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ON THE SOLUTION OF A MULTI-VARIABLE BI-ADDITIVE FUNCTIONAL EQUATION I

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ABSTRACT. We investigate the relation between the multi-variable bi-additive functional equation

$$f(x+y+z,u+v+w)=f(x,u)+f(x,v)+f(x,w)+f(y,u)+f(y,v)+f(y,w)+f(z,u)+f(z,v)+f(z,w)$$

and the multi-variable quadratic functional equation

$$g(x+y+z)+g(x-y+z)+g(x+y-z)+g(-x+y+z)=4g(x)+4g(y)+4g(z).$$

Furthermore, we find out the general solution of the above two functional equations.

1. INTRODUCTION

Throughout this paper, let n be a positive integer greater than 1 and let X and Y be vector spaces.

Definition. A mapping $f: X\times X\to Y$ is called *bi-additive* if f satisfies the system of equations

$$(1)\qquad \begin{aligned} f(x+y,z)&=f(x,z)+f(y,z),\\ f(x,y+z)&=f(x,y)+f(x,z). \end{aligned}$$

for all $x,y,z\in X$.

When $X=Y=\mathbb{R}$, the function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by $f(x,y):=cxy$ is a solution of (1). In particular, letting $x=y$, we get a quadratic function $g:\mathbb{R}\to\mathbb{R}$ in one variable given by $g(x):=f(x,x)=cx^2$.

For a mapping $f: X\times X\to Y$, consider the bi-additive functional equation:

$$(2)\qquad f(x+y+z,u+v+w)=f(x,u)+f(x,v)+f(x,w)+f(y,u)+f(y,v)+f(y,w)+f(z,u)+f(z,v)+f(z,w).$$

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For a mapping $g : X \to Y$, consider the quadratic functional equation:

$$\begin{aligned} & g(x+y+z) + g(x-y+z) + g(x+y-z) + g(-x+y+z) \\ (3) \qquad & = 4g(x) + 4g(y) + 4g(z). \end{aligned}$$

J.-H. Bae, K.-W. Jun and S.-M. Jung [1] solved the solution of the equation (3) in vector spaces. There are numerous results about various functional equations ([2, 3, 4, 5]).

In this paper, we investigate the relation between (2) and (3). And we find out the general solution of (2).

2. Results

Theorem 1. *Let $f : X \times X \to Y$ be a mapping satisfying* (2) *and let $g : X \to Y$ be the mapping given by*

$$(4) \qquad g(x) := f(x, x)$$

for all $x \in X$. Then g satisfies (3).

Proof. Letting $x = y = z = u = v = w = 0$ in (2) and then using (4), we have $g(0) = 0$. Setting $y = z = u = v = w = 0$ in (2), we have

$$f(x, 0) = 0$$

for all $x \in X$. Similarly, $f(0, u) = 0$ for all $u \in X$. Letting $z = w = 0$ in (2), we have

$$(5) \qquad f(x+y, u+v) = f(x, u) + f(x, v) + f(y, u) + f(y, v)$$

for all $x, y, u, v \in X$. Let $y = u = v = x$ in (5). Then we have by (4),

$$(6) \qquad g(2x) = 4g(x)$$

for all $x \in X$. By (4) and (5), we obtain

$$\begin{aligned} (7) \qquad & g(x+y+u+v) - g(x+y) - g(u+v) \\ & = g(x+u) - g(x) - g(u) + g(x+v) - g(x) - g(v) \\ & \quad + g(y+u) - g(y) - g(u) + g(y+v) - g(y) - g(v) \end{aligned}$$

for all $x, y, u, v \in X$. Putting $v = 0$ in (7), we see that

$$(8) \qquad g(x+y+u) = g(x+y) + g(x+u) + g(y+u) - g(x) - g(y) - g(u)$$

for all $x, y, u \in X$. Replacing x, y and u by $2x$, $2y$ and $-x-y$ in (8), respectively and by (6),

$$3g(x+y) - g(-x-y) + g(x-y) + g(y-x) = 4g(x) + 4g(y)$$

for all $x, y \in X$. Setting $y = x$ in the above equality and by (6),

$$g(x) = g(-x)$$

for all $x \in X$. Taking $u = -y$ in (8) and by the above equality,

$$g(x+y) + g(x-y) = 2g(x) + 2g(y) \tag{9}$$

for all $x, y \in X$. By [1], g satisfies (3). □

Example 1. Let X be a real algebra and $D : X \to X$ a derivation on X. Define a mapping $f : X \times X \to X$ by

$$f(x,y) := D(xy) = xD(y) + D(x)y$$

for all $x, y \in X$. Then f satisfies (2). Define a mapping $g : X \to X$ by

$$g(x) := D(x^2) = xD(x) + D(x)x$$

for all $x \in X$. Then g satisfies (4). By Theorem 1, g satisfies (3).

Theorem 2. *Let $g : X \to Y$ be a mapping satisfying (3) and let $f : X \times X \to Y$ be the mapping given by*

$$f(x,y) = \frac{1}{2}[g(x+y) - g(x) - g(y)] \tag{10}$$

for all $x, y \in X$. Then f satisfies (2) and (4).

Proof. By [1], g satisfies (9). So g also satisfies (6). Setting $y = x$ in (10) and then using (6), the equality (4) holds. By (9), the equalities

$$g(x-y+z) + g(x-y-z) = 2g(x-y) + 2g(z),$$
$$g(y-z+x) + g(y-z-x) = 2g(y-z) + 2g(x),$$
$$g(z-x+y) + g(z-x-y) = 2g(z-x) + 2g(y)$$

hold for all $x, y, z \in X$. Since g is even, by (3) and the above equalities,

$$g(x+y+z) + g(x-y) + g(y-z) + g(z-x) = 3g(x) + 3g(y) + 3g(z) \tag{11}$$

for all $x, y, z \in X$. By (10) and (11),

$$
\begin{aligned}
(12) \qquad & 2f(x+y+z, u+v+w) \\
&= g(x+y+z+u+v+w) \\
&\quad +g(x-y)+g(y-z)+g(z-x)-3g(x)-3g(y)-3g(z) \\
&\quad +g(u-v)+g(v-w)+g(w-u)-3g(u)-3g(v)-3g(w)
\end{aligned}
$$

for all $x, y, z, u, v, w \in X$. By (9),

$$
\begin{aligned}
& 2g(x-y)+2g(u-v)+2g(y-z)+2g(v-w)+2g(z-x)+2g(w-u) \\
(13) \qquad &= g(x-y+u-v)+g(x-y-u+v)+g(y-z+v-w) \\
&\quad +g(y-z-v+w)+g(z-x+w-u)+g(z-x-w+u)
\end{aligned}
$$

for all $x, y, z, u, v, w \in X$. By (9),

$$
(14) \qquad g(x-y+u-v) = 2g(x+u)+2g(y+v)-g(x+y+u+v)
$$

for all $x, y, u, v \in X$. By (13) and (14),

$$
\begin{aligned}
& g(x-y)+g(u-v)+g(y-z)+g(v-w)+g(z-x)+g(w-u) \\
&= \frac{1}{2}\big[g(x-y+u-v)+g(y-z-v+w)+g(z-x+w-u) \\
&\quad +2g(x+v)+2g(y+u)-g(x+y+u+v) \\
&\quad +2g(y+v)+2g(z+w)-g(y+z+v+w) \\
&\quad +2g(z+u)+2g(x+w)-g(z+x+w+u)\big]
\end{aligned}
$$

for all $x, y, z, u, v, w \in X$. By (9), (12) and the above equality,

$$
\begin{aligned}
(15) \qquad & 2f(x+y+z, u+v+w) \\
&= g(x+y+z+u+v+w)+\frac{1}{2}\big[g(x-y+u-v) \\
&\quad +g(y-z-v+w)+g(z-x+w-u)-g(x+y+u+v) \\
&\quad -g(y+z+v+w)-g(z+x+w+u)\big] \\
&\quad +g(x+v)+g(y+u)+g(y+v)+g(z+w)+g(z+u)+g(x+w) \\
&\quad -3[g(x)+g(y)+g(z)+g(u)+g(v)+g(w)] \\
&= g(x+y+z+u+v+w) \\
&\quad +\frac{1}{2}\big[g(x-y+u-v)+g(y-z-v+w)+g(z-x+w-u)
\end{aligned}
$$

$$\begin{aligned}
&+g(x+y+u+v)+g(y+z+v+w)+g(z+x+w+u)]\\
&-g(x+y+u+v)-g(y+z+v+w)-g(z+x+w+u)\\
&+g(x+v)+g(y+u)+g(y+v)+g(z+w)+g(z+u)+g(x+w)\\
&-3[g(x)+g(y)+g(z)+g(u)+g(v)+g(w)]\\
=\ &g(x+y+z+u+v+w)\\
&+g(x+u)+g(y+v)+g(y+w)+g(z+v)+g(z+w)+g(x+u)\\
&-g(x+y+u+v)-g(y+z+v+w)-g(z+x+w+u)\\
&+g(x+v)+g(y+u)+g(y+v)+g(z+w)+g(z+u)+g(x+w)\\
&-3[g(x)+g(y)+g(z)+g(u)+g(v)+g(w)]\\
=\ &g(x+y+z+u+v+w)+g(x+u)+g(y+v)+g(z+w)\\
&-g(x+y+u+v)-g(y+z+v+w)-g(z+x+w+u)\\
&+g(x+u)+g(x+v)+g(x+w)+g(y+u)+g(y+v)+g(y+w)\\
&+g(z+u)+g(z+v)+g(z+w)\\
&-3[g(x)+g(y)+g(z)+g(u)+g(v)+g(w)]
\end{aligned}$$

for all $x, y, z, u, v, w \in X$. By (9),

$$\begin{aligned}
&2g(x+y+u+v)+2g(y+z+v+w)+2g(x+z+u+w)\\
&\quad+2g(x+u)+2g(y+v)+2g(z+w)\\
&=3g(x+y+z+u+v+w)+g(y+z+v+w-x-u)\\
&\quad+g(x+z+u+w-y-v)+g(x+y+u+v-z-w)
\end{aligned} \tag{16}$$

for all $x, y, z, u, v, w \in X$. Replacing $x+u$, $y+v$ and $z+w$ instead of x, y, and z, respectively, in (3), we obtain that

$$\begin{aligned}
&g(x+y+z+u+v+w)+g(x+u-y-v+z+w)\\
&\quad+g(x+u+y+v-z-w)+g(-x-u+y+v+z+w)\\
&=4g(x+u)+4g(y+v)+4g(z+w)
\end{aligned}$$

for all $x, y, z, u, v, w \in X$. By (16) and the above equality,

$$\begin{aligned}
&2g(x+y+z+u+v+w)+4g(x+u)+4g(y+v)+4g(z+w)\\
&=2g(x+y+u+v)+2g(y+z+v+w)+2g(x+z+u+v)\\
&\quad+2g(x+u)+2g(y+v)+2g(z+w),
\end{aligned}$$

or equivalently,

$$\begin{aligned}&g(x+y+z+u+v+w)+g(x+u)+g(y+v)+g(z+w)\\&=g(x+y+u+v)+g(y+z+v+w)+g(x+z+u+v)\end{aligned}$$

for all $x, y, z, u, v, w \in X$. By (10), (15) and the above equality,

$$\begin{aligned}(17)\qquad &2f(x+y+z,u+v+w)\\&=g(x+u)+g(x+v)+g(x+w)+g(y+u)+g(y+v)+g(y+w)\\&\quad+g(z+u)+g(z+v)+g(z+w)\\&\quad-3[g(x)+g(y)+g(z)+g(u)+g(v)+g(w)]\\&=2[f(x,u)+f(x,v)+f(x,w)+f(y,u)+f(y,v)+f(y,w)\\&\quad+f(z,u)+f(z,v)+f(z,w)]\end{aligned}$$

for all $x, y, z, u, v, w \in X$. □

Next we solve the solution of (2).

Theorem 3. *A mapping $f : X \times X \to Y$ satisfies* (1) *if and only if it satisfies* (2).

Proof. If f satisfies (1), then

$$\begin{aligned}f(x+y+z,u+v+w)=&f(x,u)+f(x,v)+f(x,w)+f(y,u)+f(y,v)\\&+f(y,w)+f(z,u)+f(z,v)+f(z,w)\end{aligned}$$

for all $x, y, z, u, v, w \in X$.

Conversely, assume that f satisfies (2). Choosing $x = y = z = u = v = w = 0$ in (2), $f(0,0) = 0$. Letting $y = z = u = v = w = 0$ in (2), we have $f(x,0) = 0$ for all $x \in X$. Setting $x = y = z = v = w = 0$ in (2), we obtain $f(0,u) = 0$ for all $u \in X$. Putting $z = v = w = 0$ in (2), we get

$$f(x+y,u)=f(x,u)+f(y,u)$$

for all $x, y, u \in X$. Taking $y = z = w = 0$ in (2), we see that

$$f(x,u+v)=f(x,u)+f(x,v)$$

for all $x, u, v \in X$. □

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