

VOLUME OF $C^{1,\alpha}$ -BOUNDARY DOMAIN IN EXTENDED HYPERBOLIC SPACE

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ABSTRACT. We consider the projectivization of Minkowski space with the analytic continuation of the hyperbolic metric and call this an extended hyperbolic space. We can measure the volume of a domain lying across the boundary of the hyperbolic space using an analytic continuation argument. In this paper we show this method can be further generalized to find the volume of a domain with smooth boundary with suitable regularity in dimension 2 and 3. We also discuss that this volume is invariant under the group of hyperbolic isometries and that this regularity condition is sharp.

1. Introduction and preliminaries

In [1] we considered an extended model of hyperbolic space and studied how we can define a volume of a domain which lies beyond the infinity of the hyperbolic space. Such investigation gives us a natural way of studying various geometric objects in Lorentz geometry in a manner consistent with those in hyperbolic geometry. The method of calculating volume of such domain is essentially an analytic continuation argument and works very well with a domain with analytic boundary. But if the boundary is smooth or just continuous, then the volume problem turns out to be very delicate and the required regularity of the boundary necessary for finiteness of volume depends on the dimension. We discuss this phenomenon in detail in this paper focusing especially on dimension two or three. Then we discuss the invariance of the volume of domains which has boundary with necessary regularity in these dimensions. We keep the same notations used in [1], but we provide necessary materials

Received March 7, 2005.

2000 Mathematics Subject Classification: 51M10, 51M25, 53C50.

Key words and phrases: hyperbolic space, volume, analytic continuation.

The first author was supported by Research Fund 2004 of University of Seoul.

The second author was supported by grant no.(R01-1999-000-00002-0) from the Basic Research Program of the Korea Science & Engineering Foundation.

in detail so that the paper is as self-contained as possible and can be read independently from [1]. And here we do not intend to mention why the extended model is natural and what applications we can obtain using this model. We refer the reader to the paper [1] for all these explanations and other references as well.

Let $\mathbb{R}^{n,1}$ denote the Minkowski space, i.e., \mathbb{R}^{n+1} with the inner product of signature $(n, 1)$ given by

$$\langle x, y \rangle = -x_0y_0 + x_1y_1 + \cdots + x_ny_n.$$

The hyperbolic space, Lorentz space and the light cone are defined as the sets $\{x \in \mathbb{R}^{n,1} | \langle x, x \rangle = \alpha\}$ with $\alpha = -1, 1, 0$ respectively together with the induced metric. If we project these sets radially to an affine subspace $\mathbb{K}^n := \{1\} \times \mathbb{R}^n \subset \mathbb{R}^{n,1}$, then we obtain a unit ball as Kleinian projective model for hyperbolic space \mathbb{H}^n , Lorentz space of constant sectional curvature 1 outside the ball and the light cone as the common boundary $\partial\mathbb{H}^n$ of these two spaces.

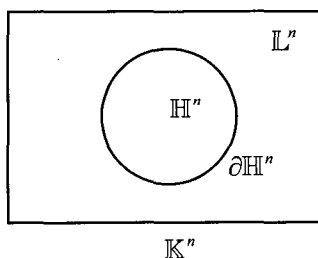


FIGURE 1

If we change the sign of the induced metric on the Lorentz space, then the new Lorentz space denoted by \mathbb{L}^n has constant sectional curvature -1 and the metrics on both parts \mathbb{H}^n and \mathbb{L}^n have the exactly same formula on \mathbb{K}^n :

$$ds_K^2 = \left(\frac{\sum x_i dx_i}{1 - |x|^2} \right)^2 + \frac{\sum dx_i^2}{1 - |x|^2}.$$

And the induced volume form is given by

$$dV_K = \frac{dx_1 \wedge \cdots \wedge dx_n}{(1 - |x|^2)^{\frac{n+1}{2}}}.$$

Now for a domain U in \mathbb{H}^n , the volume of U will be simply given by the integration of dV_K on U . For a domain U lying across the boundary

of \mathbb{H}^n , we formally calculate the volume of U using the polar coordinates as follows:

$$\begin{aligned}\text{vol}(U) &= \int_U \frac{dx_1 \cdots dx_n}{(1 - |x|^2)^{\frac{n+1}{2}}} \\ &= \int_{G^{-1}(U)} \frac{r^{n-1}}{(1 - r^2)^{\frac{n+1}{2}}} dr d\theta \\ &= \int_a^b \frac{r^{n-1} F(r)}{(1 - r^2)^{\frac{n+1}{2}}} dr, \quad F(r) = \int_{G^{-1}(U) \cap S^{n-1}(r)} d\theta,\end{aligned}$$

where $G : (r, \theta) \mapsto (x_1, \dots, x_n)$ is the polar coordinates, $S^{n-1}(r)$ is the Euclidean sphere of radius r and $d\theta$ is the volume form of the Euclidean unit sphere S^{n-1} .

Now this integral with respect to r does not make sense in general, but for a domain U with analytic boundary transversal to $\partial\mathbb{H}^n$ we may use contour integral to define a volume of U :

$$\text{vol}(U) := \int_{\gamma} \frac{r^{n-1} F(r)}{(1 - r^2)^{\frac{n+1}{2}}} dr,$$

where γ is a contour given by

$$\gamma(t) = \begin{cases} t, & a \leq t \leq 1 - \delta, \\ 1 + \delta e^{\frac{i(1-t)\pi}{\delta}}, & 1 - \delta \leq t \leq 1, \\ t + \delta, & 1 \leq t \leq b - \delta. \end{cases}$$

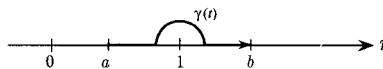


FIGURE 2

Note that the analyticity and transversality of the boundary of U was needed to make sure $F(r)$ is an analytic function of r near $r = 1$. For a domain U in the Lorentz part, our choice of the contour γ naturally determines the sign of $\text{vol}(U)$ as i^{n+1} and so is determined the sign of dV_K (see [1]).

In [1], it is shown that $\text{vol}(U)$ can also be obtained through a complex approximation. Let

$$ds_{\epsilon}^2 = \left(\frac{\sum x_i dx_i}{d_{\epsilon}^2 - |x|^2} \right)^2 + \frac{\sum dx_i^2}{d_{\epsilon}^2 - |x|^2},$$

where $d_\epsilon = 1 - \epsilon i$ with $\epsilon > 0$ and $i = \sqrt{-1}$, so that $ds_K^2 = \lim_{\epsilon \rightarrow 0} ds_\epsilon^2$. Then the induced volume form is given by

$$dV_\epsilon = \frac{d_\epsilon dx_1 \wedge \cdots \wedge dx_n}{(d_\epsilon^2 - |x|^2)^{\frac{n+1}{2}}}$$

and let $\mu(U) := \lim_{\epsilon \rightarrow 0} \int_U dV_\epsilon$. Here the choice of sign of dV_ϵ is determined by the continuity on $\epsilon \geq 0$ and the sign of dV_K . Then it was shown in [1, Proposition 2.1 and 3.2] that μ is finitely additive and $\mu(U) = \text{vol}(U)$ for a domain U with an analytic boundary transversal to $\partial\mathbb{H}^n$. We actually show this fact in the next section in a different model. The finite additivity follows easily from the definition of μ . Also notice that if U is a domain lying solely in \mathbb{H}^n or \mathbb{L}^n , then

$$\mu(U) = \lim_{\epsilon \rightarrow 0} \int_U dV_\epsilon = \int_U \lim_{\epsilon \rightarrow 0} dV_\epsilon = \int_U dV_K$$

by the Lebesgue dominated convergence theorem and coincides with the usual volume.

The measure theory for μ seems to be very delicate and it is not easy to find a large enough class of μ -measurable sets, that is, Lebesgue measurable sets with $\mu(U) < \infty$. The present work reflects the effort of finding and explaining more about μ -measurable sets and we find that a domain with $C^{1,\alpha}$ boundary in dimension 3 ($C^{0,\frac{1}{2}+\alpha}$ boundary for dimension 2, respectively) is actually μ -measurable, and also show that this regularity condition is in fact sharp.

2. A flattened model for computation

We prove the results stated in the previous section by computing various integrals. But computing the integral whose singularities lies on the unit sphere in \mathbb{K}^n is certainly inconvenient and we want to introduce a new model to facilitate the computation. In this model, we want the singularity sets of our volume form is a hyperplane. The immediate choice is a Cayley transformation or a reflection σ with respect to a sphere of radius $\sqrt{2}$ with the center at $e_n = (0, \dots, 0, 1) \in \mathbb{K}^n$.

We see immediately that under the reflection σ , \mathbb{H}^n is sent to the lower half space and \mathbb{L}^n to the upper half space.

From the obvious identities,

$$\begin{cases} y - e_n = \lambda(x - e_n), & \lambda \in \mathbb{R}, \\ |y - e_n||x - e_n| = 2, \end{cases}$$

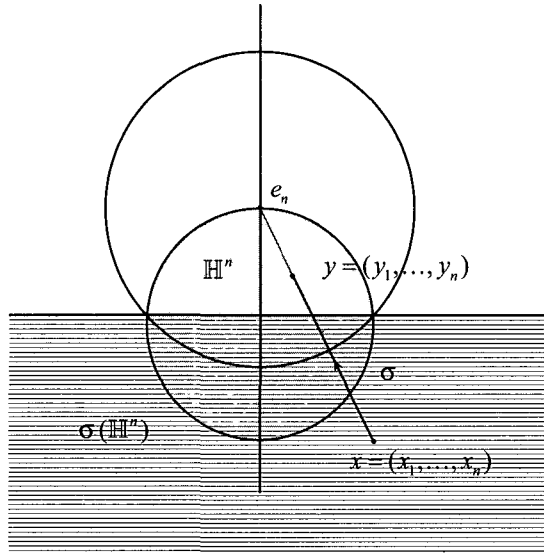


FIGURE 3

we easily obtain that $y = \sigma(x)$ is given by

$$\sigma : \begin{cases} y_i = \frac{2x_i}{|x - e_n|^2}, & i = 1, \dots, n-1, \\ y_n = \frac{2(x_n - 1)}{|x - e_n|^2} + 1. \end{cases}$$

We compute directly using this formula that the metric ds_K^2 is pulled back by σ to

$$ds^2 = \sigma^*(ds_K^2) = \left(\frac{\alpha dx_n - x_n d\alpha}{2\alpha x_n} \right)^2 - \frac{\sum dx_i^2}{\alpha x_n},$$

where $\alpha = |x - e_n|^2 = x_1^2 + \dots + x_{n-1}^2 + (x_n - 1)^2$ so that $d\alpha = 2(\sum x_i dx_i - dx_n)$.

Also the volume form dV_K is pulled back to

$$dV = \sigma^*(dV_K) = - \frac{dx_1 \wedge \dots \wedge dx_n}{2(-x_n)^{\frac{n+1}{2}} \alpha^{\frac{n-1}{2}}}.$$

Here notice that the first negative sign appears since σ is orientation reversing and we can ignore this when we compute the integrals for volume. If $x_n > 0$, that is, if $x \in \mathbb{L}^n$, we need to determine the sign of

$(-1)^{\frac{n+1}{2}}$, and this should be determined as $(-i)^{n+1}$ in order to give the sign of dV as i^{n+1} as given in the previous section.

This new model \mathbb{E}^n is of course quite different from the Poincaré half space model. It is clear from the construction that the geodesics in this model are the circles (including lines viewed as a special case of circles passing through the infinity) passing through the point e_n , and more generally spheres (including planes) passing through e_n are the totally geodesic submanifolds.

Let's consider first the volume of a domain U with analytic boundary transversal to $\partial\mathbb{H}^n$ in the new model \mathbb{E}^n . Note that $\sigma^{-1} = \sigma$.

$$\mu(U) = \lim_{\epsilon \rightarrow 0} \int_{\sigma(U)} dV_\epsilon = \lim_{\epsilon \rightarrow 0} \int_U d\tilde{V}_\epsilon,$$

where

$$d\tilde{V}_\epsilon = \sigma^*(dV_\epsilon) = -\frac{1-\epsilon i}{2} \frac{dx_1 \wedge \cdots \wedge dx_n}{\left(\frac{-\epsilon^2-2\epsilon i}{4}\alpha - x_n\right)^{\frac{n+1}{2}} \alpha^{\frac{n-1}{2}}}.$$

We also ignore negative sign in the above formula of $d\tilde{V}_\epsilon$ when we compute integrals. The induced volume form $d\tilde{V}_\epsilon$ has a complicated formula, and instead we use a different simple volume approximation $d\mu_\epsilon$ which gives us the same μ -measure of U .

THEOREM 2.1. *Let U be a bounded domain with analytic boundary transversal to $\partial\mathbb{H}^n$ in \mathbb{E}^n and let*

$$d\mu_\epsilon = \frac{dx_1 \wedge \cdots \wedge dx_n}{2(-x_n - \epsilon i)^{\frac{n+1}{2}} \alpha^{\frac{n-1}{2}}}, \quad \alpha = |x - e_n|^2.$$

Then

$$\mu(U) = \lim_{\epsilon \rightarrow 0} \int_U d\mu_\epsilon.$$

Furthermore for the domain U with $-\delta < x_n < \delta$,

$$\mu(U) = \int_\gamma \int \frac{dx_1 \wedge \cdots \wedge dx_{n-1}}{2(-x_n)^{\frac{n+1}{2}} \alpha^{\frac{n-1}{2}}} dx_n,$$

where γ is a contour given below in Figure 4.

Proof. First observe that the volume of a domain lying completely inside of \mathbb{H}^n or \mathbb{L}^n , the same statement holds. This can be easily checked from Lebesgue dominated convergence theorem using $|x_n + \epsilon i| \geq |x_n|$ and from that $d\mu_0$ is just dV . Now by the finite additivity of the volume, we may assume that U lies in the domain $-\delta < x_n < \delta$ for a sufficiently small $\delta > 0$. We will prove the theorem in the following two steps:

Step 1:

$$\mu(U) := \lim_{\epsilon \rightarrow 0} \int_U \tilde{V}_\epsilon = \lim_{\epsilon \rightarrow 0} \int_{-\delta}^{\delta} \int d\tilde{V}_\epsilon = \lim_{\epsilon \rightarrow 0} \int_{\gamma} \int d\tilde{V}_\epsilon$$

and

$$\lim_{\epsilon \rightarrow 0} \int_U d\mu_\epsilon = \lim_{\epsilon \rightarrow 0} \int_{-\delta}^{\delta} \int d\mu_\epsilon = \lim_{\epsilon \rightarrow 0} \int_{\gamma} \int d\mu_\epsilon.$$

Here for the double integral $\int_{-\delta}^{\delta} \int$ we integrate first with respect to the variables (x_1, \dots, x_{n-1}) and then with respect to the variable x_n .

The contour integral \int_{γ} is an integration with respect to complex variable x_n and γ is a contour given below in Figure 4.

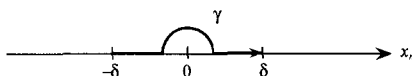


FIGURE 4

Step 2:

$$\lim_{\epsilon \rightarrow 0} \int_{\gamma} \int d\tilde{V}_\epsilon = \int_{\gamma} \int d\tilde{V}_0$$

and

$$\lim_{\epsilon \rightarrow 0} \int_{\gamma} \int d\mu_\epsilon = \int_{\gamma} \int d\mu_0.$$

Note that

$$d\tilde{V}_0 := \lim_{\epsilon \rightarrow 0} d\tilde{V}_\epsilon = dV = \lim_{\epsilon \rightarrow 0} d\mu_\epsilon =: d\mu_0,$$

and hence the theorem follows from Step 1 and Step 2.

Proof of Step 1: We can show that $\int_{-\delta}^{\delta} \int d\tilde{V}_\epsilon = \int_{\gamma} \int d\tilde{V}_\epsilon$ if we can show that the pole of $\int d\tilde{V}_\epsilon$ as a function of x_n has negative imaginary part for all $\epsilon > 0$. This looks intuitively so because

$$dV_\epsilon = \frac{d_\epsilon r^{n-1} dr d\theta_1 \cdots d\theta_{n-1}}{(d_\epsilon^2 - r^2)^{\frac{n+1}{2}}}$$

in spherical coordinates has pole with negative imaginary part near $r = 1$ and r corresponds essentially to x_n under the coordinate change map which is real.

To be more precise, let g be the coordinate change map $x=(x_1, \dots, x_n) = g(r, \theta_1, \dots, \theta_{n-1})$ given by a composite of spherical coordinates and the reflection σ :

$$\begin{cases} y_1 = r \sin \theta_1 \cdots \sin \theta_{n-2} \sin \theta_{n-1}, \\ y_2 = r \sin \theta_1 \cdots \sin \theta_{n-2} \cos \theta_{n-1}, \\ \vdots \\ y_{n-1} = r \sin \theta_1 \cos \theta_2, \\ y_n = r \cos \theta_1, \end{cases}$$

with $r > 0$, and $x_1 = \frac{2}{\alpha} y_1, \dots, x_{n-1} = \frac{2}{\alpha} y_{n-1}, x_n = \frac{2}{\alpha}(y_n - 1) + 1$, with $\alpha = |y - e_n|^2 = y_1^2 + \cdots + y_{n-1}^2 + (y_n - 1)^2$.

Write

$$d\tilde{V}_\epsilon = \frac{1}{f_\epsilon(x_1, \dots, x_n)} dx_1 \wedge \cdots \wedge dx_n$$

and consider zeroes of $f_\epsilon(c_1, \dots, c_{n-1}, x_n), c_i \in \mathbb{R}$. We claim that $f_\epsilon(c_1, \dots, c_{n-1}, x_n)$ has no real zeroes. Indeed if it had, $f_\epsilon \circ g$ would have real zeroes since g is real and hence $dV_\epsilon = \frac{d_\epsilon r^{n-1} dr d\theta}{(d_\epsilon^2 - r^2)^{\frac{n+1}{2}}} = \frac{1}{f_\epsilon} \circ g(\det g') dr d\theta$ would have real poles, which is absurd.

Therefore the imaginary part of zeroes of $f_\epsilon(c_1, \dots, c_{n-1}, x_n)$ is either positive or negative on a connected open set consisting of parameters (c_1, \dots, c_{n-1}) by continuity, and we can determine the sign by checking at one point. Notice that the r -axis given by $\theta_1 = \pi$ is sent to x_n -axis ($x_1 = \cdots = x_{n-1} = 0$) under g . In fact, $x_n = \frac{r-1}{r+1}$, $r > 0$ and this is an increasing function of r . If we complexify the real analytic function g , the complex analytic function $g_{\mathbb{C}}$ will preserve the negative imaginary parts and send $\{\text{im } r < 0\}$ to $\{\text{im } x_n < 0\}$ by the orientation reasoning. In this argument, the point $(0, \dots, 0, x_n)$ does not belong to the natural domain, i.e., the image under g of a maximal connected open domain where g is 1-1, but it is a boundary point of such domain, and the negativity of imaginary part of zeroes still follows.

Now since $d\tilde{V}_\epsilon = \frac{1}{f_\epsilon(x_1, \dots, x_n)} dx_1 \wedge \cdots \wedge dx_n$ has poles with negative imaginary part for all $x_1 = c_1, \dots, x_{n-1} = c_{n-1}, c_i \in \mathbb{R}$, therefore $\int \frac{1}{f_\epsilon(x_1, \dots, x_n)} dx_1 \wedge \cdots \wedge dx_{n-1}$ as a function of x_n is analytic near $x_n = 0$ with the poles only in the negative imaginary part. Here the analyticity comes from the condition that U has an analytic boundary transversal to $\partial\mathbb{H}^n$.

The proof of (2) is immediate by the same pole argument.

Proof of Step 2: For this part, we use Lebesgue dominated convergence theorem and it suffices to show when U is a compact set, say $U = D \times \gamma \subset \mathbb{R}^{n-1} \times \mathbb{C}$ with D compact domain. We essentially are integrating on a domain near $r = 1$ in Kleinian model \mathbb{K}^n which is symmetric with respect to the rotations and hence we may assume the corresponding domain U in \mathbb{E}^n is a small compact set near $x_n = 0$ without loss of generality. Since $dV_\epsilon = \frac{d_\epsilon r^{n-1} dr d\theta}{(d_\epsilon^2 - r^2)^{\frac{n+1}{2}}}$ is clearly uniformly bounded (with respect to $\epsilon > 0$) on $g^{-1}(U)$, its pull back $d\tilde{V}_\epsilon = g^{-1*}dV_\epsilon$, only differing by Jacobian determinant, is also uniformly bounded. Hence Lebesgue dominated convergence theorem applies.

The proof of (2) is clear by the same argument. \square

REMARK 2.2. The boundedness condition for a domain U in the statement of Theorem 2.1 is rather superficial. For a domain in the extended hyperbolic space, the finiteness of the volume depends only on how it crosses the boundary of \mathbb{H}^n . And by the finite additivity of μ , it suffices to consider a small domain near $\partial\mathbb{H}^n$ in \mathbb{K}^n , which we may assume is bounded in \mathbb{E}^n by considering rotation in \mathbb{K}^n if necessary before applying Cayley transformation to \mathbb{E}^n .

REMARK 2.3. The analyticity is required only to guarantee that the integral first with respect to variables (x_1, \dots, x_{n-1}) viewed as a function of x_n is analytic to replace the second integral by the contour integral. Therefore as far as this first integral on a domain U is an analytic function of x_n , the proof works equally well. In fact, Theorem 2.1 can be generalized to the case when U has a piecewise analytic boundary transversal to $\partial\mathbb{H}^n$ (see [1, Proposition 3.2]).

3. Volume of a domain with $C^{1,\alpha}$ -boundary

In this section we want to show that a domain U passing through $\partial\mathbb{H}^n$ with suitable regularity has a finite volume, i.e., $\mu(U) < \infty$ by computing in the flattened model. We first consider the case of dimension 2 and then the more complicated case of dimension 3.

A domain U in \mathbb{K}^n will be said to be $C^{k,\alpha}$ -transversal to $\partial\mathbb{H}^n$ if the boundary of U is given locally as a $C^{k,\alpha}$ function near $\partial\mathbb{H}^n$ and transversal to $\partial\mathbb{H}^n$ in the usual sense if $k \geq 1$. Namely for each point p in the intersection of U^b , the boundary of U , and $\partial\mathbb{H}^n$, there is a neighborhood V of p such that $U^b \cap V$ can be written as a zero set of a single $C^{k,\alpha}$ -function which is transversal to $\partial\mathbb{H}^n$. In the case of

dimension 2, a domain U in \mathbb{K}^2 is $C^{0,\alpha}$ -transversal to $\partial\mathbb{H}^2$ if locally the boundary of U near $\partial\mathbb{H}^2$ can be written as $\theta = g(r)$ for a $C^{0,\alpha}$ function g .

In the following discussions, we will say for the sake of convenience that an integral $\int f$ is equivalent to $\int g$, denoted by $\int f \sim \int g$, if $\int f < \infty$ holds if and only if $\int g < \infty$.

THEOREM 3.1. *In the two-dimensional extended hyperbolic space, the area of a domain U which is $C^{0,\frac{1}{2}+\alpha}$ -transversal to $\partial\mathbb{H}^2$ is finite.*

Proof. We will compute in the flattened model and transversality condition for the boundary may be written as $x_1 = g(x_2)$ for a $C^{0,\frac{1}{2}+\alpha}$ function g . It suffices to consider the $C^{0,\frac{1}{2}+\alpha}$ -transversal domain U in the flattened model which can be divided into pieces, one parallel strip perpendicular to x_1 -axis and other pieces (at most four pieces) with only one vertex lying in x_1 -axis.

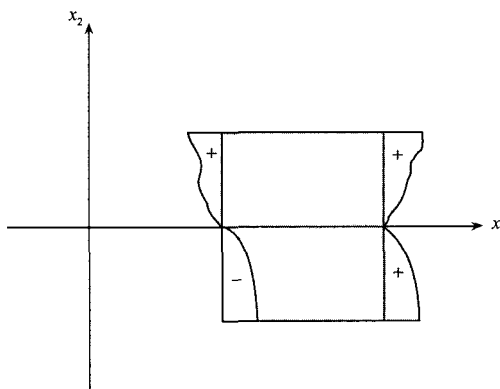


FIGURE 5

The transversal strip has finite area by the Theorem 2.1 and it suffices to show that each one vertex domain has also finite area. This can be done if we can show for a function $x_1 = g(x_2)$ with $g(0) = 0$ that

$$\int_0^\delta \int_0^{g(x_2)} \frac{dx_1}{x_2^{\frac{3}{2}}(x_1^2 + (x_2 - 1)^2)^{\frac{1}{2}}} dx_2 < \infty.$$

This integral is clearly equivalent to $\int_0^\delta \frac{g(x_2)}{x_2^{\frac{3}{2}}} dx_2$ and by $C^{0, \frac{1}{2} + \alpha}$ condition

of $g(x_2)$, we have $|g(x_2)| \leq C|x_2|^{\frac{1}{2} + \alpha}$ and hence

$$\int_0^\delta \frac{|g(x_2)|}{x_2^{\frac{3}{2}}} dx_2 \leq C \int_0^\delta \frac{1}{x_2^{1-\alpha}} dx_2 < \infty.$$

Thus $\int_0^\delta \frac{g(x_2)}{x_2^{\frac{3}{2}}} dx_2$ is finite. \square

So every polygonal domain transversal to $\partial\mathbb{H}^2$ has finite area trivially.

THEOREM 3.2. *In the three-dimensional extended hyperbolic space, the volume of a domain U which is $C^{1,\alpha}$ -transversal to $\partial\mathbb{H}^3$ is finite.*

Proof. We work in the flattened model as before. We first explain our strategy for the proof schematically in dimension 2 since the three dimensional picture is more complicated. Since all the difficulties arise near the boundary and near the hyperplane $\partial\mathbb{H}^n = \{x|x_n = 0\}$, we first localize the problem by taking a small rectangle near boundary in $\partial\mathbb{H}^n = \{x|x_n = 0\}$ and we want to show that the volume of the shaded domain in following picture is finite. We prove this by showing each of the following three types of integrals ((1), (2) and (3) in Figure 6) have finite values.

In dimension 3, the basic idea is the same as above and we use boxes instead of rectangles. We still have the above three types of integrals and show each of three is finite. We have to prove the following three integrals (1), (2) and (3) are finite.

For a compact domain B with C^1 boundary in the plane $x_3 = 0$, the volume of a cylindrical domain $B \times [-\delta, \delta]$ is represented by $\lim_{\epsilon \rightarrow 0} \int_{-\delta}^\delta \int_B d\tilde{V}_\epsilon$, and

$$(1) \quad \lim_{\epsilon \rightarrow 0} \int_B \int_{-\delta}^\delta d\tilde{V}_\epsilon = \lim_{\epsilon \rightarrow 0} \int_B \int_\gamma d\tilde{V}_\epsilon = \int_B \int_\gamma d\mu_0 < \infty.$$

This follows from the pole argument used in the proof of Theorem 2.1 and uniform boundedness of F_ϵ on compact set, where $d\tilde{V}_\epsilon = F_\epsilon(x_1, x_2, x_3)dx_1dx_2dx_3$.

For the type (2) integral, consider typically the domains $U_+ = \{(x_1, x_2, x_3) \in \mathbb{E}^3 | a \leq x_1 \leq b, 0 \leq x_3 \leq \delta, c(x_1) \leq x_2 \leq c(x_1) + d(x_1)x_3\}$ and $U_- = \{(x_1, x_2, x_3) \in \mathbb{E}^3 | a \leq x_1 \leq b, -\delta \leq x_3 \leq 0, c(x_1) + d(x_1)x_3 \leq x_2 \leq c(x_1)\}$ as given in Figure 7.

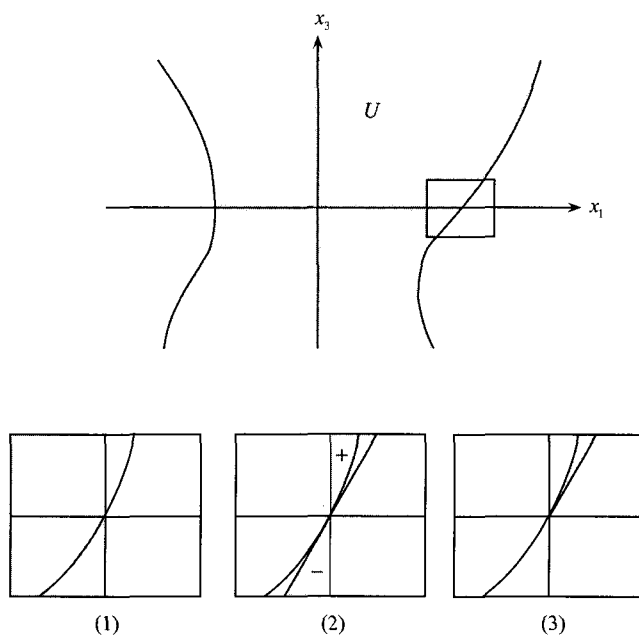


FIGURE 6

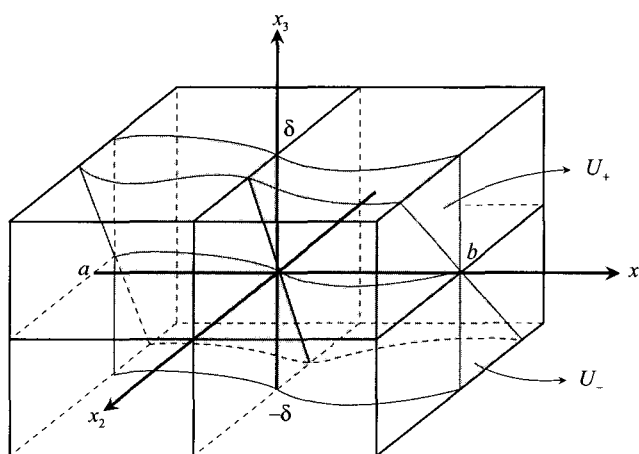


FIGURE 7

Then the $\text{vol}(U_+) - \text{vol}(U_-)$ is represented by

$$\int_a^b \int_0^\delta \int_{c(x_1)}^{c(x_1)+d(x_1)x_3} d\tilde{V}_\epsilon - \int_a^b \int_{-\delta}^0 \int_{c(x_1)+d(x_1)x_3}^{c(x_1)} d\tilde{V}_\epsilon,$$

and simplified to

$$(2) \quad \int_a^b \int_{-\delta}^{\delta} \int_{c(x_1)}^{c(x_1)+d(x_1)x_3} F_{\epsilon}(x_1, x_2, x_3) dx_2 dx_3 dx_1.$$

As we have shown in the proof of Step 1 of Theorem 2.1, the pole of $\int_c^{c+d \cdot x_3} F_{\epsilon} dx_2$, as a function of x_3 , has negative imaginary part and is analytic on $\{x_3 = \alpha_3 + \beta_3 i | \beta_3 \geq 0\}$, and hence we have

$$\int_{-\delta}^{\delta} \int_c^{c+d \cdot x_3} F_{\epsilon} dx_2 dx_3 = \int_{\gamma} \int_c^{c+d \cdot x_3} F_{\epsilon} dx_2 dx_3.$$

From the uniform boundedness of F_{ϵ} , it follows that

$$\lim_{\epsilon \rightarrow 0} \int_a^b \int_{\gamma} \int_c^{c+d \cdot x_3} d\tilde{V}_{\epsilon} = \int_a^b \int_{\gamma} \int_c^{c+d \cdot x_3} \lim_{\epsilon \rightarrow 0} d\tilde{V}_{\epsilon} = \int_a^b \int_{\gamma} \int_c^{c+d \cdot x_3} d\mu_0 < \infty.$$

Let us think about the third type integral. In this case we integrate on the domain lying only in one side \mathbb{H}^3 or \mathbb{L}^3 , and the integral becomes

$$(3) \quad \int_a^b \int_0^{\delta} \int_{c(x_1)+d(x_1)x_3}^{c(x_1)+d(x_1)x_3+g(x_1,x_3)} F_0(x_1, x_2, x_3) dx_2 dx_3 dx_1,$$

where $g(x_1, 0) = \frac{\partial g}{\partial x_3}(x_1, 0) = 0$ and $g \in C^{1,\alpha}$ from the hypothesis of $C^{1,\alpha}$ -transversality of the boundary of U and implicit function theorem for $C^{1,\alpha}$ function. The finiteness of (3) follows from the finiteness of $\int_0^{\delta} \int_{c+d \cdot x_3}^{c+d \cdot x_3+g(x_3)} F_0 dx_2 dx_3$, where “ $g(x_3)$ ” = $g(x_1, x_3)$ for each fixed x_1 abusing the notation for g . And this integral is equivalent to

$$\int_0^{\delta} \int_{c+d \cdot x_3}^{c+d \cdot x_3+g(x_3)} \frac{1}{x_3^2} dx_2 dx_3 = \int_0^{\delta} \frac{g(x_3)}{x_3^2} dx_3.$$

The $C^{1,\alpha}$ condition gives us $|g(x_3)| \leq C|x_3|^{1+\alpha}$ and hence $\int_0^{\delta} \frac{|g(x_3)|}{x_3^2} dx_3 \leq \int_0^{\delta} \frac{1}{x_3^{1-\alpha}} dx_3 < \infty$, and therefore $\int_0^{\delta} \frac{g(x_3)}{x_3^2} dx_3 < \infty$.

We have shown that the local volumes are finite. But this is not enough in dimension 3. For this type of finitely additive measure μ is very subtle and we can not say in general that the volume of the intersection of two domains with finite volumes is also finite.

Let's arrange boxes carefully as in the following picture around the boundary of U and (x_1, x_2) -coordinate plane. The picture is the intersections of boxes with (x_1, x_2) -coordinate plane and shows the wedge shaped domains obtained as intersections (G_i) of two boxes and discrepancies (F_i) not covered by boxes.

Notice that $F_i \cap U$, the domain not covered by boxes S_i , is contained in the tetrahedron T which is bounded by the sides of the boxes and the tangent plane of ∂U (or a suitable plane so that the tetrahedron T contains $F_i \cap U$). The domain $G_i \cap U$, overlapped by two boxes, is contained in the prism minus tetrahedron. We have already shown that the volume of prism is finite as it is a type (1) integral. Hence if we can show that the volume of tetrahedron T is finite, then we can complete the proof of the theorem. But T lies completely in \mathbb{H}^3 or \mathbb{L}^3 and also being a subset of a cone, it suffices to show that the cone type domain $E = \{(x_1, x_2, x_3) | 0 \leq x_3 \leq \delta, x_3 \geq k\sqrt{x_1^2 + x_2^2}\}$ has finite volume. Because the measure in \mathbb{H}^n or \mathbb{L}^n is essentially positive measure.

$$\begin{aligned}
 \text{vol}(E) &= \int_0^\delta \int_0^{\frac{x_3}{k}} \int_0^{2\pi} \frac{rd\theta dr dx_3}{2x_3^2(r^2 + (x_3 - 1)^2)} \\
 (4) \quad &\sim \int_0^\delta \int_0^{\frac{x_3}{k}} \frac{r}{x_3^2} dr dx_3 \\
 &= \int_0^\delta \frac{1}{2k^2} dx_3 < \infty
 \end{aligned}$$

□

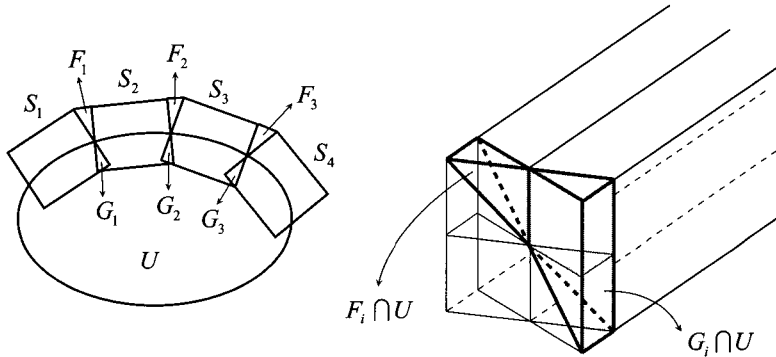


FIGURE 8

REMARK 3.3. The regularity condition for ∂U is sharp in the theorem, and in fact there exists a C^1 -transversal domain U with infinite volume.

Let $U := \{(x_1, x_2, x_3) | a \leq x_1 \leq b, -\delta \leq x_3 \leq \delta < 1, -1 \leq x_2 \leq h(x_3)\}$, where $h(x_3) = -x_3/\log x_3$ for $x_3 \in (0, \delta]$ and $h(x_3) = 0$ for

$x_3 \in [-\delta, 0]$, then it is easy to show that the volume of U is infinite by showing that

$$\begin{aligned}\mu(U) &\sim \int_a^b \int_0^\delta \int_0^{-\frac{x_3}{\log x_3}} \frac{dx_2 dx_3 dx_1}{x_3^2(x_1^2 + x_2^2 + (x_3 - 1)^2)} \\ &\sim \int_a^b \int_0^\delta \int_0^{-\frac{x_3}{\log x_3}} \frac{dx_2 dx_3 dx_1}{x_3^2} = \int_a^b \int_0^\delta \frac{dx_3 dx_1}{-x_3 \log x_3} = \infty.\end{aligned}$$

REMARK 3.4. For the higher dimensional case, if we can handle a domain U as in the proof Theorem 3.2, from the formula of volume form, we may expect the necessary regularity condition for the finiteness as follows:

$$\begin{aligned}n = 4, & \quad C^{1, \frac{1}{2} + \alpha}\text{-transversal} \\ n = 5, & \quad C^{2, \alpha}\text{-transversal} \\ & \quad \vdots\end{aligned}$$

The necessary regularity increases by $1/2$ for each dimension increase. We do not pursue this issue here any more. But note that this condition is sharp in the sense that we can find a domain with infinite volume as in Remark 3.3 if $\alpha = 0$.

REMARK 3.5. In the proof of Theorem 3.2, we used the volume form $d\tilde{V}_\epsilon$ in the computation of integrals. But we can use $d\mu_\epsilon$ as well instead of $d\tilde{V}_\epsilon$. Indeed for the integrals (3) and (4), both $d\tilde{V}_\epsilon$ and $d\mu_\epsilon$ will converge to the singular volume form $d\mu_0$ and gives the same value for the integrals. For the integrals (1) and (2), the replacement by $d\mu_\epsilon$ leads to the same integral equation by the same pole argument and uniform boundedness, and eventually get the same integration value.

As a final results, we will show that the volume of $C^{1,\alpha}$ -transversal 3-dimensional domain U is invariant under hyperbolic isometries. Of course we can obtain the same result in dimension 2 for $C^{0, \frac{1}{2} + \alpha}$ -transversal domain U similarly and more easily.

THEOREM 3.6. *The volume of $C^{1,\alpha}$ -transversal domain U is invariant under isometry.*

Proof. Since the hyperbolic isometries are generated by reflections, we show the theorem for a reflection g . Furthermore it suffices to show $\text{vol}(U) = \text{vol}(gU)$ for each of the four types domain appeared as (1), (2), (3), and (4) in the proof of Theorem 3.2.

For types (1), we can write as follows:

$$\begin{aligned}\text{vol } (U) &= \lim_{\epsilon \rightarrow 0} \int_B \int_{-\delta}^{\delta} d\tilde{V}_\epsilon = \int_B \int_\gamma d\mu_0 = \int_B \int_\gamma g^*(d\mu_0) \\ &= \lim_{\epsilon \rightarrow 0} \int_B \int_{-\delta}^{\delta} g^*(d\tilde{V}_\epsilon) = \text{vol } (gU).\end{aligned}$$

Here it is enough to give the proof of the fourth equality, which requires the pole argument and uniform boundedness as we have used several times before. Indeed notice that $g^*d\tilde{V}_\epsilon$ never has a real pole for all reflections g since g is real and $d\tilde{V}_\epsilon$ does not have a real pole. Hence the poles of $g^*d\tilde{V}_\epsilon$ have either positive imaginary parts or negative imaginary parts for all g by continuity with respect to g . Now it suffices to check the sign of imaginary part for a particular reflection g_0 that fixes (x_2, x_3) -coordinate plane. This fixes x_3 -axis and hence its complexification preserves negative imaginary part of complex x_3 -axis and poles of $g_0^*d\tilde{V}_\epsilon$ has negative imaginary part since $d\tilde{V}_\epsilon$ does. The uniform boundedness on a compact set follows by the same argument as in the proof of Theorem 2.1.

The invariance of type (2) integral follows similarly.

The domain appeared in the integrals of type (3) and (4) are either in hyperbolic or Lorentzian space and the integrals are usual volumes which of course are isometry invariant. \square

References

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