{\partial^2 v}{\partial x^2}\right)_B = 0 \tag{6a}$$

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_A - \left(\frac{\partial^2 w}{\partial x^2}\right)_B = 0 \tag{6b}$$

Subtraction of the z derivative of Eq. (2e) from the y derivative of Eq. (2c) gives Eq. (7), from which Eq. (8) is derived subsequently.

$$\left(\frac{\partial^2 v}{\partial z^2}\right)_A - \left(\frac{\partial^2 v}{\partial z^2}\right)_B = 0 \tag{7}$$

$$v_A - v_B = c_5 z + c_6 \tag{8}$$

The coefficients in the equation can be evaluated in terms of the positions and displacements of the reference points (L,3), (R,3), (L,4), and (R,4) in Fig. 2 to obtain Eq. (9).

$$v_A - v_B = \frac{1}{z_4 - z_3} \lfloor z_4 - z \quad z - z_3 \rfloor \begin{Bmatrix} v_3^L - v_3^R \\ v_4^L - v_4^R \end{Bmatrix} \tag{9}$$

Following the similar procedure as the above, the difference in w's between the corresponding points can be described as Eq. (10) by using the positions and displacements of the reference points (L,1), (R,1), (L,2), and (R,2) in Fig. 2.

$$w_A - w_B = \frac{1}{y_2 - y_1} \lfloor y_2 - y \quad y - y_1 \rfloor \begin{Bmatrix} w_1^L - w_1^R \\ w_2^L - w_2^R \end{Bmatrix} \tag{10}$$

The set of equations (5), (9) and (10) completes the periodicity constraints to be imposed on the nodes on the faces shared by the adjacent unit cells. With the help of these constraints on the FE model, the strain components are guaranteed to be periodic.

As the number of the FE nodes to be constrained grows, it may be time consuming to implement all the equations into the analysis data file. In this study, a short FORTRAN program has been coded so that the corresponding nodes are identified from the unit cell model data and the proper constraint equations are generated automatically according to the input file format for the FE solver.

3. Effective Stiffness of Beam

3.1 Unit Cell Model

As a verification example of the homogenization technique in the former section, a beam with periodically arrayed voids has been

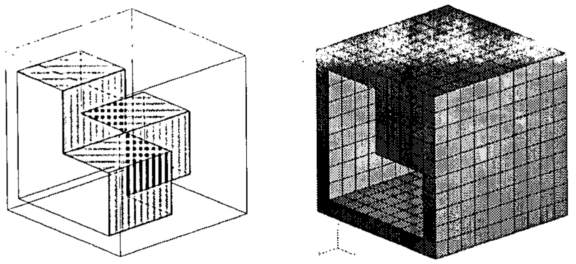

Fig.3 Geometry and FE model of a unit cell

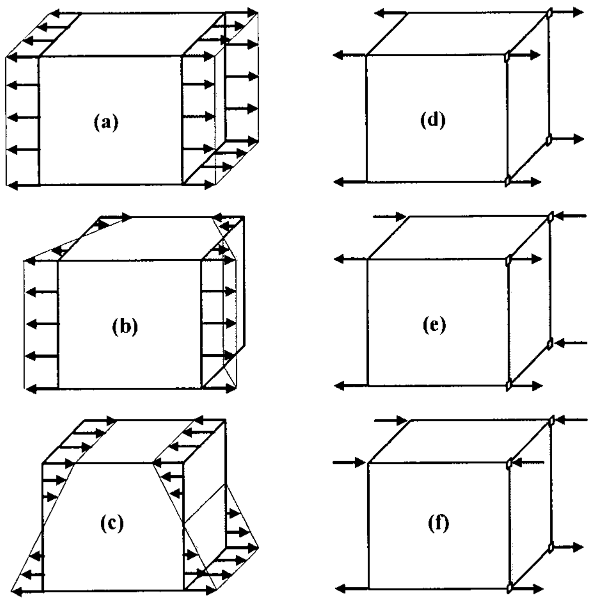

Fig. 4 Boundary conditions to evaluate effective stiffness from analysis of a unit cell: (a) & (d) extension, (b) & (e) bending in y-dir., (c) & (d) bending in z-dir (Periodicity constraint imposed in (d), (e) & (f))

analyzed. The shape and FE model of the unit cell are illustrated in Fig. 3. Due to the geometry of the voids, the internal structure of the cell is not symmetrical with any of the *xy*, *yz* and *zx* planes. All the edge lengths were 1. The cell model was subdivided into 680 brick elements, and the number of the nodes was 3988. On each of the constrained boundary faces, 341 nodes were located. Among them, the corner and mid-edge nodes were selected to be the reference points. The elastic modulus and Poisson's ratio were 1 and 0.25, respectively.

3.2 Effective Stiffness

The simplest method of homogenization is to apply a uniform stress on the boundary of a unit cell and to evaluate the average deformation, or to apply a uniform displacement and to evaluate the average stress. The approximate stiffness or flexibility could be obtained from the structural response from the analysis. In the case of a beam, uniform axial deformation or rotation may be applied as shown in Fig. 4(a-c). The reaction forces on the boundary nodes are summed up to evaluate the axial force F_x and the bending moments M_y and M_z. Then, the stiffness matrix in Eq. (1a) can be computed. This method, however, is based on an over-simplified displacement field without taking the periodic behavior into account, and the effective properties, in consequence, can not avoid errors to a certain degree.

On the other hand, all the nodes on the boundary faces are constrained in the present study, in a way that the periodic behavior is maintained. Only the selected reference nodes are displaced so that a desired extension or bending is approximated, as shown in Fig 4(d-f).

Reaction forces were calculated from the unit cell shown in Fig. 3, under each of the deformation condition given in Fig. 4(d-f). The

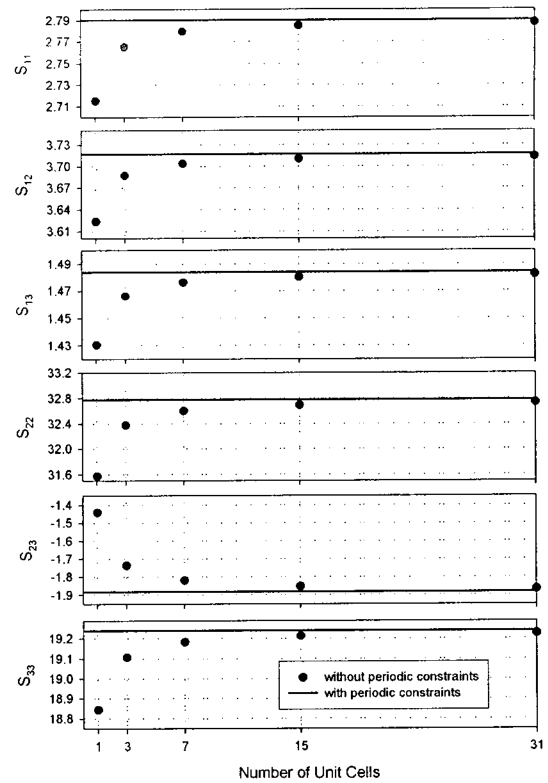

Fig.5 Comparison of flexibility matrix components from uniform displacement and rotation boundary conditions with those from a single cell analysis with periodicity constraints

stiffness matrix in Eq. (1a) was evaluated and inverted to obtain the flexibility matrix in Eq. (1b) as Eq. (11).

$$[S_{ij}] = \begin{bmatrix} 2.7901 & 3.7166 & 1.4838 \\ 3.7164 & 32.7742 & -1.8827 \\ 1.4838 & -1.8823 & 19.2384 \end{bmatrix} \tag{11}$$

Numerical errors are observed, but the flexibility matrix is very close to being symmetric. Also, none of the off-diagonal terms are null due to the aforementioned coupling effects.

The flexibility components computed from the conditions in Fig. 4 (a-c) without the periodicity constraints were compared with those in Eq. (11), and are shown in Fig. 5. Using a single unit cell, the differences of S_{11}, S_{12}, S_{13}, S_{22}, S_{23}, S_{33} from the periodicity solution are 2.68%, 2.50%, 3.59%, 3.65%, 23.45% and 2.04%, respectively. As the number of cells is increased, the difference is reduced. However, even 31 cell model employing 21080 elements still carry some errors. The cause of the difference is in the over-simplified boundary conditions. The boundary faces do not remain plane after the deformation in reality. As more cells are appended in the model, the inside cells deform in the periodic manner, and therefore, the influence of the wrong boundary conditions diminishes relatively.

The above difference is highly dependent on the in-homogeneities included, and may get worse than the example case in this study. For an in-plane problem with square arrayed circular holes, Meguid et al.[10], in their asymptotic homogenization study, showed that the error grows as the void ratio increases. It is because the edges of the cell are distorted severely as the hole grows. Also in the case of a beam, the error would grow as the deformed boundary faces of the cell depart from a plane.

Deformed shapes and the distributions of von Mises stress of the unit cell in Fig. 3 are shown in Fig. 6. The loading condition was the bending in *y* direction. Single cell results without and with periodicity constraints are compared in Fig. 6(a) and Fig. 6(b). The dissimilarities between them are obvious. Fig. 6(c) is the result from a mid-cell of 31 cell model without periodicity constraints. Because it is very far from the uniformly rotated cell faces, the deformation behavior is expected to be close to the periodic one. A very close similarity to Fig. 6(b) is

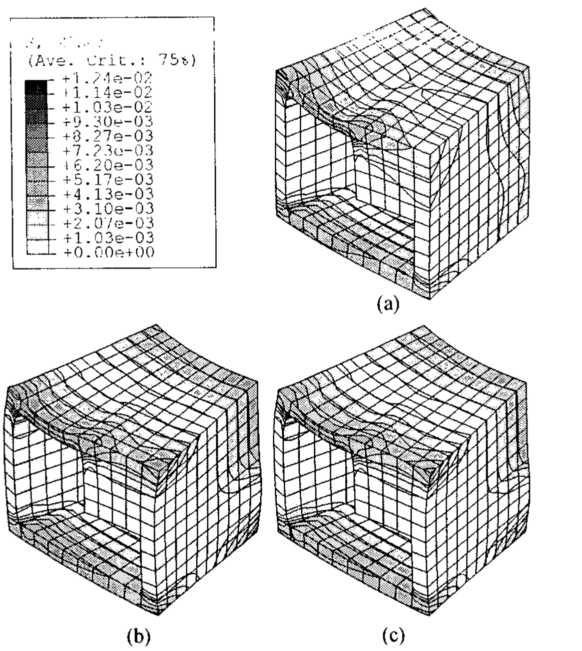

Fig.6 Deformed shape and distribution of von Mises stress under bending in y-dir.: (a) single cell under uniform rotation, (b) single cell with periodicity constraint, (c) mid-cell of 31 cell model

observed.

4. Deflection Analysis of Beam with Effective Flexibility

Employing the effective properties of the beam in the previous section, the deflection analysis of a cantilever beam composed of 31 cells was attempted. Its left end was clamped, while the other end was loaded by an axial force P_x, shear forces P_y and P_z, and concentrated moments M_y and M_z. From Eq. (1b), the displacement of beam sections are derived as follows.

$$u^{th}=\left\{S_{11}P_x+S_{12}M_y+S_{13}M_z+\frac{S_{12}P_z-S_{13}P_y}{2}(x-2l)\right\}x$$

$$v^{th}=\left\{\frac{S_{13}P_x+S_{23}M_y+S_{33}M_z}{2}+\frac{S_{23}P_z-S_{33}P_y}{6}(x-3l)\right\}x^2$$

$$w^{th}=-\left\{\frac{S_{12}P_x+S_{22}M_y+S_{23}M_z}{2}+\frac{S_{22}P_z-S_{23}P_y}{6}(x-3l)\right\}x^2$$

The length of the beam, designated as l, was 31, and x is the distance of a section from the fixed end. Normalized displacements of the beam sections are illustrated in Fig. 7, for each load components. They were normalized by the displacements at the loaded end. The symbols in Fig. 7 are for the detailed model analysis results. In the model, all the voids were modeled as they were. The displacements of a center node on each section were normalized by the above beam solutions at the loaded end.

For all the loading conditions and most of the span-wise locations, the theoretical prediction using the effective properties is in very good agreement with the detailed FE analysis. One thing to note from Fig. 7(a), though, is that the difference in displacement increases at the loaded end under P_x. This is due to the localized severe deformation induced by a concentrated force. The degree of this type of discrepancy depends on the internal structure around the loading point. This would be an inherent weakness of homogenization itself. However, the homogenization technique including the present study still does not lose its practical importance, considering that structural designers try to avoid concentration of force and resulting stresses. If the concentration is the real case, only the loaded part may be modeled in detail and combined with the homogenized model for the

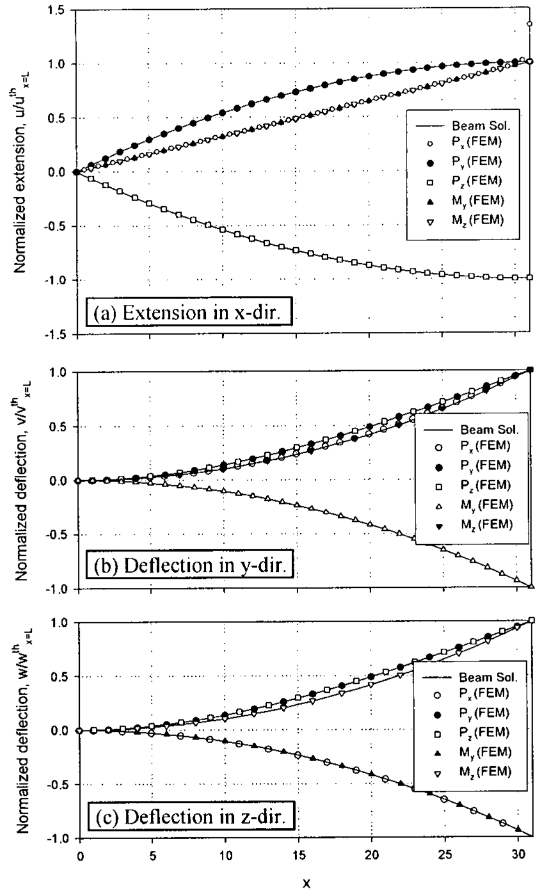

Fig.7 Displacements normalized by those at the free end of 31 cell cantilever beam subject to end loads

remainder.

5. Conclusions

A method to evaluate the homogenized structural properties has been studied for three dimensional beams which are constructed by repeating a representative unit structure in one direction.

Under a uniform section load, the resulting strain fields are periodically repeated. Based on this fact, periodicity constraint equations have been derived. All the nodes on the faces shared by adjacent unit cells are required to satisfy them. Imposing the constraints is time consuming especially when the unit cell is subdivided into a number of elements, but this was avoidable by coding a short program.

Uniform stress, strain or rotation boundary condition is often employed in many homogenization works, with no discretion on the periodic and localized behavior of the real structures. They can be imposed very easily on FE models, but the accuracy is lost as a consequence. The error can be reduced by modeling a larger domain with a larger number of cells. A good alternative for this is to use the periodicity constraints. Only a single cell was found to be enough to yield accurate effective properties of three dimensional beams-in-space.

The effective stiffness and flexibility components have been computed for a beam including unsymmetrical voids, which were periodically arrayed in the beam axis direction. Using the computed effective properties, a cantilever beam was analyzed. Agreement with the deformation behavior from the detailed FE model analysis was excellent, which validates the present method.

Strictly speaking, the periodicity assumption of section load does not hold true in beams if the sectional shear load exists. It is because uniform shear force field translates directly into non-uniform bending moment distribution. Therefore, it is thought that the present method does not guarantee the accuracy for the shear flexible beams. However, this is still a very effective method for Euler-Bernouilli beams, and even for beams with a certain degree of shear flexibility such as the example case of this study.

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