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WEAK AXIOM OF CHOICE ON THE CATEGORY FUZ

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ABSTRACT. Category *Fuz* of fuzzy sets has a similar function to the topos *Set*. But Category *Fuz* forms a weak topos. We show that supports split weakly (SSW) and with some properties, implicity axiom of choice (IAC) holds in weak topos *Fuz*. So weak axiom of choice (WAC) holds in weak topos *Fuz*. Also we show that weak extensionality principle for arrow holds in weak topos *Fuz*.

1. INTRODUCTION

Category *Fuz* of fuzzy sets has a similar function to the topos *Set*. Yuan and Lee [4] showed that *Fuz* has finite products, middle object, equalizers, exponentials and weak subobject classifier. But *Fuz* is not a topos, it forms a weak topos. Yuan [3] also made some comparison between weak topos *Fuz* and topos *Set*. In this paper, we investigate weak axiom of choice and weak extensionality principle of weak topos *Fuz*. First we show that supports split weakly (SSW) and with some condition, implicity axiom of choice (IAC) holds in weak topos *Fuz*. So weak axiom of choice (WAC) holds in weak topos *Fuz*. Secondly we show that *Fuz* satisfies weak extensionality principle for arrow.

2. PRELIMINARIES

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

Definition 2.1. An *elementary topos* is a category $\mathcal{E}$ that satisfies the following:

(T1) $\mathcal{E}$ is finitely complete,

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(T2) $\mathcal{E}$ has exponentiation,

(T3) $\mathcal{E}$ has a subobject classifier.

(T2) means that for every object A in $\mathcal{E}$, endofunctor $(-) \times A$ has its right adjoint $(-)^A$. Hence for every object A in $\mathcal{E}$, there exists an object B^A, and a morphism $ev_A : B^A \times A \to B$, called the evaluation map of A, such that for any Y and $f : Y \times A \to B$ in $\mathcal{E}$, there exists a unique morphism g such that $ev_A \circ (g \times i_A) = f$;

$$\begin{array}{ccc} Y \times A & \xrightarrow{f} & B \\ {\scriptstyle g \times i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\ B^A \times A & \xrightarrow[ev_A]{} & B \end{array}$$

And subobject classifier in (T3) is an $\mathcal{E}$-object Ω, together with a morphism $\top : 1 \to \Omega$ such that for any monomorphism $h : D \to C$, there is unique morphism $\chi_h : C \to \Omega$, called the character of $h : D \to C$ that makes the following diagram a pull-back;

$$\begin{array}{ccc} D & \xrightarrow{!} & 1 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \top} \\ C & \xrightarrow[\chi_h]{} & \Omega \end{array}$$

Example 2.2. Category *Set* is a topos. $*$ is a terminal object. $\Omega = \{0, 1\}$ and $\top : \{*\} \to \Omega$ with $\top(*) = 1$ is a subobject classifier. If we define

$\chi_h = 1$ if $c = h(d)$ for some $d \in D$,

$\chi_h = 0$ otherwise

then χ_h is a characteristic function of D.

Category *Fuz* of fuzzy sets is a category whose object is (A, α_A) where A is an object and $\alpha_A : A \to I$ is a morphism with $I = (0, 1]$ in *Set* and morphism from (A, α_A) to (B, α_B) is a morphism $f : A \to B$ in *Set* such that $\alpha_A(a) \leqq \alpha_B \circ f(a)$. An object (A, α_A) is called normal if A has at least one element whose membership degree is 1.

Definition 2.3. We say that an object (I, α_I) is a *middle object* of *Fuz* if there exists a unique morphism $f : A \to I$ such that $\alpha_A(a) = \alpha_I \circ f(a)$ for all (A, α_A) and $a \in A$.

Definition 2.4. We say that $((J, \alpha_J), i)$ is a *weak subobject classifier* of *Fuz* if there exists a unique morphism $\alpha_f : (A, \alpha_A) \to (J, \alpha_J)$ for all monomorphism $f : (B, \alpha_B) \to (A, \alpha_A)$ where $J = [0, 1]$ and $\alpha_J(j) = 1$ for all $j \in J$ such that $\alpha_f(a) \leqq \alpha_A(a)$ and the following diagram

$$\begin{array}{ccc} (B, \alpha_B) & \xrightarrow{\alpha_B} & (I, \alpha_I) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i} \\ (A, \alpha_A) & \xrightarrow[\alpha_f]{} & (J, \alpha_J) \end{array}$$

is a pull-back.

Definition 2.5. A *weak topos* is a category $\mathcal{E}$ that satisfies the following:

(WT1) $\mathcal{E}$ has equalizer, finite product and exponentiation.
(WT2) $\mathcal{E}$ has a middle object.
(WT3) $\mathcal{E}$ has a weak subobject classifier.

Proposition 2.6. *Category Fuz is a weak topos.*

For the proof see Yuan and Lee [4].

Definition 2.7. We say that *supports split weakly* (SSW) for every $\mathcal{E}$-object A in the weak topos $\mathcal{E}$ if, for the middle object (I, α_I), the epic part of the epi-monic factorization of $! : A \to I$ has a right inverse.

Definition 2.8. We say that an object A of a weak topos $\mathcal{E}$ is *internally projective* if the functor $(-)^A : \mathcal{E} \to \mathcal{E}$ preserves epimorphisms. And we say that *the implicit axiom of choice* (IAC) holds in a weak topos $\mathcal{E}$ if every object of $\mathcal{E}$ is internally projective.

Definition 2.9. We say that weak topos $\mathcal{E}$ satisfies the *weak axiom of choice* (WAC) if supports split weakly in $\mathcal{E}/X$ for every X.

Definition 2.10. A weak topos satisfies the *weak extentionality principle for arrow* if $f, g : A \to B$ are a pair of distinct parallel morphisms, then for the middle object (I, α_I) there is a morphism $a : I \to A$ of A such that $f \circ a \neq g \circ a$.

3. Main Parts

Theorem 3.1. *Supports split weakly in Fuz.*

Proof. Let $\alpha_A : (A, \alpha_A) \to (I, \alpha_I)$ then $\alpha_A = m \circ e$ where $e : A \to Im(\alpha_A) \equiv K$ is an epimorphism and $m : Im(\alpha_A) \to I$ is an monomorphism. Let $\alpha_K : K \to I$ is a restriction of α_I on K. Since $\alpha_A[A] = e[A]$ and m is an inclusion, we get that $\alpha_K \circ e \geqq \alpha_A$ and $\alpha_I \circ m \geqq \alpha_K$. So the arrow $\alpha_K : Im(\alpha_A) \to I$ is an object in *Fuz.*

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & I \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_I} \\ I & = & I \end{array}$$

Construct $g : K \to A$ defined by $g(k) = a \in A' \subseteq A$ with $e[A'] = k$. Then $e \circ g(k) = e(a) = k$. We only claim that $\alpha_A \circ g \geqq \alpha_K$. For all $a \in A$, $\alpha_A \circ g(k) = \alpha_A(a) = m \circ e(a) = m(k) = k = \alpha_K(k)$. So $\alpha_A \circ g \geqq \alpha_K$. □

Proposition 3.2. *The implicit axiom of choice does not hold in Fuz.*

Proof. Let $f : (A = \{a, b\}, \alpha_A) \to (Y = \{y\}, \alpha_Y)$ be an epimorphism defined by $f(a) = f(b) = y$, $\alpha_A(a) = 0.1, \alpha_A(b) = o.2$ and $\alpha_Y(y) = 0.3$ then $\alpha_Y \circ f \geqq \alpha_A$. And let $(X = \{x\}, \alpha_X)$ be any object where $\alpha_X(x) = 0.3$ in *Fuz.* Assume that $f^X : A^X \to B^X$ is an epimorphism. Then for any $t : X \to B$ such that $\alpha_Y \circ t \geqq \alpha_X$ there exist a morphism $s : X \to A$ such that $f \circ s = t$. But for any $s : X \to A$ defined by $s(x) = a$ or $s(x) = b$, it does not hold that $\alpha_A \circ s \geqq \alpha_X$. □

Theorem 3.3. *The implicit axiom of choice hold in Fuz for any epimorphism $f : (A, \alpha_A) \to (B, \alpha_B)$ such that $\alpha_B(b) = Max\{\alpha_A[A'], A' \subseteq A\}$ where $b = f[A']$.*

Proof. Let $f : (A, \alpha_A) \to (B, \alpha_B)$ be an epimorphism and (X, α_X) be an object in *Fuz.* We only claim that $f^X : A^X \to B^X$ is an epimorphism. For any $t : X \to B$, there exists $t(x) \in B$. Since f is an epimorphism, there exists $A' \subseteq A$ such that $f[A'] = t(x)$. We construct a morphism $s : X \to A$ defined by $s(x) = a \in A'$ such that $\alpha_A(a)$ is maximal in $\alpha_A[A']$. Then we get $f \circ s(x) = f(a) = t(x)$. By $f \circ s = t$, $\alpha_B \circ f \geqq \alpha_A$ and $\alpha_Y B \circ t \geqq \alpha_X$, we get $\alpha_B \circ t \geqq \alpha_A \circ s$ and $\alpha_B \circ t \geqq \alpha_X$. So, by definition of α_B, $\alpha_A \circ s \geqq \alpha_X$.

$$\begin{array}{ccc} X & = & X \\ \downarrow{\scriptstyle s} & & {\scriptstyle t}\downarrow \\ A & \xrightarrow{f} & B \end{array}$$

□

Corollary 3.4. *The weak axiom of choice hold in Fuz for any epimorphism $f : (A, \alpha_A) \to (B, \alpha_B)$ such that $\alpha_B(b) = Max\{\alpha_A[A'], A' \subseteq A\}$ where $b = f[A']$.*

Theorem 3.5. *The weak extentionality principle for arrow holds in Fuz with normal object.*

Proof. Let $f \neq g : (A, \alpha_A) \to (B, \alpha_B)$, then there is an $b \in A$ such that $f(b) \neq g(b)$ and $\alpha_A(b) \in (0,1]$. Consider a partial element $(0, \alpha_A(b)] \subseteq I$. We can construct $\beta : I \to A$ defined by $\beta(x) = b$ for all $x \in (0, \alpha_A(b)]$ and $\beta(x) = c$ for all $x \in (\alpha_A(b), 1]$ where $\alpha_A(c) = 1$. This yields $\alpha_A \circ \beta(x) \geqq x$. Thus we get $\alpha_A \circ \beta \geqq \alpha_I$.

$$\begin{array}{ccccc} I & \xrightarrow{\beta} & A & \xrightarrow{f,g} & B \\ {\scriptstyle \alpha_I}\downarrow & & \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ I & = & I & = & I \end{array}$$

□

Theorem 3.6. *For any normal object (A, α_A) in Fuz, there exists a morphism $\beta : I \to A$ such that $\alpha_A \circ \beta \geqq \alpha_I$, where $\alpha_I : I \to I$ is the middle object.*

Proof. Let A be a disjoint union of A_i such that $\alpha_A(a_i) = c_i$ for all $a_i \in A_i$ where $c_i \in I$. Then we have that $0 < \alpha_A[A_1] < \cdots < \alpha_A[A_j] < \cdots < \alpha_A[A_m] = 1$, where $\alpha_A[A_i]$ is a image value in I. Consider a morphism $\beta : I \to A$ defined by, for all $0 < i \leqq 1$,

$$\beta(i) = a_1 \in A_1, \ \ for\ all\ i \in (0, \alpha_A[A_1]]$$

$$\vdots$$

$$\beta(i) = a_j \in A_j, \ \ for\ all\ i \in (\alpha_A[A_{j-1}], \alpha_A[A_j]]$$

$$\vdots$$

$$\beta(i) = a_m \in A_m, \ \ for\ all\ i \in (\alpha_A[A_{m-1}], \alpha_A[A_m]].$$

Then $\alpha_A \circ \beta \geqq \alpha_I$. □

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