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ON SPIN ALTERNATING GROUP ACTIONS ON SPIN 4-MANIFOLDS

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Abstract. Let X be a smooth, closed, connected spin 4-manifold with $b_1(X) = 0$ and signature $\sigma(X)$. In this paper we use Seiberg-Witten theory to prove that if X admits a spin alternating A_4 action, then $b_2^+(X) \geq |\sigma(X)|/8 + 3$ under some non-degeneracy conditions.

1. Introduction

Let X be a smooth, closed, connected spin 4-manifold. We denote by $b_2(X)$ the second Betti number and denote by $\sigma(X)$ the signature of X. In [12], Y. Matsumoto conjectured the following inequality

$$b_2(X) \geq \frac{11}{8}|\sigma(X)|. \tag{1}$$

This conjecture is well known and has been called the $\frac{11}{8}$-conjecture (see also [7]). All complex surfaces and their connected sums satisfy the conjecture (see [11]).

From the classification of unimodular even integral quadratic forms and the Rochlin's theorem, for the choice of orientation with non-positive signature the intersection form of a closed spin 4-manifold X is

$$-2kE_8 \oplus mH, \qquad k \geq 0,$$

where E_8 is the 8×8 intersection form matrix and H is the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Thus, $m = b_2^+(X)$ and $k = -\sigma(X)/16$ and so the inequality (1) is equivalent to $m \geq 3k$. Since $K3$ surface satisfies the equality with $k = 1$ and $m = 3$, the coefficient $\frac{11}{8}$ is optimal, if the $\frac{11}{8}$-conjecture is true.

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Donaldson has proved that if $k > 0$ then $m \geq 3$ [4]. In early 1995, using the Seiberg-Witten theory introduced by Seiberg and Witten [16], Furuta [8] proved that

$$b_2(X) \geq \frac{5}{4}|\sigma(X)| + 2. \tag{2}$$

This estimate has been dubbed the $\frac{10}{8}$-theorem. In fact, if the intersection form of X is definite, i.e., $m = 0$, then Donaldson proved that $b_2(X)$ and $\sigma(X)$ are zero [4, 5]. Thus, Furuta assumed that m is not zero. Inequality (2) follows by a surgery argument from the non-positive signature, $b_1(X) = 0$ case:

THEOREM 1.1. (Furuta [8]). *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ with non-positive signature. Let $k = -\sigma(X)/16$ and $m = b_2^+(X)$. Then*

$$2k + 1 \leq m$$

if $m \neq 0$.

His key idea is to use a finite dimensional approximation of the monopole equation. Later Furuta and Kametani [9] used equivariant e-invariants and improved the above $\frac{10}{8}$-theorem as following.

THEOREM 1.2. (Furuta and Kametani [9]). *Suppose that X is a closed oriented spin 4-manifold. If $\sigma(X) < 0$, then*

$$b_2^+(X) \geq \begin{cases} 2(-\sigma(X)/16) + 1, & -\sigma(X)/16 \equiv 0, 1 \ mod\ 4, \\ 2(-\sigma(X)/16) + 2, & -\sigma(X)/16 \equiv 2 \ mod\ 4, \\ 2(-\sigma(X)/16) + 1, & -\sigma(X)/16 \equiv 3 \ mod\ 4. \end{cases}$$

The above inequality was also proved by N. Minami [13] by using an equivariant join theorem to reduce the inequality to a theorem of Stolz [15].

Throughout this paper we will assume that m is not zero and $b_1(X) = 0$, unless stated otherwise.

A $Z/2^p$-action is called a spin action if the generator of the action $\tau : X \to X$ lifts to an action $\hat{\tau} : P_{Spin} \to P_{Spin}$ of the Spin bundle P_{Spin}. Such an action is of even type if $\hat{\tau}$ has order 2^p and is of odd type if $\hat{\tau}$ has order 2^{p+1}.

In [2], Bryan (see also [6]) used Furuta's technique of "finite dimensional approximation" and the equivariant K-theory to improve the above bound by p under the assumption that X has a spin odd type $Z/2^p$-action satisfying some non-degeneracy conditions analogous to the condition $m \neq 0$. More precisely, he proved

THEOREM 1.3. (Bryan [2]). *Let X be a smooth, closed, connected spin 4-manifold with $b_1(X) = 0$. Assume that $\tau : X \to X$ generates a spin smooth $Z/2^p$-action of odd type. Let X_i denote the quotient of X by $Z/2^i \subset Z/2^p$. Then*

$$2k + 1 + p \leq m$$

if $m \neq 2k + b_2^+(X_1)$ and $b_2^+(X_i) \neq b_2^+(X_j) > 0$ for $i \neq j$.

In the paper [10], Kim gave the same bound for smooth, spin, even type $Z/2^p$-action on X satisfying some non-degeneracy conditions analogous to Bryan's.

In this article, we follow a suggestion of Furuta and study the spin alternating group A_4 actions on spin 4-manifolds, we prove that if X admits a spin alternating A_4 action, then $b_2^+(X) \geq |\sigma(X)|/8 + 3$ under some non-degeneracy conditions.

The organization of the remainder of this paper is as follows. In section 2, we give some preliminaries to prove the main theorem. We refer the readers to the Bryan's excellent exposition [2] for more details. We introduce the representation ring and the the character table of alternating group A_4 in section 3. In section 4, we we use equivariant K-theory and representation theory to study the G-equivariant properties of the moduli space. In the last section we give our main results.

2. Notations and preliminaries

We assume that we have completed every Banach spaces with suitable Sobolev norms. Let $S = S^+ \oplus S^-$ denote the decomposition of the spinor bundle into the positive and negative spinor bundles. Let $D : \Gamma(S^+) \to \Gamma(S^-)$ be the Dirac operator, and $\rho : \Lambda_C^* \to \mathrm{End}_C(S)$ be the Clifford multiplication. The Seiberg-Witten equations are for a pair $(a, \phi) \in \Omega^1(X, \sqrt{-1}R) \times \Gamma(S^+)$ and they are

$$D\phi + \rho(a)\phi = 0, \qquad \rho(d^+a) - \phi \otimes \phi^* + \frac{1}{2}|\phi|^2 id = 0, \qquad d^*a = 0.$$

Let

$$V = \Gamma(\sqrt{-1}\Lambda^1 \oplus S^+),$$
$$W' = \Gamma(S^- \oplus \sqrt{-1}su(S^+) \oplus \sqrt{-1}\Lambda^0).$$

We can think of the equation as the zero set of a map

$$\mathcal{D} + \mathcal{Q} : V \to W,$$

where $\mathcal{D}(a,\phi) = (D\phi, \rho(d^+a), d^*a))$, $\mathcal{Q}(a,\phi) = (\rho(a)\phi, \phi\otimes\phi^* - \frac{1}{2}|\phi|^2 id$, $0)$, and W is defined to be the orthogonal complement to the constant functions in W'.

Now it is time to describe the group of symmetries of the equations. Define $\mathrm{Pin}(2) \subset SU(2)$ to be the normalizer of $S^1 \subset SU(2)$. Regarding $SU(2)$ as the group of unit quaternions and taking S^1 to be elements of the form $e^{\sqrt{-1}\theta}$, Pin(2) then consists of the form $e^{\sqrt{-1}\theta}$ or $e^{\sqrt{-1}\theta}J$. We define the action of Pin(2) on V and W as follows: Since S^+ and S^- are $SU(2)$ bundles, Pin(2) naturally acts on $\Gamma(S^{\pm})$ by multiplication on the left. $Z/2$ acts on $\Gamma(\Lambda^*_C)$ by multiplication by ± 1 and this pulls back to an action of Pin(2) by the natural map $\mathrm{Pin}(2) \to Z/2$. A calculation shows that this pullback also describes the induced action of Pin(2) on $\sqrt{-1}su(S^+)$. Both $\mathcal{D}$ and $\mathcal{Q}$ are seen to be Pin(2) equavariant maps.

Let X be a smooth closed spin 4-manifold. Suppose that X admits a spin structure preserving action by a compact Lie group (or finite group) G. We may assume a Riemannian metric on X so that G acts by isometries. If the action is of even type, Both $\mathcal{D}$ and $\mathcal{Q}$ are $\tilde{G} = \mathrm{Pin}(2)\times G$ equavariant maps.

Now we define V_λ to be the subspace of V spanned by the eigenspaces $\mathcal{D}^*\mathcal{D}$ with eigenvalues less than or equal to $\lambda \in R$. Similarly, define W_λ using $\mathcal{D}\mathcal{D}^*$. The virtual G-representation $[V_\lambda\otimes C] - [W_\lambda \otimes C] \in R(\tilde{G})$ is the $\tilde{G}$-index of $\mathcal{D}$ and can be determined by the $\tilde{G}$-index and is independent of $\lambda \in R$, where $R(\tilde{G})$ is the complex representation of $\tilde{G}$. In particular, since $V_0 = \mathrm{Ker}D$ and $W_0 = \mathrm{Coker}D \oplus \mathrm{Coker}d^+$, we have

$$[V_\lambda\otimes C] - [W_\lambda\otimes C] = [V_0\otimes C] - [W_0\otimes C] \in R(\tilde{G}).$$

Note that $\mathrm{Coker}d^+ = H^2_+(X,R)$.

3. The alternating group A_4

The alternating group A_4 is the group of even permutations of a set $\{a,b,c,d\}$ having 4 elements; it is isomorphic to the group of rotations in R^3 which stabilize a regular tetrahedron with barycenter the origin. It has the following 12 elements:

(a) the identity element 1;

(b) 3 elements of order 2, $x = (ab)(cd)$, $y = (ac)(bd)$, $z = (ad)(bc)$, which correspond to reflections of the tetrahedron through lines joining the midpoints of two edges.

(c) 8 elements of order 3: (abc), (acb), ..., (bcd), which correspond to rotations of $\pm\frac{2\pi}{3}$ with respect to lines joining a vertex to the barycenter of the opposite face.

Set $t = (abc)$, $K = \{1, t, t^2\}$ and $H = \{1, x, y, z\}$. We have

$$txt^{-1} = z, \qquad tzt^{-1} = y, \qquad tyt^{-1} = x;$$

moreover H and K are subgroups of A_4, H is normal, and $H \cap K = \{1\}$. It is easy to see that each element of A_4 can be written uniquely as a product $h \cdot k$ with $h \in H$ and $k \in K$.

There are the following 4 conjugacy classes in $A_4 : \{1\}, \{x, y, z\}, \{t, tx, ty, tz\}$, and $\{t^2, t^2x, t^2y, t^2z\}$, hence 4 irreducible characters. There are three characters of degree 1, corresponding to the three characters χ_0, χ_1, and χ_2 of the group K extended to A_4 by setting $\chi_i(h \cdot k) = \chi_i(k)$ for $h \in H$ and $k \in K$. The last character ψ is the character of the natural representation of A_4 in R^3 (extended to C^3 by linearity). Thus we have the following character table for A_4 [14]:

	1	x	t	t^2
χ_0	1	1	1	1
χ_1	1	1	ω	ω^2
χ_2	1	1	ω^2	ω
ψ	3	-1	0	0

where $\omega = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Let X be a smooth closed spin 4-manifold. Suppose that X admits a spin structure preserving action by a compact Lie group (or finite group) G. We may assume a Riemannian matric on X so that G acts by isometries. This G-action can always be lifted to $\hat{G}$-actions on the spinor bundles, where $\hat{G}$ is the following extension

$$1 \to Z_2 \to \hat{G} \to G \to 1.$$

Recall that the G-action is of even type if $\hat{G}$ contains a subgroup isomorphic to G, and in turn is of odd type, otherwise. For alternating group A_4, the extension of A_4 by Z_2 is isomorphic to $Z_2 \times A_4$, that is any spin alternating group A_4 action on spin 4-manifolds is of even type.

4. The index of $\mathcal{D}$ and the character formula for the K-theory degree

The virtual representation $[V_{\lambda,C}] - [W_{\lambda,C}] \in R(\tilde{G})$ is the same as $\text{Ind}(\mathcal{D}) = [\ker\mathcal{D}] - [\text{Coker}\mathcal{D}]$. Furuta determines $\text{Ind}(\mathcal{D})$ as a Pin(2) representation; denoting the restriction map $r : R(\tilde{G}) \to R(\text{Pin}(2))$, Furuta shows

$$r(\text{Ind}(\mathcal{D})) = 2kh - m\tilde{1},$$

where $k = -\sigma(X)/16$ and $m = b_2^+(X)$. Thus $\text{Ind}(\mathcal{D}) = sh - t\tilde{1}$, where s and t are polynomials such that $s(1) = 2k$ and $t(1) = m$. For a spin A_4 action, $\tilde{G} = \text{Pin}(2) \times A_4$, we can write

$$s(\xi, \eta) = a_0 + b_0\xi + c_0\xi^2 + d_0\eta$$

and

$$t(\xi, \eta) = a_1 + b_1\xi + c_1\xi^2 + d_1\eta$$

such that $a_0 + b_0 + c_0 + 3d_0 = 2k$ and $a_1 + b_1 + c_1 + 3d_1 = m = b_2^+(X)$.

For any element $g \in A_4$, denote by $< g >$ the subgroup of A_4 generated by g. Then we have

$$\begin{aligned}
\dim (H^+(X)^{A_4}) &= a_1 = b_2^+(X/A_4),\\
\dim (H^+(X)^{<(abc)>}) &= a_1 + d_1 = b_2^+(X/ < (abc) >),\\
\dim (H^+(X)^{<(ab)(cd)>}) &= a_1 + b_1 + c_1 + d_1 = b_2^+(X/ < (ab)(cd)) >),\\
\dim (H^+(X)^H) &= a_1 + b_1 + c_1 = b_2^+(X/H).
\end{aligned}$$

The Thom isomorphism theory in equivariant K-theory for a general compact Lie group is a deep theory proved using elliptic operator [1]. The subsequent character formula of this section uses only elementary properties of the Bott class.

Let V and W be complex Γ representations for some compact Lie group Γ. Let BV and BW denote balls in V and W and let $f : BV \to BW$ be a Γ-map preserving the boundaries SV and SW. $K_\Gamma(V)$ is by definition $K_\Gamma(BV, SV)$, and by the equivariant Thom isomorphism theorem, $K_\Gamma(V)$ is a free $R(\Gamma)$ module with generator the Bott class $\lambda(V)$. Applying the K-theory functor to f we get a map

$$f^* : K_\Gamma(W) \to K_\Gamma(V)$$

which defines a unique element $\alpha_f \in R(\Gamma)$ by the equation $f^*(\lambda(W)) = \alpha_f \cdot \lambda(V)$. The element α_f is called the K-theory degree of f.

Let V_g and W_g denote the subspaces if V and W fixed by an element $g \in \Gamma$ and let $V_g^\perp$ and $W_g^\perp$ be the orthogonal complements. Let $f^g : V_g \to W_g$ be the restriction of f and let $d(f^g)$ denote the ordinary topological degree of f^g (by definition, $d(f^g) = 0$ if $\dim V_g \neq \dim W_g$). For any $\beta \in R(\Gamma)$, let $\lambda_{-1}\beta$ denote the alternating sum $\Sigma(-1)^i\lambda^i\beta$ of exterior powers.

T. tom Dieck proved the following character formula for the degree α_f:

THEOREM. ([3]). *Let $f : BV \to BW$ be a Γ-map preserving boundaries and let $\alpha_f \in R(\Gamma)$ be the K-theory degree. Then*

$$\mathrm{tr}_g(\alpha_f) = d(f^g)\mathrm{tr}_g(\lambda_{-1}(W_g^\perp - V_g^\perp)),$$

where tr_g is the trace of the action of an element $g \in \Gamma$.

This formula is very useful in the case where $\dim V_g \neq \dim W_g$ so that $d(f^g) = 0$.

Recall that $\lambda_{-1}(\Sigma_i a_i r_i) = \prod_i (\lambda_{-1} r_i)^{a_i}$ and that for a one dimensional representation r, we have $\lambda_{-1} r = (1 - r)$. A two dimensional representation such as h has $\lambda_{-1} h = (1 - h + \Lambda^2 h)$. In this case, since h comes from an $SU(2)$ representation, $\Lambda^2 h = \det h = 1$ so $\lambda_{-1} h = (2 - h)$.

In the following by using the character formula to examine the K-theory degree α_{f_λ} of the map $f_\lambda : BV_{\lambda,C} \to BW_{\lambda,C}$ coming from the Seiberg-Witten equations. We will abbreviate α_{f_λ} as α and $V_{\lambda,C}$ and $W_{\lambda,C}$ as just V and W. Let $\phi \in S^1 \subset \mathrm{Pin}(2) \subset G$ be the element generating a dense subgroup of S^1, and recall that there is the element $J \in \mathrm{Pin}(2)$ coming from the quaternion. Note that the action of J on h has two invariant subspaces on which J acts by multiplication with $\sqrt{-1}$ and $-\sqrt{-1}$.

5. The main results

Consider $\alpha = \alpha_{f_\lambda} \in R(\mathrm{Pin}(2) \times A_4)$, it has the following form

$$\alpha = \alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty} \alpha_i h_i,$$

where $\alpha_i = m_i + n_i\xi + l_i\xi^2 + f_i\eta$, $i \geq 0$ and $\tilde{\alpha_0} = \tilde{m_0} + \tilde{n_0}\xi + \tilde{l_0}\xi^2 + \tilde{f_0}\eta$.

Since ϕ acts non-trivially on h and trivially on $\tilde{1}$, so

$$\dim(V(\xi,\eta))_\phi - \dim(W(\xi,\eta))_\phi = -(a_1 + b_1 + c_1 + 3d_1) = -m = -b_2^+(X).$$

So if $b_2^+(X) > 0$, $\mathrm{tr}_\phi\alpha = 0$.

Since ϕt acts non-trivially on $V(\xi,\eta)h$, ϕ trivially on $\tilde{1}$ and t acts trivially on a_1 and non-trivially on $b_1\xi$ and $c_1\xi^2$, the action of t on $d_1\eta$ has a one-dimensional invariant subspace. So we have

$$\dim(V(\xi,\eta))_{\phi t} - \dim(W(\xi,\eta))_{\phi t} = -(a_1+d_1) = -b_2^+(X/<(abc)>).$$

Similarly,

$$\dim(V(\xi,\eta))_{\phi t^2} - \dim(W(\xi,\eta))_{\phi t^2} = -(a_1+d_1) = -b_2^+(X/<(abc)>).$$

So if $a_1+d_1 = b_2^+(X/<(abc)>) \neq 0$, $\mathrm{tr}_{\phi t}\alpha = \mathrm{tr}_{\phi t^2}\alpha = 0$.

Since ϕx acts non-trivially on $V(\xi,\eta)h$, ϕx acts trivially on $\tilde{1}$, $b_1\xi\tilde{1}$ and $c_1\xi^2\tilde{1}$. The action of ϕx on $d_1\eta\tilde{1}$ has a one-dimensional invariant subspace. So

$$\begin{aligned}\dim(V(\xi,\eta))_{\phi x} - \dim(W(\xi,\eta))_{\phi x} &= -(a_1+b_1+c_1+d_1)\\ &= -b_2^+(X/<(ab)(cd)>).\end{aligned}$$

So if $a_1+b_1+c_1+d_1 = b_2^+(X/<(ab)(cd)>) \neq 0$, $\mathrm{tr}_{\phi x}\alpha = 0$.

If $b_2^+(X) - b_2^+(X/<(ab)(cd)>) \neq 0$, that is $d_1 \neq 0$, we have $\mathrm{tr}_{\phi}\alpha = \mathrm{tr}_{\phi t}\alpha = \mathrm{tr}_{\phi t^2}\alpha = \mathrm{tr}_{\phi x}\alpha = 0$ which implies that

$$\begin{aligned}0 &= \mathrm{tr}_{\phi}\alpha = \mathrm{tr}_{\phi}(\alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty}\alpha_i h_i)\\ &= \mathrm{tr}_{\phi}\alpha_0 + \mathrm{tr}_{\phi}\tilde{\alpha_0} + \sum_{i=1}^{\infty}\mathrm{tr}_{\phi}\alpha_i(\phi^i+\phi^{-i})\\ &= (m_0+n_0+l_0+3f_0) + (\tilde{m_0}+\tilde{n_0}+\tilde{l_0}+3\tilde{f_0})\\ &\quad + \sum_{i=1}^{\infty}\mathrm{tr}_{\phi}\alpha_i(\phi^i+\phi^{-i}),\end{aligned}$$

$$\begin{aligned}0 &= \mathrm{tr}_{\phi t}\alpha = \mathrm{tr}_t(\alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty}\alpha_i(\phi^i+\phi^{-i}))\\ &= (m_0+n_0\omega+l_0\omega^2) + (\tilde{m_0}+\tilde{n_0}\omega+\tilde{l_0}\omega^2)\\ &\quad + \sum_{i=1}^{\infty}\mathrm{tr}_t\alpha_i(\phi^i+\phi^{-i}),\end{aligned}$$

and so on. From these equations we have $\alpha_0 = -\tilde{\alpha_0}$ and $\alpha_i = 0, i > 0$, that is $\alpha = \alpha_0(1-\tilde{1})$.

Next we calculate $\mathrm{tr}_J\alpha$. Since J acts non-trivially on both h and $\tilde{1}$, $\dim V_J = \dim W_J = 0$, so $d(f^J) = 1$ and the character formula gives

$\mathrm{tr}_J(\alpha) = \mathrm{tr}_J(\lambda_{-1}(m\tilde{1} - 2kh) = \mathrm{tr}_J((1-\tilde{1})^m(2-h)^{-2k}) = 2^{m-2k}$ using $\mathrm{tr}_J h = 0$ and $\mathrm{tr}_J\tilde{1} = -1$.

Now we calculate $\mathrm{tr}_{Jt}\alpha$. Since Jt acts non-trivially on both $V(\xi,\eta)h$ and $W(\xi,\eta)\tilde{1}$, so $d(f^{Jt}) = 1$. By tom Dieck formula, we have

$$\begin{aligned}
\mathrm{tr}_{Jt}(\alpha) &= \mathrm{tr}_{Jt}[\lambda_{-1}(a_1 + b_1\xi + c_1\xi^2 + d_1\eta)\tilde{1} \\
&\quad -\lambda_{-1}(a_0 + b_0\xi + c_0\xi^2 + d_0\eta)h] \\
&= \mathrm{tr}_{Jt}[(1-\tilde{1})^{a_1}(1-\xi\tilde{1})^{b_1}(1-\xi^2\tilde{1})^{c_1}(1-\eta\tilde{1})^{d_1} \\
&\quad \times(1-h)^{-a_0}(1-\xi h)^{b_0}(1-\xi^2 h)^{-c_0}(1-\eta h)^{-d_0}] \\
&= 2^{a_1}(1+\omega)^{b_1}(1+\omega^2)^{c_1}\cdot 2^{d_1}\cdot 2^{-a_0}\cdot(1+\omega^2)^{-b_0} \\
&\quad \times(1+\omega)^{-c_0}2^{-d_0} \\
&= 2^{(a_1+d_1)-(a_0+d_0)}(1+\omega)^{b_1-c_0}(1+\omega^2)^{c_1-b_0} \\
&= 2^{(a_1+d_1)-(a_0+d_0)}(1+\omega)^{(b_0+b_1)-(c_0+c_1)}.
\end{aligned}$$

Here the 2-dimensional representation h decomposes into two complex line on which J acts as $\sqrt{-1}$ and $-\sqrt{-1}$. And the 3 dimensional representation η decomposes into three complex line on which t acts as 1, ω and ω^2. J acts on $\tilde{1}$ as -1.

Similarly we could get

$$\mathrm{tr}_{Jt^2}(\alpha) = 2^{(a_1+d_1)-(a_0+d_0)}(1+\omega^2)^{(b_0+b_1)-(c_0+c_1)}.$$

The 3-dimensional representation η decomposes into three complex line on which x acts as 1, -1 and -1.

Since Jx acts non-trivially on $V(\xi,\eta)h$, and Jx acts non-trivially on $\tilde{1}$, $a_1\xi\tilde{1}$ and $c_1\xi^2\tilde{1}$, but the action of Jx on $d_1\eta\tilde{1}$ has two-dimensional invariant subspace. So we have

$$\dim(V(\xi,\eta))_{Jx} - \dim(W(\xi,\eta))_{Jx} = -2d_1.$$

Then if $d_1 \neq 0$, $\mathrm{tr}_{Jx}\alpha = 0$.

By direct calculation, we have

$$\mathrm{tr}_J\alpha_0 = m_0 + n_0 + l_0 + 3f_0 = 2^{m-2k-1}. \tag{3}$$

$$\mathrm{tr}_t\alpha_0 = m_0 + n_0\omega + l_0\omega^2 = \frac{2^{(a_1+d_1)-(a_0+d_0)}(1+\omega)^{(b_0+b_1)-(c_0+c_1)}}{2}. \tag{4}$$

$$\mathrm{tr}_{t^2}\alpha_0 = m_0 + n_0\omega^2 + l_0\omega = \frac{2^{(a_1+d_1)-(a_0+d_0)}(1+\omega^2)^{(b_0+b_1)-(c_0+c_1)}}{2}. \tag{5}$$

$$\mathrm{tr}_x\alpha_0 = m_0 + n_0 + l_0 - f_0 = 0. \tag{6}$$

Here we use $0 = \mathrm{tr}_{Jx}\alpha = \mathrm{tr}_x(2\cdot\alpha_0) = 2\cdot\mathrm{tr}_x\alpha_0$ and so on.

From (3) and (6) we get $f_0 = 2^{m-2k-3}$. So we have the following main result.

THEOREM 1. *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. Let $k = -\sigma(X)/16$ and $m = b_2^+(X)$. If X admits a spin alternating group A_4 action, then $2k+3 \leq m$ if $b_2^+(X) - b_2^+(X/<(ab)(cd)>) \neq 0$.*

As an element of $R(A_4)$, we know that $\mathrm{Ind}_{A_4}D = \overline{\mathrm{Ind}_{A_4}D}$, so from $\mathrm{Ind}_{A_4}D = a_0 + b_0\xi + c_0\xi^2 + d_0\eta$ we have $b_0 = c_0$. Similarly since $H^+(X,C) = \overline{H^+(X,C)}$, so from $H^+(X,C) = a_1 + b_1\xi + c_1\xi^2 + d_1\eta$ we have $b_1 = c_1$.

Since a_0 and d_0 are even numbers, we could denote $a_0 = 2a_0'$ and $d_0 = 2d_0'$, then we have

$$
\begin{pmatrix} m_0 \\ n_0 \\ l_0 \end{pmatrix} = \frac{2^{(a_1+d_1)-2(a_0'+d_0')-1}}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} 2^{2((b_1+d_1)-(b_0+2d_0'))-2} \\ 1 \\ 1 \end{pmatrix}. \tag{7}
$$

From above we get

$$m_0 = \frac{2^{(a_1+d_1)-2(a_0'+d_0')-1}}{3}(2^{2((b_1+d_1)-(b_0+2d_0'))-2} + 2),$$

and

$$n_0 = l_0 = \frac{2^{(a_1+d_1)-2(a_0'+d_0')-1}}{3}(2^{2((b_1+d_1)-(b_0+2d_0'))-2} - 1).$$

PROPOSITION 1. *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. If X admits a spin alternating group A_4 action, then*

$$\dim\left((\mathrm{Ind}_{A_4}D)^{<(abc)>}\right) + 1 \leq b_2^+(X/<(abc)>)$$

if $b_2^+(X) - b_2^+(X/<(ab)(cd)>) \neq 0$.

Proof. Since $l_0, n_0 \in Z$, so we have

$$\frac{2^{(a_1+d_1)-2(a_0'+d_0')-1}}{3}(2^{2((b_1+d_1)-(b_0+2d_0'))-2} - 1) \in Z,$$

from this we have $2^{(a_1+d_1)-2(a_0'+d_0')-1}(2^{2((b_1+d_1)-(b_0+2d_0'))-2} - 1) \in 3Z \subset Z$. But

$$((a_1+d_1)-2(a_0'+d_0')-1)+(2((b_1+d_1)-(b_0+2d_0'))-2) = m-2k-3 \geq 0,$$

that is,

$$2^{((a_1+d_1)-2(a_0'+d_0')-1)+2((b_1+d_1)-(b_0+2d_0'))-2}\in Z.$$

From this we have $2^{(a_1+d_1)-2(a_0'+d_0')-1}\in Z$, i.e., $2(a_0'+d_0')+1\leq a_1+d_1$, that is, $\dim((\mathrm{Ind}_{A_4}D)^{<(abc)>})+1\leq b_2^+(X/<(abc)>)$ and this completes the proof of Proposition 1. □

Next we assume that $b_2^+(X/A_4)>0$ and $b_2^+(X)=b_2^+(X/\langle(ab)(cd)\rangle)$, that is $a_1>0$ and $d_1=0$.

From tom Dieck formula, we have

$$\mathrm{tr}_J\alpha=2^{m-2k}.$$

We also have

$$\mathrm{tr}_J\alpha=2(m_0+n_0+l_0+3f_0),$$

$$\mathrm{tr}_{Jt}\alpha=2^{-a_0-d_0+a_1}(1+\omega^2)^{-b_0-d_0+c_1}(1+\omega)^{-c_0-d_0+b_1},$$

and

$$\mathrm{tr}_{Jt}\alpha=2(m_0+n_0\omega+l_0\omega^2),\quad \mathrm{tr}_{Jt}\alpha=\overline{\mathrm{tr}_{Jt^2}\alpha}.$$

Now we look at $\mathrm{tr}_{Jx}\alpha$:

x acts on h as $\begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix}$, so $\dim(Vh)_{Jx}-\dim(W\tilde{1})_{Jx}=-2d_1=0$, and hence $d(f^{Jx})=1$. From tom Dieck formula, we have

$$\begin{aligned}\mathrm{tr}_{Jx}\alpha &= \mathrm{tr}_{Jx}[\lambda_1(a_1+b_1\xi+c_1\xi^2)\tilde{1}-\lambda_1(a_0+b_0\xi+c_0\xi^2+d_0\eta)h]\\ &= \mathrm{tr}_{Jx}[(1-\tilde{1})^{a_1}(1-\xi\tilde{1})^{b_1}(1-\xi^2\tilde{1})^{c_1}(1-h)^{-a_0}\cdot\\ &\quad (1-\xi h)^{-b_0}(1-\xi^2h)^{-c_0}(1-\eta h)^{-d_0}]\\ &= 2^{-a_0-b_0-c_0-3d_0+a_1+b_1+c_1}=2^{m-2k}.\end{aligned}$$

On the other hand, we have $\mathrm{tr}_{Jx}\alpha=2(m_0+n_0+l_0-f_0)$, so $m_0+n_0+l_0-f_0=2^{m-2k-1}$, but $m_0+n_0+l_0+3f_0=2^{m-2k-1}$. So we have the following result

PROPOSITION 2. *Let X be a smooth spin 4-manifold with $b_1(X)=0$ and non-positive signature. If X admits a spin alternating group A_4 action, then the K-theory degree $\alpha=\alpha_0(1-\tilde{1})$ for some $\alpha_0=m_0+n_0\xi+l_0\xi^2$ if $b_2^+(X/A_4)>0$ and $b_2^+(X)=b_2^+(X/<(ab)(cd)>)$.*

Finally we assume that $b_2^+(X/A_4)=0$ and $b_2^+(X)=b_2^+(X/\langle(ab)(cd)\rangle)$, that is, $a_1=d_1=0$.

Case 1. $b_2^+(X/A_4)=0$ and $b_2^+(X)=b_2^+(X/\langle(ab)(cd)\rangle)$, but $b_2^+(X)\neq 0$.

We know that ϕ acts on h as $\begin{pmatrix} \phi & \\ & \phi^{-1} \end{pmatrix}$. Now we consider the action of ϕt.

The action of ϕt on h, ξh, $\xi^2 h$ and ηh has no invariant space, and the action of ϕt on $\xi\tilde{1}$ and $\xi^2\tilde{1}$ has no invariant space either, but ϕt acts on $\tilde{1}$ trivially, and the action of ϕt on $\eta\tilde{1}$ has 1-dimensional invariant space. So $\dim(Vh)_{\phi t} - \dim(W\tilde{1})_{\phi t} = -(a_1 + d_1) = 0$ and $d(f^{t\phi}) = 1$.

The eigenvalues of the action t on η are 1, ω and ω^2.

$$\begin{aligned} \mathrm{tr}_{\phi t}\alpha &= d(f^{\phi t})[(1-\phi)(1-\phi^{-1})]^{-a_0-d_0}[(1-\omega\phi)(1-\omega\phi^{-1})]^{-b_0-d_0}\cdot \\ &\quad [(1-\omega^{\phi})(1-\omega^2\phi^{-1})]^{-c_0-d_0}(1+\omega)^{b1}(1+\omega^2)^{c_1}. \end{aligned}$$

Since $\mathrm{tr}_{t\bullet}\alpha :\ U(1) \to C$ is a C^0-function, ϕ is a generic element, so $-a_0 - d_0 \geq 0$, $-b_0 - d_0 \geq 0$, and $-c_0 - d_0 \geq 0$.

On the other hand, $\mathrm{Ind}D = -\frac{\sigma}{8} \in Z$, but we have $\mathrm{Ind}D = a_0 + b_0 + c_0 + 3d_0 \leq 0$, so $a_0 + d_0 = b_0 + d_0 = c_0 + d_0 = 0$, and X is homotopic to $\sharp_n S^2 \times S^2$ for some even integer n.

Case 2. $H_2^+(X) = 0$, that is, $H^*(X) \cong H^*(S^4)$.

x acts on 1, ξ and ξ^2 trivially, and x acts on η as $\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$, the actions of ϕx on $\tilde{1}$, $\xi\tilde{1}$, $\xi^2\tilde{1}$ and $\eta\tilde{1}$ all have a 1-dimensional invariant space.

If $H_2^+(X) = 0$, that is, $a_1 = b_1 = c_1 = d_1 = 0$, we have $\dim(Vh)_{\phi x} - \dim(W\tilde{1})_{\phi x} = 0$, so $d(f^{\phi x}) = 1$. Then from tom Dieck formula, we have

$$\mathrm{tr}_{\phi x}\alpha = [(1-\phi)(1-\phi^{-1})]^{-a_0-b_0-c_0-d_0}[(1+\phi)(1+\phi^{-1})]^{-2d_0}.$$

Since $\mathrm{tr}_{x\bullet}\alpha :\ U(1) \to C$ is a C^0-function, ϕ is a generic element, so $a_0 + b_0 + c_0 + d_0 \leq 0$ and $2d_0 \leq 0$.

On the other hand, $\mathrm{ind}D = -\frac{\sigma}{8} \in Z$, but $\mathrm{ind}D = a_0+b_0+c_0+3d_0 \leq 0$, so $\mathrm{ind}D = 0$. Moreover $a_0 + b_0 + c_0 + d_0 = 0$ and $2d_0 = 0$, that is, $a_0 = b_0 = c_0 = d_0 = 0$, so we have $\mathrm{Ind}_{A_4}D = 0 \in R(A_4)$.

In summary, we have the following result:

PROPOSITION 3. *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. If X admits a spin alternating group A_4 action, if $b_2^+(X/A_4) = 0$ and $b_2^+(X) = b_2^+(X/ < (ab)(cd) >)$, then as an element of $R(A_4)$, $\mathrm{Ind}_{A_4}D$ is a multiple of $1 + \xi + \xi^2 - \eta$, and X is homotopic to $\sharp_n S^2 \times S^2$ for some even integer n. Moreover if $b_2^+(X) = 0$, that is, $H^*(X) \cong H^*(S^4)$, then $\mathrm{Ind}_{A_4}D = 0 \in R(A_4)$.*

In fact we can apply the following equivalent version of Furuta's $\frac{10}{8}$-theorem to obtain the following result similar to Proposition 1.

PROPOSITION 4. (see also [6]). *Let X be a smooth closed spin G-manifold of dimension 4, where G is compact. Suppose that $b_1(X) = 0$ and $\sigma(X) \leq 0$. If the G-action is of even type so that $\operatorname{ind}^G(D) \neq 0$, then*

$$b_2^+(X/G) \geq \operatorname{ind}^G(D) + 1,$$

where $\operatorname{ind}^G(D) = \dim(\ker D)^G - \dim(\operatorname{coker} D)^G$.

For the subgroup $< (abc) >$ of A_4, applying above proposition we can get

PROPOSITION 5. *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. If X admits a spin alternating group A_4 action, then*

$$\dim\left((\operatorname{Ind}_{A_4} D)^{<(abc)>}\right) + 1 \leq b_2^+(X/ < (abc) >)$$

if $\dim\left((\operatorname{Ind}_{A_4} D)^{<(abc)>}\right) \neq 0$.

REMARK. If $\dim\left((\operatorname{Ind}_{A_4} D)^{<(abc)>}\right) \neq 0$, that is, $a_0 + d_0 \neq 0$, then from the discussion of Case 1, we know $a_1 \neq 0$ or $d_1 \neq 0$. So the condition in Proposition 1 is still a little different from the condition in Proposition 5.

Now we look at a concrete example of A_4-action. Let X be the $K3$ surface of Fermat type, that is, $X = \{[z_0, z_1, z_2, z_3] \in CP^3 | \sum_{i=0}^3 z_i^4 = 0\}$, the smooth "standard action" of A_4 on X is given as permutations of variables. For this action it is easy to get that $\operatorname{Ind}_{A_4} D = 2 \in R(A_4)$ and $H_2(X, Z) = 6 + 2(\xi + \xi^2) + 4\eta \in R(A_4)$, so $b_2^+(X/A_4) = 3 > 0$, then by Theorem 1, we must have $b_2^+(X) = b_2^+(X/ < (ab)(cd) >)$, that is, $d_1 = 0$.

Applying Proposition 4 to the "standard action" of A_4 on the above $K3$ surface of Fermat type, we have $\dim(H_2^+(X)^{A_4}) \geq \dim((\operatorname{Ind}_{A_4} D)^{A_4}) + 1 = 2 + 1 = 3$, but $\dim(H_2^+(X)) = 3$, so $(H_2^+(X))^{A_4} = H_2^+(X)$.

For homotopy $K3$ surface we have

PROPOSITION 6. *Let X be a homotopy $K3$ surface. If X admits a spin alternating group A_4 action, then as an element of $R(A_4)$, $H_2^+(X, C) = 3 \cdot 1$ or $H_2^+(X, C) = 1 + \xi + \xi^2$.*

Proof. If $H_2^+(X, C)$ contains η term, that is, $d_1 > 0$, by Theorem 1, it is impossible, so $H_2^+(X, C) = a + b\xi + c\xi^2$, but since $\dim H_2^+(X) = 3$, so the proposition follows. $\square$

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