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NONDEGENERATE AFFINE HOMOGENEOUS DOMAIN OVER A GRAPH

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ABSTRACT. The affine homogeneous hypersurface in $\mathbb{R}^{n+1}$, which is a graph of a function $F : \mathbb{R}^n \to \mathbb{R}$ with $|\det DdF| = 1$, corresponds to a complete unimodular left symmetric algebra with a nondegenerate Hessian type inner product. We will investigate the condition for the domain over the homogeneous hypersurface to be homogeneous through an extension of the complete unimodular left symmetric algebra, which is called the graph extension.

1. Introduction

An affine homogeneous domain Ω in $\mathbb{R}^{n+1}$ is an open subset on which a Lie subgroup of affine transformation group $\mathrm{Aff}(n+1)$ acts transitively.

J. L. Koszul and E. Vinberg studied the homogeneous domain Ω which is convex and proper, that is, Ω does not contain any full line [12, 19]. Vinberg showed that there is a one-to-one correspondence between the set of homogeneous proper convex domains and the set of clans, the left symmetric algebras (*abbreviated to* LSA's) whose left multiplication operator has only real eigenvalues and which admit a Lie algebra homomorphism s into $\mathbb{R}$ such that the induced bilinear form $\langle x, y\rangle := s(x \cdot y)$ is positive definite. In a clan, there exists a principal idempotent, from which he obtained the principal decomposition of the clan. And this eventually leads to the structure theory of the clan.

The homogeneous Hessian domain, which admits a Riemannian Hessian metric, was studied by Shima [18]. In fact, the homogeneous Hessian domain turns out to be the convex domain which may contain a full line, that is, need not be proper. For the non-convex homogeneous

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domain, H. Kim studied through the LSA which is obtained from the simply transitive Lie group action [10]. The LSA structure represents a left invariant, torsion free, flat affine structure on the Lie group.

In [13], A. Mizuhara studied the left symmetric algebra with a principal idempotent. He shows that the LSA with a principal idempotent gives a domain over the graph of a polynomial on $\mathbb{R}^n$. Here the notion of principal idempotent is generalized from Vinberg's, so the left multiplication operator of the principal idempotent can have another eigen values besides $\frac{1}{2}$ and 1. In this context, he called the domain over a graph associated to the LSA of this type as the *interior of the generalized paraboloid.* He also showed that certain interiors of the generalized paraboloids have the constant sectional curvature $-\frac{1}{4}$ with respect to the nondegenerate metric induced from a Hessian type inner product on the LSA.

In this paper, we will consider the *graph domain* Ω over the nondegenerate homogeneous hypersurface $\Sigma = \partial\Omega$, where Σ is the graph of a function on $\mathbb{R}^n$, which admits a simply transitive action of a unimodular subgroup $\mathbb{A} \subset \mathrm{Aff}(n+1)$. We will study the Hessian structure of the graph domain $\Omega \in \mathbb{R}^{n+1}$ and the automorphism group $\mathrm{Aut}(\Omega) \subset \mathrm{Aff}(n+1)$. Then we will show that if a Lie subgroup $\mathbb{G} \subset \mathrm{Aut}(\Omega)$ acts simply transitively on Ω, the associated LSA on the Lie algebra of $\mathbb{G}$ has an idempotent and it induces the decomposition of the LSA as in the case of the clan. Conversely, for a complete unimodular LSA with a nondegenerate Hessian type inner product, we will introduce the *graph extension* of the LSA with a compatible derivation. Then we will show that the graph extension gives the graph domain over the hypersurface which is determined by the complete unimodular LSA and the compatible derivation. The following is the one of our main results:

THEOREM 4.11. *Let Σ be a graph of a function $F : \mathbb{R}^n \to \mathbb{R}$ with $|\det DdF| = 1$, and let Ω be the domain over the graph Σ. Suppose a Lie subgroup $\mathbb{G} \subset \mathrm{Aut}(\Omega)_0$ acts simply transitively on Ω as affine transformations and it contains an unimodular equiaffine subgroup $\mathbb{A}$ which acts on Σ simply transitively. Then the set of homogeneous domain $(\mathbb{G}, \Omega)$ over Σ is in one to one correspondence with the set of graph extensions $\mathcal{G}$ of $\mathcal{A}$, the complete unimodular LSA corresponding to $(\mathbb{A}, \Sigma)$. In this case, there is a nondegenerate Hessian metric on Ω, or equivalently, there is a nondegenerate Hessian type inner product on $\mathcal{G}$ which is defined by the trace form of the right multiplication operator.*

Note that the interiors of generalized paraboloids in [13] are all the homogeneous graph domain, or equivalently, the LSA with principal idempotent in [13] are all the graph extension of a complete unimodular LSA with a Hessian type inner product. But the converse is not yet clear to author. The deciding factor of converse argument could be the eigen values of compatible derivation.

As an example, the graph domain over the paraboloid in $\mathbb{R}^{n+1}$ is isometric to the hyperbolic space H^{n+1} of sectional curvature $-\frac{1}{4}$, and it is affine homogeneous. Moreover, from [13], many other graph domains also have the constant negative sectional curvature with respect to the semi-Riemannian Hessian metric and they are homogeneous. So it would be interesting to ask which graph domains are homogeneous, and to study the curvature property and the completeness of the graph domain.

The improper affine hypersphere was studied by many authors. With some conditions, the hypersphere becomes homogeneous carrying simple transitive action of a unimodular Lie group. So the structure of these hypersphere could be re-treated with the algebraic tools which are introduced in our paper. Indeed, on the Cayley hypersurface the abelian Lie group acts simply transitively, and in [3], the authors gave an affirmative answer to the complete case of the Eastwood and Ezhov conjecture about the Cayley hypersurface using the structure of the complete abelian LSA. For another example, the hypersurface with parallel difference tensor ∇K such that $K^{n-1} \neq 0$ which is given in [4] admits a simply transitive action of a nilpotent unimodular Lie group, more precisely, it is the graph of a polynomial associated to a non-abelian filiform LSA.

Let us sketch the contents of the paper: In Section 2, we will introduce some results about the homogeneous hypersurface from [3]. In Section 3, we will study the Hessian structure and the automorphism group of graph domain. In Section 4, we will study the associated LSA structure of a Lie group which acts simply transitively on the graph domain and the algebraic condition for the graph domain to be homogeneous. In Section 5, we will give a classification of graph extension of low dimensional complete unimodular left symmetric algebras determining the corresponding affine homogeneous graph domains in these dimensions.

2. Preliminary

Let Σ be a graph of a function $F : \mathbb{R}^n \to \mathbb{R}$ with $|\det DdF| = 1$, and let Ω be the domain over Σ. In this case, the affine normals of Σ must be equal to $\xi = (0, \ldots, 0, 1)$. By the affine transformation in

$\mathbb{R}^n$ and by the parallel translation with respect to the last direction in $\mathbb{R}^{n+1}$, we may assume that $F(0) = 0$ and $dF_0 = 0$. Denote $\mathrm{Aut}(\Omega)$ (resp. $\mathrm{Aut}(\Sigma)$) as the subgroup of all the affine transformation of $\mathbb{R}^{n+1}$ preserving Ω(resp. Σ). $\mathrm{Aut}(\Omega)_0$ (resp. $\mathrm{Aut}(\Sigma)_0$) means the identity component of $\mathrm{Aut}(\Omega)$ (resp. $\mathrm{Aut}(\Sigma)$). Then we have the following results from [3]:

PROPOSITION 2.1. ([3]) $\mathrm{Aut}(\Omega) \subset \mathrm{Aut}(\Sigma)$ *and* $\mathrm{Aut}(\Sigma)_0 = \mathrm{Aut}(\Omega)_0$.

LEMMA 2.2. ([3]) *The affine automorphism* $g \in \mathrm{Aut}(\Omega)_0$ *is represented as the following:*

$$g = \left(\begin{pmatrix} A & 0 \\ c' & s \end{pmatrix}, \begin{pmatrix} a \\ F(a) \end{pmatrix} \right) \in \mathrm{Aff}(n+1), \tag{2.1}$$

where $A \in \mathrm{GL}(n, \mathbb{R})$, $a, c \in \mathbb{R}^n$, $s = (\det A)^{\frac{2}{n}} \in \mathbb{R}_+$ *and* c' *denotes the transpose of* c *as a column vector.*

Let's denote $q : \Sigma \to \mathbb{R}^n$, $\begin{pmatrix} a \\ F(a) \end{pmatrix} \mapsto a$ as a projection and by abusing the notation, let's denote $q : \mathrm{Aut}(\Sigma) \to \mathrm{Aff}(n)$ given by

$$q\left(\begin{pmatrix} A & 0 \\ c' & s \end{pmatrix}, \begin{pmatrix} a \\ F(a) \end{pmatrix} \right) = (A, a)$$

as a group homomorphism. In the following, we will suppose that $\mathrm{Aut}(\Omega)$ contains an unimodular subgroup $\mathbb{A}$ which acts on Σ simply transitively. The image $\bar{\mathbb{A}} = q(\mathbb{A})$ of the subgroup $\mathbb{A}$ by q is an n dimensional subgroup of $\mathrm{Aff}(n)$ and acts on $\mathbb{R}^n$ simply transitively. The induced left invariant flat affine connection on $\bar{\mathbb{A}}$ will be denoted by ∇. Then it defines a complete LSA structure on the unimodular Lie algebra $\bar{\mathfrak{a}} = \mathrm{Lie}\,\bar{\mathbb{A}}$ whose product is given by $a \cdot b = \nabla_a b$ for $a, b \in \bar{\mathfrak{a}}$. In fact, since ∇ is flat and torsion free, we have

$$\nabla_a \nabla_b c - \nabla_b \nabla_a c - \nabla_{[a,b]} c = a(bc) - b(ac) - [a,b]c = 0, \tag{2.2}$$

$$\nabla_a b - \nabla_b a - [a,b] = ab - ba - [a,b] = 0, \tag{2.3}$$

and we obtain the left symmetry of the associator $(a,b,c) = (ab)c - a(bc)$. We will denote the LSA as $\mathcal{A} = (\bar{\mathfrak{a}}, \cdot)$ and the left multiplication by $a \in \mathcal{A}$ as λ_a so that $\lambda_a(b) = ab$. By identifying $\mathcal{A}$ with $\mathbb{R}^n$, λ induces a representation of $\bar{\mathfrak{a}}$ into $\mathrm{aff}(n) = gl(n) + \mathbb{R}^n$ which maps a to (λ_a, a). Exponentiating this representation gives us the representation of $\bar{\mathbb{A}}$ into $\mathrm{Aff}(n) = \mathrm{GL}(n) \ltimes \mathbb{R}^n$ so that the group

$$\bar{\mathbb{A}} = \{(\exp \lambda_a,\ e^a - 1) \mid a \in \bar{\mathfrak{a}}\} \subset \mathrm{Aff}(n),$$

where $e^a - 1 = a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \cdots$ and $a^k = (\lambda_a)^{k-1}a$, $(k = 1, 2, \ldots)$, acts on $\mathbb{R}^n$ simply transitively(see [10] for details). Generally, the developing image of an LSA $\mathcal{A}$ at $x_0 \in \mathbb{R}^n$ means that the orbit $\bar{\mathbb{A}} \cdot x_0$ of the corresponding Lie group $\bar{\mathbb{A}}$. Note that, since the developing image of $\mathcal{A}$ is the whole space $\mathbb{R}^n$, the induced LSA $\mathcal{A}$ must be complete. The following theorem about the equivalent conditions of the completeness is well known:

THEOREM 2.3. ([7, 16]) *Let $\mathcal{A}$ be an LSA. Then the following statements are equivalent:*

(a) *$\mathcal{A}$ is complete.*
(b) *$\operatorname{tr} \rho_a = 0$ for all $a \in \mathcal{A}$, where ρ_a is the right multiplication of a.*
(c) *$\det(I + \rho_a) = 1$ for all $a \in \mathcal{A}$.*

Note that in our case, since the associated Lie algebra $\bar{\mathfrak{a}}$ is unimodular,

$$\operatorname{tr} \lambda_a = \operatorname{tr} \rho_a = 0, \qquad \text{for all } a \in \mathcal{A}.$$

PROPOSITION 2.4. ([3]) *Let $\mathbb{A}$ be an unimodular subgroup of $\mathrm{Aut}(\Omega)_0$ which acts on Σ simply transitively. For any element $g \in \mathbb{A}$, there exists a unique $a \in \mathbb{R}^n$ such that $g = g_a$, where g_a is represented by*

$$g_a = \left(\begin{pmatrix} M_a & 0 \\ dF_a M_a & 1 \end{pmatrix}, \begin{pmatrix} a \\ F(a) \end{pmatrix} \right),$$

where $M_a \in \mathrm{GL}(n)$ is given by the equations: for $b \in \mathcal{A}$

$$M_a = \exp \lambda_b \ \text{ and } \ a = e^b - 1 \in \mathbb{R}^n.$$

Moreover (M_a, a) act on $\mathbb{R}^n$ as isometries with respect to the Hessian metric DdF.

The Hessian DdF of the function F defines a left invariant metric on $\bar{\mathbb{A}}$, and hence it induces an inner product $H = DdF_0$ on the Lie algebra $\bar{\mathfrak{a}}$. We will frequently identify $\bar{\mathbb{A}}(\mathcal{A}$, resp.) and $\mathbb{R}^n(T_0\mathbb{R}^n = \mathbb{R}^n$, resp.) via the evaluation map at 0(its differential, resp.) in the following so that the left invariant vector fields on $\bar{\mathbb{A}}$ becomes a vector field on $\mathbb{R}^n$.

PROPOSITION 2.5. ([3]) *The induced inner product H on $\mathcal{A}$ is of Hessian type, that is, H satisfies*

$$H(a, bc) - H(ab, c) = H(b, ac) - H(ba, c), \tag{2.4}$$

for all $a, b, c \in \mathcal{A}$.

Ultimately, the homogeneous affine hypersurface, which is a graph of a function $F : \mathbb{R}^n \to \mathbb{R}$ with $|\det DdF| = 1$ and whose automorphism group contains an unimodular simply transitive subgroup, gives a complete LSA with the Hessian type inner product.

Conversely, let $\mathcal{A}$ be a complete unimodular LSA with a Hessian type inner product H with $|\det H| = 1$. (Abusing the notation, we denote by H in this paper both the inner product and its associated symmetric matrix with respect to the standard basis on $\mathbb{R}^n$.) Let $\bar{\mathfrak{a}}$ be the associated Lie algebra of $\mathcal{A}$. Define a map $\psi : \bar{\mathfrak{a}} \to \mathrm{aff}(n+1)$ by $\psi(a) = \left(\begin{pmatrix} \lambda_a & 0 \\ a'H & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} \right)$, where a' is the transpose of $a \in \bar{\mathfrak{a}}$. Then ψ is the Lie algebra homomorphism(see [3]). Since $\mathfrak{a} = \psi(\bar{\mathfrak{a}})$ is a Lie subalgebra of $\mathrm{aff}(n+1)$, we have a corresponding Lie subgroup $\mathbb{A} = \exp \mathfrak{a} \subset \mathrm{Aff}(n+1)$ whose elements are given by

$$\left(\begin{pmatrix} e^{\lambda_a} & 0 \\ a'H(I + \frac{\lambda_a}{2!} + \frac{(\lambda_a)^2}{3!} + \cdots) & 1 \end{pmatrix} \begin{pmatrix} e^a - 1 \\ a'H(\frac{a}{2!} + \frac{a^2}{3!} + \cdots) \end{pmatrix} \right), \tag{2.5}$$

for $a \in \bar{\mathfrak{a}}$. Note that the orbit space of the Lie group $\bar{\mathbb{A}} = \{(e^{\lambda_a}, e^a - 1) \in \mathrm{Aff}(n) \mid a \in \mathcal{A}\}$ at the origin is the whole space $\mathbb{R}^n$ because $\mathcal{A}$ is complete. Put $x = e^a - 1 \in \mathbb{R}^n$ and let's define a function $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(x) = F(e^a - 1) = a'H(\frac{a}{2!} + \frac{a^2}{3!} + \cdots). \tag{2.6}$$

Then the homogeneous hypersurface $\Sigma = \{(x, F(x)) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n\}$ contains the origin of $\mathbb{R}^{n+1}$ since $F(0) = F(e^0 - 1) = 0$ and $\mathbb{A}$ acts simply transitively on Σ. Differentiating (2.6)(see [3]), we have

$$\begin{aligned} DdF_0 &= H, \\ DdF_x(e^{\lambda_a}, e^{\lambda_a}) &= DdF(e^a - 1)(e^{\lambda_a}, e^{\lambda_a}) \\ &= DdF(0)(\cdot,\cdot) = DdF_0(\cdot,\cdot). \end{aligned}$$

Since $\det e^{\lambda_a} = 1$, we have $\det DdF_x = \det DdF_0$ and hence $|\det DdF_x| = |\det H| = 1$ for all $x \in \mathbb{R}^n$. Therefore the affine normals of the hypersurface Σ are equal to $\xi = (0, \ldots, 0, 1)$, so they are parallel. Now we can summarize as follows.

THEOREM 2.6. ([3]) *There is a one-to-one correspondence between the set of the graph Σ of $F : \mathbb{R}^n \to \mathbb{R}$ with $|\det DdF| = 1$ on which an unimodular Lie subgroup $\mathbb{A}$ of affine transformations acts simply* *transitively and the set of the complete unimodular LSA $\mathcal{A}$ with a non-degenerate Hessian type inner product H, $|\det H| = 1$ with respect to the standard basis of $\mathbb{R}^n = \mathcal{A}$.*

REMARK 2.7. On a graph of $F : \mathbb{R}^n \to \mathbb{R}$ with $|\det DdF| = 1$, non-isomorphic unimodular Lie subgroups of affine transformations could act simply transitively. In this case, the induced complete unimodular LSA's are not necessarily isomorphic. In section 5, we will give two unimodular Lie subgroups acting on a parabola in $\mathbb{R}^4$ simply transitively, one is abelian and the other is non-abelian.

DEFINITION 2.8. Let $\mathfrak{L}$ be an n-dimensional LSA with a Hessian type inner product $\langle , \rangle$.

(a) An automorphism A of $\mathfrak{L}$ is called *isometry* if it satisfies

$$\langle Ax, Ay\rangle = \langle x, y\rangle \quad \text{for all } x, y \in \mathfrak{L}.$$

The set of all isometries will be denoted by $\mathrm{IAut}(\mathfrak{L})$.

(b) A derivation B of $\mathfrak{L}$ is called *infinitesimal similarity* if it satisfies

$$\langle Bx, y\rangle + \langle x, By\rangle = \frac{2}{n}\operatorname{tr} B\langle x, y\rangle \quad \text{for all } x, y \in \mathfrak{L}.$$

The set of all infinitesimal similarities will be denoted by $\mathrm{sDer}(\mathfrak{L})$.

(c) An infinitesimal similarity B of $\mathfrak{L}$ is called *compatible* if $\operatorname{tr} B = \frac{n}{2}$, that is, it satisfies the following,

$$\langle Bx, y\rangle + \langle x, By\rangle = \langle x, y\rangle \quad \text{for all } x, y \in \mathfrak{L}.$$

The set of all compatible derivations will be denoted by $\mathrm{cDer}(\mathfrak{L})$.

3. Domain over the graph

Let Σ be a graph of a function $F : \mathbb{R}^n \to \mathbb{R}$ with $|\det DdF| = 1$, $F(0) = 0$ and $dF_0 = 0$, that is, Σ is the nondegenerate hypersurface such that the affine normals are $\xi = (0, \ldots, 0, 1)$.

With the given function $F : \mathbb{R}^n \to \mathbb{R}$, satisfying the above assumptions, we can define a function $p(a, s)$ from $\mathbb{R}^{n+1}$ into $\mathbb{R}$ by

$$p(a, s) = s - F(a),$$

and let $\Omega = \{(a, s) \in \mathbb{R}^{n+1} | \, p(a, s) > 0\}$. Let's denote by Σ_t the t-level surface $p^{-1}(t)$ for all $t \in \mathbb{R}_+$. Since $p(a, s) > 0$ for $(a, s) \in \Omega$, we can define a function $\phi(a, s)$ on Ω (*cf.* [8]) by

$$\phi(a, s) = -\ln p(a, s) \tag{3.1}$$

Differentiating (3.1), we have

$$d\phi = -\frac{1}{p}(-dF, 1) = \frac{1}{p}(dF, -1)$$

$$Dd\phi = \frac{1}{p^2}\begin{pmatrix} pDdF + (dF)'(dF) & -(dF)' \\ -dF & 1 \end{pmatrix},$$

where $(dF)'$ is the transpose of dF as a row vector. Then we see that $\left|\det Dd\phi\right| = \frac{1}{p^{n+2}}\left|\det DdF\right| = \frac{1}{p^{n+2}} > 0$ on Ω. This says:

PROPOSITION 3.1. ([8]) *The Hessian $Dd\phi$ of ϕ is a nondegenerate metric on Ω.*

Consider a canonical vector field E on Ω given by

$$Dd\phi(E, X) = d\phi(X) \tag{3.2}$$

for all $X \in \mathfrak{X}(\Omega)$, the space of the vector fields on Ω. The vector field E is the dual of $d\phi$ with respect to the nondegenerate metric $Dd\phi$ on Ω. Using the matrix form of the inverse of $Dd\phi$, which is given by,

$$(Dd\phi)^{-1} = p\begin{pmatrix} (DdF)^{-1} & (DdF)^{-1}(dF)' \\ (dF)(DdF)^{-1} & p + (dF)(DdF)^{-1}(dF)' \end{pmatrix},$$

we can obtain a explicit form of the canonical vector field E on Ω:

$$E' = (d\phi)(Dd\phi)^{-1} = (0, \ldots, 0, -p), \text{ i.e., } E = -p\frac{\partial}{\partial x^{n+1}}, \tag{3.3}$$

where E' is the transpose of the vector field E. From this, we have the following properties of the vector field E:

PROPOSITION 3.2. *Let E be the canonical vector field on Ω. Then,*
(a) *$d\phi(E) = 1$.*
(b) *$D_X E = 0$ for all $X \in \mathfrak{X}(\Omega)$ such that $X|_{\Sigma_t} \in \mathfrak{X}(\Sigma_t)$.*
(c) *$D_E E = -E$.*

Proof. (a) Using the matrix forms of $d\phi$ and E, we have

$$d\phi(E) = \frac{1}{p}(dF, -1) \cdot (0, \ldots, 0, -p)' = 1.$$

(b) From (3.3), E is parallel on the level surface Σ_t of p. So $D_X E = 0$ for all $X \in \mathfrak{X}(\Sigma_t)$.

(c) From (3.3), we have

$$D_E E = D_{-p\frac{\partial}{\partial x^{n+1}}}(-p\frac{\partial}{\partial x^{n+1}}) = p(\frac{\partial p}{\partial x^{n+1}})\frac{\partial}{\partial x^{n+1}} = p\frac{\partial}{\partial x^{n+1}} = -E.$$

□

Let $\tau : \mathrm{Aut}(\Omega)_0 \to \mathbb{R}_+$ be the group homomorphism defined by the following: With the matrix form (2.1) of $g \in \mathrm{Aut}(\Omega)_0$,

$$\tau : g = \left(\begin{pmatrix} A & 0 \\ c' & s \end{pmatrix}, \begin{pmatrix} a \\ F(a) \end{pmatrix} \right) \mapsto s.$$

PROPOSITION 3.3. *For any $x \in \Omega$ and $g \in \mathrm{Aut}(\Omega)_0$, we have*:

(a) $p(g \cdot x) = \tau(g)p(x)$.

(b) $d\phi_{g\cdot x} \cdot g_* = d\phi_x$.

(c) $Dd\phi_{g\cdot x}(g_*\cdot,\ g_*\cdot) = Dd\phi_x(\cdot,\ \cdot)$.

Proof. (a) With the matrix form of g and $x = \begin{pmatrix} b \\ t + F(b) \end{pmatrix} \in \Omega$, where $p(x) = t \in \mathbb{R}_+$ and $\begin{pmatrix} b \\ F(b) \end{pmatrix} \in \Sigma$, we have

$$g \cdot x = \begin{pmatrix} Ab + a \\ c'b + st + sF(b) + F(a) \end{pmatrix},$$

so $p(g \cdot x) = c'b + st + sF(b) + F(a) - F(Ab + a)$. Since $g \in \mathrm{Aut}(\Sigma)$ and hence $F(Ab + a) = c'b + sF(b) + F(a)$, $p(g \cdot x) = st$. Because $\tau(g) = s$ and $p(x) = t$, we conclude that $p(gx) = \tau(g)p(x)$.

(b) and (c) follow from (a) by differentiating the following function with respect to x, $\phi(g \cdot x) = -\ln p(g \cdot x) = -\ln \tau(g)p(x) = -\ln \tau(g) + \phi(x)$. □

Above Proposition 3.3 says that $d\phi$ and $Dd\phi$ are $\mathrm{Aut}(\Omega)_0$-invariant.

COROLLARY 3.4. *The canonical vector field E is $\mathrm{Aut}(\Omega)_0$-invariant.*

Proof. For any $g \in \mathrm{Aut}(\Omega)_0$ and $X \in \mathfrak{X}(\Omega)$,

$$\begin{aligned} Dd\phi_{g\cdot x}(g_*E, g_*X) &= Dd\phi_x(E, X) = d\phi_x(X) = d\phi_{g\cdot x}(g_*X) \\ &= Dd\phi_{g\cdot x}(E, g_*X). \end{aligned}$$

Since $Dd\phi$ is nondegenerate, we have $g_*E = E$. □

Let $\tilde{\mathbb{A}} = \ker \tau$ be the normal subgroup of $\mathrm{Aut}(\Omega)_0$, so the elements of $\tilde{\mathbb{A}}$ are represented by

$$\left(\begin{pmatrix} A & 0 \\ c' & 1 \end{pmatrix}, \begin{pmatrix} a \\ F(a) \end{pmatrix} \right)$$

with $\det A = 1$. Then $\tilde{\mathbb{A}}$ leave invariant each level hypersurfaces Σ_t's for all $t \geq 0$. Furthermore, the elements of $\mathrm{Aut}(\Omega)_0$ preserving each level surface must be contained in $\tilde{\mathbb{A}}$. If $\mathrm{Aut}(\Omega)_0$ acts transitively on Ω, then τ must be an epimorphism and $\tilde{\mathbb{A}}$ acts on $\Sigma_t (t \geq 0)$ transitively.

Now, let $\mathbb{G}$ be a subgroup of $\mathrm{Aut}(\Omega)_0$ which acts on Ω simply transitively. Then $\mathbb{A} = \tilde{\mathbb{A}} \cap \mathbb{G}$ is a normal subgroup of $\mathbb{G}$ and it acts simply transitively on Σ_t for all $t \geq 0$. Note that $\mathbb{A}$ is an unimodular subgroup because $\mathbb{A}$ is a subgroup of $\tilde{\mathbb{A}}$, that is, $\det A = 1$ for all $A \in \mathbb{A}$. Moreover, $q(\mathbb{A})$ acts on $\mathbb{R}^n$ simply transitively. Since $\mathbb{G}$ leave Σ invariant, $\mathbb{G}$ is decomposed by $\mathbb{G} = \mathbb{A} \rtimes \mathbb{J}$, where $\mathbb{J}$ is the isotropy subgroup at $(0,0)$. In this case, $\tau|_{\mathbb{J}}$ must be an isomorphism onto $\mathbb{R}_+$ because $\dim \mathbb{J} = 1$. In fact, any element $j \in \mathbb{J}$ is represented as the following:

$$j = \begin{pmatrix} A & 0 \\ 0 & s \end{pmatrix} \in \mathrm{GL}(n+1, \mathbb{R}),$$

where $A \in \mathrm{GL}(n, \mathbb{R})$ and $s \in \mathbb{R}_+$, since $j \cdot (0,0) = (0,0)$ and $j^* dF_0 = 0$.

The following theorem is the summary of this section.

THEOREM 3.5. *For a nondegenerate hypersurface Σ given by a function, $F : \mathbb{R}^n \to \mathbb{R}$ with $|\det DdF| = 1$, $F(0) = 0$ and $dF_0 = 0$, let Ω be the domain over Σ. Suppose that $\mathrm{Aut}(\Omega)_0$ contains a subgroup $\mathbb{G}$ which acts simply transitively on Ω. Then we have the following:*

(a) *$Dd\phi$ is a nondegenerate metric on Ω, where $\phi(a, s) = -\ln(s - F(a))$ is a function from Ω to $\mathbb{R}$.*

(b) *There exist a canonical vector field $E \in \mathfrak{X}(\Omega)$ satisfying*

$$Dd\phi(E, X) = d\phi(X) \qquad \text{for all } X \in \mathfrak{X}(\Omega).$$

Moreover, $d\phi$, $Dd\phi$ and E are all $\mathrm{Aut}(\Omega)_0$-invariant.

(c) *There is a Lie group homomorphism $\tau : \mathbb{G} \to \mathbb{R}_+$ such that $\ker \tau$ is equal to the normal subgroup $\mathbb{A}$ which acts on Σ simply transitively. In this case, $\mathbb{A}$ is unimodular.*

(d) *There exists a 1-dimensional Lie subgroup $\mathbb{J}$, which is the isotropy subgroup at $(0,0)$, such that $\mathbb{G} = \mathbb{A} \rtimes \mathbb{J}$.*

4. LSA structure of the graph domain

Let's denote the evaluation map $\mathrm{ev} : \mathbb{G} \to \Omega$ with $\mathrm{ev}(g) = g \cdot e_{n+1}$, where $e_{n+1} = (0, \ldots, 0, 1) \in \Omega \subset \mathbb{R}^{n+1}$. Since the action of $\mathbb{G}$ on Ω is simply transitive, the evaluation map is a diffeomorphism, so it induces a left invariant affine connection $D(= \mathrm{ev}^* D)$ on $\mathbb{G}$ from the standard affine connection on Ω which is torsion free and flat. Let $\mathfrak{g} = \mathrm{Lie}\,\mathbb{G}$ be the set of left invariant vector fields on $\mathbb{G}$, so $\mathrm{ev}_*(\mathfrak{g}) \subset \mathfrak{X}(\Omega)$ is the set of $\mathbb{G}$-invariant vector fields on Ω. The left invariant affine connection gives a LSA structure on $\mathfrak{g}$ by $x \cdot y = D_x y$ for any $x, y \in \mathfrak{g}$, which is compatible with the Lie structure, that is,

$$x \cdot y - y \cdot x = [x, y] \qquad \text{for all } x, y \in \mathfrak{g}. \tag{4.1}$$

For the canonical vector field $E \in \mathfrak{X}(\Omega)$, we can choose $e \in \mathfrak{g}$ such that

$$\mathrm{ev}_*(e) = -E$$

since E is $\mathbb{G}$-invariant from Corollary 3.4. Let $\mathfrak{j} = \mathrm{Lie}\,\mathbb{J}$ be the Lie subalgebra of $\mathfrak{g}$.

PROPOSITION 4.1. *The element e is an idempotent and it belongs to $\mathfrak{j}$, so $\mathfrak{j} = \mathrm{span}\{e\}$ is a subalgebra of $(\mathfrak{g}, \cdot)$.*

Proof. From Proposition 3.2 (c) and the map ev_*,

$$\begin{aligned} \mathrm{ev}_*(e \cdot e) &= \mathrm{ev}_*(D_e e) = D_{\mathrm{ev}_*(e)}\,\mathrm{ev}_*(e) = D_{-E}(-E) \\ &= -E = \mathrm{ev}_*(e), \end{aligned}$$

so we have $e \cdot e = e$. Since $\mathrm{ev}(\mathbb{J}) = \{(0, \ldots, 0, s) \in \Omega \mid s > 0\}$, we have that $\mathrm{ev}_*(\mathfrak{j})|_{e_{n+1}} = \{(0, \ldots, 0, t) \in \mathbb{R}^{n+1} \mid t \in \mathbb{R}\}$. But $\mathrm{ev}_*(e)|_{e_{n+1}} = -E|_{e_{n+1}} = (0, \ldots, 0, 1) \in \mathrm{ev}_*(\mathfrak{j})|_{e_{n+1}}$. Because everything is all $\mathbb{G}$-invariant, we have that $\mathfrak{j} = \{te \mid t \in \mathbb{R}\}$. □

The idempotent e in Proposition 4.1 is called a *principal idempotent* of $\mathfrak{g}$. Let $\mathfrak{a}$ be the Lie algebra of the unimodular subgroup $\mathbb{A}$ of $\mathbb{G}$. Since $\mathbb{A}$ is a normal subgroup, $\mathfrak{a}$ is a Lie ideal of $\mathfrak{g}$. Then we have:

PROPOSITION 4.2. (a) *The Lie ideal $\mathfrak{a}$ is equal to the kernel of the right multiplication of e, ρ_e, that is,*

$$\mathfrak{a} = \{a \in \mathfrak{g} \mid \rho_e(a) = ae = 0\}.$$

(b) *The left multiplication of e, λ_e, leave invariant the Lie ideal $\mathfrak{a}$.*

Proof. (a) For any $a \in \mathfrak{a}$ and $t \in \mathbb{R}_+$, $\mathrm{ev}_*(a)|_{\Sigma_t} \in \mathfrak{X}(\Sigma_t)$ because $\mathbb{A}$ leave invariant Σ_t. Then from Proposition 3.2 (b),

$$\mathrm{ev}_*(a \cdot e) = \mathrm{ev}_*(D_a e) = D_{\mathrm{ev}_* a}(-E) = 0.$$

Since the map ev_* is one-to-one, we have that $a \cdot e = 0$, that is, $\mathfrak{a} \subset \ker \rho_e$. By dimension argument, they must be equal.

(b) From (a) and (4.1), $[e, a] = ea - ae = ea$ for all $a \in \mathfrak{a}$. Then, since $\mathfrak{a}$ is a Lie ideal, $\lambda_e \mathfrak{a} = [e, \mathfrak{a}] \subset \mathfrak{a}$. □

From the Lie group homomorphism $\tau : \mathbb{G} \to \mathbb{R}_+$, we obtain a 1-form $d\tau : \mathfrak{g} \to \mathbb{R}$, which is a Lie algebra homomorphism(*cf.* [19]). On the other hand, we have an induced 1-form $\mathrm{ev}^*\, d\phi$ on $\mathfrak{g}$, where $\phi = -\ln p$ is the function on Ω given by (3.1).

Lemma 4.3. *For two 1-forms on $\mathfrak{g}$ given in the above, we have*

$$d\tau = -\,\mathrm{ev}^*\, d\phi.$$

Proof. Recall that $\phi(e_{n+1}) = -\ln p(e_{n+1}) = -\ln(1 - F(0)) = 0$ and hence $\phi(g \cdot e_{n+1}) = -\ln p(g \cdot e_{n+1}) = -\ln \tau(g)p(e_{n+1}) = -\ln \tau(g)$ for any $g \in \mathbb{G}$, where $e_{n+1} = (0, \ldots, 0, 1)$. This says that $\phi \circ \mathrm{ev} = -\ln \tau$ as a function from $\mathbb{G}$ to $\mathbb{R}$. By differentiating the functions at the identity $\mathbb{id}$ of $\mathbb{G}$,

$$d\phi|_{e_{n+1}} d(\mathrm{ev})|_{\mathbb{id}} = -d(\ln \tau)|_{\mathbb{id}} = -\frac{d\tau}{\tau}|_{\mathbb{id}}.$$

Since $\tau(\mathbb{id}) = 1$, we have $d\phi|_{e_{n+1}} d(\mathrm{ev})|_{\mathbb{id}} = -d\tau|_{\mathbb{id}}$. □

Using this Lemma and Proposition 3.2 (b), we have

$$d\tau(e) = -\,\mathrm{ev}^*\, d\phi(e) = -d\phi(-E) = 1.$$

From the Hessian metric $Dd\phi$ on Ω, an inner product $\langle\!\langle\, ,\, \rangle\!\rangle$ on $\mathfrak{g}$ could be induced as the following:

$$\langle\!\langle x, y \rangle\!\rangle := Dd\phi(\mathrm{ev}_*(x), \mathrm{ev}_*(y)), \qquad x, y \in \mathfrak{g}.$$

With this inner product, the length of principal idempotent is 1, that is,

$$\langle\!\langle e, e \rangle\!\rangle = Dd\phi(\mathrm{ev}_*(e), \mathrm{ev}_*(e)) = Dd\phi(-E, -E) = d\phi(E) = 1 \tag{4.2}$$

by using Proposition 3.2 (a).

Proposition 4.4. *The induced inner product $\langle\!\langle\, ,\, \rangle\!\rangle$ is related with the 1-form $d\tau$ as follows.*

(a) $\langle\!\langle e, x \rangle\!\rangle = d\tau(x)$ *for all* $x \in \mathfrak{g}$.
(b) $\langle\!\langle x, y \rangle\!\rangle = d\tau(x \cdot y)$ *for all* $x, y \in \mathfrak{g}$.

Proof. (a) Using Lemma 4.3 and (3.2), we have, for any $x \in \mathfrak{g}$,

$$\langle\!\langle e, x \rangle\!\rangle = -Dd\phi(E, \mathrm{ev}_*(x)) = -d\phi(\mathrm{ev}_*(x)) = -\,\mathrm{ev}^*\, d\phi(x) = d\tau(x).$$

(b) For any $\mathbb{G}$-invariant vector fields $X = \mathrm{ev}_*(x)$ and $Y = \mathrm{ev}_*(y)$,

$$Dd\phi(X, Y) = (D_X d\phi)(Y) = X(d\phi(Y)) - d\phi(D_X Y) = -d\phi(D_X Y),$$

since $d\phi(Y)$ is a G-invariant function, i.e., it is constant on Ω. On the other hand, $\mathrm{ev}_*(x \cdot y) = \mathrm{ev}_*(D_x y) = D_X Y$. Hence we have

$$\begin{aligned} \langle\!\langle x, y \rangle\!\rangle &= Dd\phi(X, Y) = -d\phi(D_X Y) = -d\phi(\mathrm{ev}_*(x \cdot y) \\ &= -\,\mathrm{ev}^*\, d\phi(x \cdot y) \;=\; d\tau(x \cdot y). \end{aligned}$$

□

Notice that the element e is uniquely determined as the dual of $d\tau$ with respect to the inner product $\langle\!\langle , \rangle\!\rangle$ on $\mathfrak{g}$ from Proposition 4.4 (a).

COROLLARY 4.5. (a) *The inner product $\langle\!\langle , \rangle\!\rangle$ is of Hessian type.*
(b) *The Lie ideal $\mathfrak{a}$ is perpendicular to the principal idempotent with respect to the inner product $\langle\!\langle , \rangle\!\rangle$.*
(c) *The Lie ideal $\mathfrak{a}$ is equal to $\ker d\tau$.*
(d) *For any element $a, b \in \mathfrak{a}$, $a \cdot b - \langle\!\langle a, b\rangle\!\rangle e \in \mathfrak{a}$.*

Proof. (a) Since $\langle\!\langle xy, z\rangle\!\rangle = d\tau((xy)z)$ for any $x, y, z \in \mathfrak{g}$ and $d\tau$ is linear, the inner product$\langle\!\langle , \rangle\!\rangle$ satisfies (2.4).

(b) For all $a \in \mathfrak{a}$, $\langle\!\langle a, e\rangle\!\rangle = d\tau(ae) = 0$ from Proposition 4.2 (b).

(c) For all $a \in \mathfrak{a}$, $d\tau(a) = \langle\!\langle e, a\rangle\!\rangle = 0$ from Proposition 4.2 (a). So $\mathfrak{a} \subset \ker d\tau$. Since they are the subspaces of codimension 1, they must be equal.

(d) For any $a, b \in \mathfrak{a}$, $d\tau(a \cdot b - \langle\!\langle a, b\rangle\!\rangle e) = \langle\!\langle a, b\rangle\!\rangle - \langle\!\langle a, b\rangle\!\rangle d\tau(e) = 0$. Therefor from (c), $a \cdot b - \langle\!\langle a, b\rangle\!\rangle e \in \mathfrak{a}$. □

Let's denote the restriction of the inner product $\langle\!\langle , \rangle\!\rangle$ to $\mathfrak{a}$ by $\langle , \rangle$. Note that $\langle , \rangle$ is nondegenerate from Corollary 4.5 (b) and (4.2). Moreover, for $x = \begin{pmatrix} a \\ s \end{pmatrix}$ and $y = \begin{pmatrix} b \\ t \end{pmatrix}$, we have

$$\langle\!\langle x, y\rangle\!\rangle = \langle\!\langle a, b\rangle\!\rangle + \langle\!\langle s, t\rangle\!\rangle = \langle a, b\rangle + st.$$

By using Corollary 4.5 (d), we can define a multiplication $*$ on $\mathfrak{a}$ as follows: for any $a, b \in \mathfrak{a}$

$$a * b = a \cdot b - \langle a, b\rangle e. \tag{4.3}$$

LEMMA 4.6. *The algebra $(\mathfrak{a}, *)$ is an LSA with a Hessian type inner product $\langle , \rangle$.*

Proof. For any $a, b, c \in \mathfrak{a}$, we have from (4.3)

$$\begin{aligned} (a * b) * c &= (a \cdot b - \langle a, b\rangle e) \cdot c - \langle a \cdot b - \langle a, b\rangle e,\ c\rangle \\ &= (a \cdot b) \cdot c - \langle a, b\rangle e \cdot c + \langle\!\langle a \cdot b,\ c\rangle\!\rangle e, \\[1ex] a * (b * c) &= a \cdot (b \cdot c - \langle b, c\rangle e) - \langle a,\ b \cdot c - \langle b, c\rangle e\rangle e \\ &= a \cdot (b \cdot c) + \langle\!\langle a, b \cdot c\rangle\!\rangle e. \end{aligned}$$

Then the associator $(a * b) * c - a * (b * c)$ is left symmetric since $(a \cdot b) \cdot c - a \cdot (b \cdot c)$ is left symmetric, the inner product $\langle\!\langle , \rangle\!\rangle$ is Hessian type, and $\langle , \rangle$ is symmetric. Hence $(\mathfrak{a}, *)$ is an LSA. By using (4.3) and Corollary 4.5 (b), $\langle a * b, c\rangle = \langle\!\langle a \cdot b, c\rangle\!\rangle - \langle\!\langle \langle a, b\rangle e, c\rangle\!\rangle = \langle\!\langle a \cdot b, c\rangle\!\rangle$. Therefore $\langle , \rangle$ is also Hessian type. □

Proposition 4.7. *The restricted left multiplication $\lambda_e|_{\mathfrak{a}}$ of e is the compatible derivation on $(\mathfrak{a}, *)$ with respect to the inner product $\langle , \rangle$.*

Proof. For any $a, b \in \mathfrak{a}$,

$$\begin{aligned} e \cdot (a \cdot b) &= (e \cdot a) \cdot b + a \cdot (e \cdot b) - (a \cdot e) \cdot b \\ &= (e \cdot a) \cdot b + a \cdot (e \cdot b) \\ &= (e \cdot a) * b + \langle e \cdot a, b \rangle e + a * (e \cdot b) + \langle a, e \cdot b \rangle e, \end{aligned}$$

and $e \cdot (a \cdot b) = e \cdot (a * b) + \langle a, b \rangle e$. Then from Proposition 4.2 (b),

$$e \cdot (a * b) = (e \cdot a) * b + a * (e \cdot b), \tag{4.4}$$

$$\langle a, b \rangle = \langle e \cdot a, b \rangle + \langle a, e \cdot b \rangle. \tag{4.5}$$

The equation (4.4) says that $\lambda_e|_{\mathfrak{a}}$ acts on $(\mathfrak{a}, *)$ as derivation and the equation (4.5) says that $\lambda_e|_{\mathfrak{a}}$ is compatible with respect to the inner product $\langle , \rangle$ on $\mathfrak{a}$. □

As a result, the LSA $(\mathfrak{g}, \cdot)$ can be split into the sum of a unimodular Lie ideal $\mathfrak{a}$ and a 1-dimensional Lie subalgebra $\mathfrak{j}$,

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{j} \tag{4.6}$$

in such a way that $\lambda_e|_{\mathfrak{a}}$ belongs to the space $\mathrm{cDer}(\mathfrak{a}, *)$(see Definition 2.8). The decomposition (4.6) is called the *principal decomposition* of the LSA $(\mathfrak{g}, \cdot)$.

Conversely, let $\mathcal{A} = (\mathfrak{a}, *)$ be a complete LSA with a nondegenerate Hessian type inner product $\langle , \rangle = H$ with $|\det H| = 1$, where $\mathfrak{a}$ is the unimodular Lie algebra. Note that from (2.4), H satisfies

$$[a, b]'H = a'H\lambda_b - b'H\lambda_a, \tag{4.7}$$

for all a and $b \in \mathcal{A}$ as column vectors.

Proposition 4.8. *Let $\mathcal{G} = \mathcal{A} + \mathcal{J}$ be a vector space direct sum of the complete unimodular LSA $\mathcal{A}$ and 1-dimensional vector space $\mathcal{J} = \mathrm{span}\{e\}$. Define a multiplication $\cdot$ on $\mathcal{G}$: for $a, b \in \mathcal{A}$,*

$$\begin{cases} a \cdot b = a * b + \langle a, b \rangle e, \\ e \cdot e = e, \\ a \cdot e = 0, \\ e \cdot a = \lambda_e a \in \mathcal{A}, \end{cases} \tag{4.8}$$

where $\langle , \rangle = H$ is a Hessian type inner product on $\mathcal{A}$. Then $\mathcal{G}$ is an LSA if and only if the left multiplication operator of e, λ_e, acts on the LSA $(\mathcal{A}, *)$ as a compatible derivation with respect to $\langle , \rangle$.

Proof. Let $\mathcal{G} = \mathcal{A} + \mathcal{J}$ be an algebra with (4.8), then the left multiplication operators are given as following

$$\lambda_a = \begin{pmatrix} \bar{\lambda}_a & 0 \\ a'H & 0 \end{pmatrix}, \quad \lambda_e = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix},$$

where $\bar{\lambda}_a$ is the left multiplication operator on $(\mathcal{A}, *)$ by $a \in \mathcal{A}$ and B is the restriction of λ_e on $\mathcal{A}$. Recall $\mathcal{G}$ is an LSA if and only if $\lambda_{[x,y]} = [\lambda_x, \lambda_y]$ for all $x, y \in \mathcal{G}$. By put $x = \begin{pmatrix} a \\ s \end{pmatrix}$, $y = \begin{pmatrix} b \\ t \end{pmatrix} \in \mathcal{G}$,

$$\lambda_{[x,y]} = \begin{pmatrix} \bar{\lambda}_{a*b-b*a+sBb-tBa} & 0 \\ (a*b-b*a+sBb-tBa)'H & 0 \end{pmatrix},$$

$$[\lambda_x, \lambda_y] = \begin{pmatrix} \bar{\lambda}_a\bar{\lambda}_b - \bar{\lambda}_b\bar{\lambda}_a + t\bar{\lambda}_a B - s\bar{\lambda}_b B + sB\bar{\lambda}_b - tB\bar{\lambda}_a & 0 \\ a'H\bar{\lambda}_b - b'H\bar{\lambda}_a + ta'HB - sb'HB + sb'H - ta'H & 0 \end{pmatrix}.$$

So, the LSA condition of $\mathcal{G}$ is equivalent to the following equations

$$(4.9) \qquad \begin{array}{rl} & \bar{\lambda}_{a*b-b*a+sBb-tBa} \\ = & \bar{\lambda}_a\bar{\lambda}_b - \bar{\lambda}_b\bar{\lambda}_a + t\bar{\lambda}_a B - s\bar{\lambda}_b B + sB\bar{\lambda}_b - tB\bar{\lambda}_a \\ & (a*b-b*a+sBb-tBa)'H \\ = & a'H\bar{\lambda}_b - b'H\bar{\lambda}_a + ta'HB - sb'HB + sb'H - ta'H. \end{array}$$

By using the left symmetric condition on $(\mathcal{A}, *)$ and the equation (4.7), the equations in (4.9) are reduced to

$$\begin{aligned} \bar{\lambda}_{sBb-tBa} &= t\bar{\lambda}_a B - s\bar{\lambda}_b B + sB\bar{\lambda}_b - tB\bar{\lambda}_a \\ (sBb - tBa)'H &= ta'HB - sb'HB + sb'H - ta'H. \end{aligned}$$

Hence we have

$$(4.10) \qquad \bar{\lambda}_{Ba} = -\bar{\lambda}_a B + B\bar{\lambda}_a,$$

$$(4.11) \qquad B'H = -HB + H$$

for all $a \in \mathcal{A}$. The equation (4.10) means that $B(= \lambda_e|_{\mathcal{A}})$ is a derivation of $(\mathcal{A}, *)$ and the equation (4.11) says that B is compatible with respect to the Hessian type inner product. □

Above Proposition says that the LSA $\mathcal{G}$ is the extension of a complete unimodular LSA $\mathcal{A}$ by an 1-dimensional LSA $\mathcal{J} = \mathrm{span}\{e\}$, so we obtain a short exact sequence

$$(4.12) \qquad 0 \to \mathcal{A} \to \mathcal{G} \to \mathcal{J} \to 0.$$

DEFINITION 4.9. In (4.12), $\mathcal{G}$ will be called a *graph extension* of $\mathcal{A}$, where the multiplication is given in (4.8) and $\lambda_e \in \mathrm{cDer}(\mathcal{A})$.

Let's identify the LSA $\mathcal{G}$ and $\mathbb{R}^{n+1}$, where the LSA $\mathcal{A}$ is identified to a subspace $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ and the element $e \in \mathcal{J}$ corresponds to $e_{n+1} = (0, \ldots, 0, 1)$. Denote the right multiplication operator of $x \in (\mathcal{G}, \cdot)$ (resp. $a \in (\mathcal{A}, *)$) by ρ_x (resp. $\bar{\rho}_a$). Then we have:

LEMMA 4.10. *With the notation in Proposition 4.8, for* $x = \begin{pmatrix} a \\ s \end{pmatrix} \in \mathbb{R}^{n+1}(= \mathcal{G})$, *the right multiplication operator is given by*

$$\rho_x = \begin{pmatrix} \bar{\rho}_a & Ba \\ a'H & s \end{pmatrix}.$$

Moreover, we have the followings:

(a) $\operatorname{tr} \rho_x = s$ *and hence* $\operatorname{tr} \rho_{xy} = a'Hb + st$ *defines a nondegenerate Hessian type inner product on* $\mathcal{G}$.
(b) $\det(I + \rho_x) = -a'H(I + \bar{\rho}_a)^{-1}Ba + s + 1$.

Proof. Since $\rho_x y = y \cdot x = \lambda_y x$ for any $y = \begin{pmatrix} b \\ t \end{pmatrix} \in \mathbb{R}^{n+1}$, we have,

$$\rho_x y = \begin{pmatrix} \bar{\lambda}_b a + tBa \\ b'Ha + st \end{pmatrix} = \begin{pmatrix} \bar{\rho}_a b + tBa \\ a'Hb + st \end{pmatrix} = \begin{pmatrix} \bar{\rho}_a & Ba \\ a'H & s \end{pmatrix} \begin{pmatrix} b \\ t \end{pmatrix}.$$

Therefore $\rho_x = \begin{pmatrix} \bar{\rho}_a & Ba \\ a'H & s \end{pmatrix}$ and $\operatorname{tr} \rho_x = \operatorname{tr} \bar{\rho}_a + s$.

(a) Since $(\mathcal{A}, *)$ is complete LSA, we have $\operatorname{tr} \bar{\rho}_a = 0$ for all $a \in \mathcal{A}$. Then we have $\operatorname{tr} \rho_x = s$ and $\operatorname{tr} \rho_{xy} = a'Hb + st$. Because H is nondegenerate, $\operatorname{tr} \rho$ is nondegenerate, and it must be of Hessian type from the left symmetric condition of multiplication.

(b) From the completeness of $(\mathcal{A}, *)$, $\det(I + \bar{\rho}_a) = 1$ for all $a \in \mathcal{A}$. So we have

$$\begin{aligned} \det(I + \rho_x) &= \det \begin{pmatrix} I + \bar{\rho}_a & Ba \\ a'H & s + 1 \end{pmatrix} \\ &= \det \begin{pmatrix} I + \bar{\rho}_a & Ba \\ 0 & -a'H(I + \bar{\rho}_a)^{-1}Ba + s + 1 \end{pmatrix} \\ &= \det(I + \bar{\rho}_a) \cdot (-a'H(I + \bar{\rho}_a)^{-1}Ba + s + 1) \\ &= -a'H(I + \bar{\rho}_a)^{-1}Ba + s + 1. \end{aligned}$$

□

From the above Lemma, the developing image $\tilde{\Omega}$ of the LSA $\mathcal{G}$ is the domain containing the origin of $\mathbb{R}^{n+1}$ and bounded by the hypersurface defined by $s = a'H(I + \bar{\rho}_a)^{-1}Ba - 1$. Moreover the induced metric on $\tilde{\Omega}$

from the Hessian type inner product $\langle\langle , \rangle\rangle := \operatorname{tr}\rho$ is nondegenerate. Put a function $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(a) = a'H(I + \bar{\rho}_a)^{-1}Ba. \tag{4.13}$$

Then we have $F(0) = 0$. Since $(\mathcal{A}, *)$ is complete, the right multiplication $\bar{\rho}_a$ is nilpotent, that is, $\bar{\rho}_a^n = 0$ for all $a \in \mathcal{A}$. So

$$(1 + \bar{\rho}_a)^{-1} = 1 - \bar{\rho}_a + \bar{\rho}_a^2 + \cdots + \bar{\rho}_a^{n-1}$$

is a finite sum and hence the function $F(a)$ must be a polynomial, whose degree is less than $n + 1$. Differentiating (4.13), for $v \in \mathbb{R}^n$

$$\begin{aligned} dF_a(v) &= v'H(I + \bar{\rho}_a)^{-1}Ba \\ &\quad -a'H(\bar{\rho}_v - (\bar{\rho}_a\bar{\rho}_v + \bar{\rho}_v\bar{\rho}_a) + \cdots)Ba \\ &\quad +a'H(I + \bar{\rho}_a)^{-1}Bv. \end{aligned}$$

From this, $dF_0 = 0$ and $DdF_0 = B^tH + HB = H$. Let $\tilde{\mathbb{G}}$(resp. $\tilde{\mathbb{A}}$) be the Lie subgroup of $\mathrm{Aff}(n + 1)$ obtained from $\mathcal{G}$(resp. $\mathcal{A}$), then $\tilde{\mathbb{G}}$ acts simply transitively on $\tilde{\Omega}$, the domain over the hypersurface $\tilde{\Sigma}$ which is the graph of the function $s = F(a) - 1$. Since $\mathcal{A}$ is a complete unimodular LSA, $\tilde{\mathbb{A}}$ must be unimodular and equiaffine. So we have $|\det DdF| = |\det H| = 1 \neq 0$ because $\tilde{\mathbb{A}}$ acts on $\tilde{\Sigma}$ transitively. Therefore $\tilde{\Omega}$ is the graph domain, translated by $(0, \ldots, 0, -1)$ from the domain, which we considered at first, over the nondegenerate hypersurface, where a unimodular equiaffine Lie subgroup acts simply transitively.

THEOREM 4.11. *Let Σ be a graph of a function $F : \mathbb{R}^n \to \mathbb{R}$ with $|\det DdF| = 1$, and let Ω be the domain over the graph Σ. Suppose a Lie subgroup $\mathbb{G} \subset \mathrm{Aut}(\Omega)_0$ acts simply transitively on Ω as affine transformations and it contains an unimodular equiaffine subgroup $\mathbb{A}$ which acts on Σ simply transitively. Then the set of homogeneous domain $(\mathbb{G}, \Omega)$ over Σ is in one to one correspondence with the set of graph extensions $\mathcal{G}$ of $\mathcal{A}$, the complete unimodular LSA corresponding to $(\mathbb{A}, \Sigma)$. In this case, there is a nondegenerate Hessian metric on Ω, or equivalently, there is a nondegenerate Hessian type inner product on $\mathcal{G}$ which is defined by the trace form of the right multiplication operator.*

REMARK 4.12. If non-isomorphic Lie subgroups $\mathbb{G}$ and $\mathbb{G}'$ of $\mathrm{Aut}(\Omega)_0 \subset \mathrm{Aff}(n + 1)$ act on a domain Ω simply transitively, they yield the non-isomorphic LSA's $\mathcal{G}$ and $\mathcal{G}'$.

5. Classification of low dimensional graph extension

Before the calculation, we will investigate the automorphism group of the graph extension since two isomorphic graph extension give the affinely equivalent domain over the graph and the affinely equivalent group action on the domain. Let $(\mathcal{A}, *)$ be a complete unimodular LSA with a Hessian type inner product $\langle , \rangle = H$, where $|\det H| = 1$, and let $(\mathcal{G}, \cdot)$ be a graph extension of $\mathcal{A}$. Let's denote $\lambda_e|_{\mathcal{A}} = B$ the compatible derivation on $\mathcal{A}$.

PROPOSITION 5.1. *The LSA automorphism $\tilde{A}$ of $\mathcal{G}$ is represented as*

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

where A is an element of $\mathrm{IAut}(\mathcal{A})$ *satisfying $AB = BA$. (see section 2 for the definition of* $\mathrm{IAut}(\mathcal{A})$*)*

Proof. From the Lemma 4.10, $\mathcal{A} = \ker \operatorname{tr} \rho$. Since

$$\operatorname{tr} \rho_{\tilde{A}x} = \operatorname{tr} \tilde{A}\rho_x\tilde{A}^{-1} = \operatorname{tr} \rho_x$$

for all $x \in \mathcal{G}$, the LSA automorphism $\tilde{A}$ leaves invariant the subspace $\mathcal{A}$. Therefore $\tilde{A}$ is represented as the following $(n+1) \times (n+1)$ matrix

$$\tilde{A} = \begin{pmatrix} A & \beta \\ 0 & t \end{pmatrix},$$

where A is a $n \times n$ matrix, β is a vector in $\mathbb{R}^n$ and t is a real number. From $\tilde{A}(e \cdot a) = \tilde{A}(e) \cdot \tilde{A}(a)$,

$$\begin{pmatrix} ABa \\ 0 \end{pmatrix} = \begin{pmatrix} \beta \\ t \end{pmatrix} \cdot \begin{pmatrix} Aa \\ 0 \end{pmatrix} = \begin{pmatrix} \beta * Aa + tBAa \\ \langle \beta, Aa \rangle \end{pmatrix}.$$

Then $\langle \beta, Aa \rangle = 0$ for all $a \in \mathcal{A}$. Because the matrix A and the inner product $\langle , \rangle$ are nondegenerate, β must be 0, so we have

$$AB = tBA. \tag{5.1}$$

Since $\tilde{e} := \tilde{A}(e) = \begin{pmatrix} 0 \\ t \end{pmatrix}$ is an idempotent, we have $t^2 = t$. Because of the nondegeneracy of $\tilde{A}$, t must be equal to 1 and hence (5.1) becomes

$$AB = BA.$$

Now from $\tilde{A}(a \cdot b) = \tilde{A}(a)\tilde{A}(b)$ for $a, b \in \mathcal{A}$, we obtain

$$\begin{pmatrix} A(a * b) \\ \langle a, b \rangle \end{pmatrix} = \begin{pmatrix} (Aa) * (Ab) \\ \langle Aa, Ab \rangle \end{pmatrix},$$

thus

$$A(a * b) = (Aa) * (Ab), \qquad \langle Aa, Ab\rangle = \langle a, b\rangle. \tag{5.2}$$

The equations in (5.2) says that A is an isometric LSA automorphism of $\mathcal{A}$, that is, $A \in \mathrm{IAut}(\mathcal{A})$. □

Above Proposition says that, on a complete unimodular LSA equipped with nondegenerate Hessian type inner product H and the compatible derivation B, the graph extension is determined by the isometric automorphism class commuting with B. By the similar argument in the proof of Proposition 5.1, for another compatible derivation $\bar{B}$, $\mathcal{G}(\mathcal{A}, *, H, B)$ and $\bar{\mathcal{G}}(\mathcal{A}, *, H, \bar{B})$ are isomorphic if and only if there exists $A \in \mathrm{IAut}(\mathcal{A})$ such that $\bar{B} = A^{-1}BA$. This says that

$$\{\text{isomorphic class of } \mathcal{G}(\mathcal{A}, *, H)\} = \mathrm{cDer}(\mathcal{A})/\!\sim,$$

where $\mathrm{cDer}(\mathcal{A}) = \{B \in \mathrm{sDer}(\mathcal{A}) \mid \operatorname{tr} B = \frac{\dim \mathcal{A}}{2}\}$ and $\sim$ is the conjugate action by $\mathrm{IAut}(\mathcal{A})$.

In the following three subsections, we calculate the Hessian type inner product and the compatible derivation of the non-isomorphic complete unimodular LSA's which are derived from the classification of low dimensional LSA in [1, 11]. Then these data will give the classification of graph extensions and, from the (4.13) we can obtain the associated polynomials which are the boundary of the domains, the developing images of the graph extensions.

5.1. $\dim \mathcal{A} = 1$

Let $\mathcal{A}$ be an 1-dimensional complete unimodular LSA, then $\mathcal{A}$ must be a trivial algebra, that is, $\mathcal{A} = \mathrm{span}\{e_1\}$ with $e_1 * e_1 = 0$. Let $\mathcal{G}$ be a graph extension of $\mathcal{A}$ with the multiplication

$$\begin{cases} e_1 \cdot e_1 = \langle e_1, e_1\rangle e \\ e \cdot e = e \\ e_1 \cdot e = 0 \\ e \cdot e_1 = s e_1 \text{ for some } s \in \mathbb{R}. \end{cases}$$

From the nondegeneracy of $\langle,\rangle$ and the compatibility condition (4.11), we have $s = \frac{1}{2}$. By normalising, put $\langle e_1, e_1\rangle = \pm 1$. Then

$$\mathcal{G} = \mathrm{span}\{e_1, e_2 = e \mid e_1 \cdot e_1 = \pm e_2,\ e_2 \cdot e_1 = \frac{1}{2} e_1,\ e_2 \cdot e_2 = e_2\}$$

and hence $\mathcal{G}$ is the LSA on the domain lying over the graph of

$$F(a) = \pm\frac{1}{2}a^2.$$

5.2. $\dim \mathcal{A} = 2$

Let $\mathcal{A}$ be a 2-dimensional complete unimodular LSA, then $\mathcal{A}$ must be one of the following up to isomorphism from [1]

$\mathcal{A}_{a1} = \mathrm{span}\{e_1, e_2\}$, that is, $\mathcal{A}_{a1}$ is trivial.
$\mathcal{A}_{a2} = \mathrm{span}\{e_1, e_2 \mid e_1e_1 = e_2\}$.

In the following table, we give the Hessian type inner product H representing the isometric class, the compatible derivation B, and the associated polynomial F of each LSA: for $a = a_1e_1 + a_2e_2 \in \mathcal{A}$

	H	B	$F(a_1, a_2)$
$\mathcal{A}_{a1}$ (2,0)	$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & b_{12} \\ -b_{12} & \frac{1}{2} \end{pmatrix}$	$\pm\frac{1}{2}(a_1^2 + a_2^2)$
$\mathcal{A}_{a1}$ (1,1)	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & b_{12} \\ b_{12} & \frac{1}{2} \end{pmatrix}$	$\frac{1}{2}(a_1^2 - a_2^2)$
$\mathcal{A}_{a2}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$	$a_1a_2 - \frac{1}{3}a_1^3$

The graph extensions of each LSA are determined by H and B from Proposition 4.8. In fact, for the basis of the graph extension $\mathcal{G}$, $\{e_1, e_2, e_3 = e\}$, H and $*$ determine $e_1 \cdot e_1$, $e_1 \cdot e_2$, $e_2 \cdot e_1$, $e_2 \cdot e_2$, and B determines $e_3 \cdot e_1$, $e_3 \cdot e_2$. The following table shows that presentations of each graph extension:

$\mathcal{G}_{a1}$ (2,0)	$e_3e_3 = e_3,\ e_1e_1 = \pm e_3 = e_2e_2,$ $e_3e_1 = \frac{1}{2}e_1 - b_{12}e_2,\ e_3e_2 = b_{12}e_1 + \frac{1}{2}e_2$
$\mathcal{G}_{a1}$ (1,1)	$e_3e_3 = e_3,\ e_1e_1 = e_3,\ e_2e_2 = -e_3,$ $e_3e_1 = \frac{1}{2}e_1 + b_{12}e_2,\ e_3e_2 = b_{12}e_1 + \frac{1}{2}e_2$
$\mathcal{G}_{a2}$	$e_3e_3 = e_3,\ e_1e_1 = e_2,\ e_1e_2 = e_3 = e_2e_1,$ $e_3e_1 = \frac{1}{3}e_1,\ e_3e_2 = \frac{2}{3}e_2$

Note that, in the case of the graph extension $\mathcal{G}_{a1}$(both $\mathcal{G}_{a1}(2,0)$ and $\mathcal{G}_{a1}(1,1)$), non-isomorphic classes give the same homogeneous graph domain. It says that the non-isomorphic Lie subgroups act on the domain simultaneously. So we guess the dimension of the automorphism group of the domain. The above table shows that $\dim \mathrm{Aut}(\Omega(\mathcal{G}_{a1})) = 4$ and

$\dim \mathrm{Aut}(\Omega(\mathcal{G}_{a2})) = 3$. On the other hand, the above classification of the graph extension of complete unimodular LSA is exactly same as the classification of 3-dimensional incomplete simple LSA's [11].

5.3. $\dim \mathcal{A} = 3$

Let $\mathcal{A}$ be a 3-dimensional complete unimodular LSA, then $\mathcal{A}$ must be one of the following up to isomorphism from [11] : two simple and nine non-simple LSA's,

$$\begin{aligned}
\mathcal{A}_{s1} &= \mathrm{span}\{e_1, e_2, e_3 \mid e_1e_2 = -e_3,\ e_1e_3 = e_2,\ e_2e_2 = e_1,\ e_3e_3 = e_1\},\\
\mathcal{A}_{s2} &= \mathrm{span}\{e_1, e_2, e_3 \mid e_1e_2 = -e_2,\ e_1e_3 = e_3,\ e_2e_3 = e_1,\ e_3e_2 = e_1\},\\
\mathcal{A}_{n1} &= \mathrm{span}\{e_1, e_2, e_3 \mid e_2e_3 = e_1,\ e_3e_2 = (1-t)e_1,\ e_3e_3 = (1+t)e_2\},\\
\mathcal{A}_{n2} &= \mathrm{span}\{e_1, e_2, e_3 \mid e_3e_2 = e_1,\ e_3e_3 = e_2\},\\
\mathcal{A}_{n3} &= \mathrm{span}\{e_1, e_2, e_3 \mid e_3e_1 = -e_2,\ e_3e_2 = e_1\},\\
\mathcal{A}_{n4} &= \mathrm{span}\{e_1, e_2, e_3 \mid e_2e_2 = e_1\},\\
\mathcal{A}_{n5} &= \mathrm{span}\{e_1, e_2, e_3 \mid e_2e_3 = e_1\},\\
\mathcal{A}_{n6} &= \mathrm{span}\{e_1, e_2, e_3 \mid e_2e_2 = e_1,\ e_3e_3 = e_1\},\\
\mathcal{A}_{n7} &= \mathrm{span}\{e_1, e_2, e_3 \mid e_2e_3 = te_1,\ e_3e_2 = (t-1)e_1\},\\
\mathcal{A}_{n8} &= \mathrm{span}\{e_1, e_2, e_3 \mid e_2e_2 = e_1,\ e_3e_2 = -e_1,\ e_3e_3 = te_1\},\\
\mathcal{A}_{n9} &= \mathrm{span}\{e_1, e_2, e_3\}, \text{ that is, } \mathcal{A} \text{ is trivial LSA.}
\end{aligned}$$

By calculating the equation (2.4) on each LSA above, we can see that the LSA's $\mathcal{A}_{s1}, \mathcal{A}_{s2}, \mathcal{A}_{n5}, \mathcal{A}_{n6}, \mathcal{A}_{n7}(t \neq -1, 2)$ and $\mathcal{A}_{n8}$ do not admit any nondegenerate Hessian type inner product. The LSA $\mathcal{A}_{n3}$ admits nondegenerate Hessian type inner products, but all the derivation of $\mathcal{A}_{n3}$ is degenerate as the following:

$$H = \begin{pmatrix} h_{11} & 0 & 0 \\ 0 & h_{11} & 0 \\ 0 & 0 & h_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & 0 \\ -b_{12} & b_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Applying the compatibility condition (4.11), the Hessian type inner products H on the LSA $\mathcal{A}_{n3}$ become degenerate. This says that the LSA $\mathcal{A}_{n3}$ does not permit a nondegenerate graph extension, though the Lie group of the associated Lie algebra of $\mathcal{A}_{n3}$ acts simply transitively on a nondegenerate hypersurface. In fact, using the equation (2.6) and $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we could see that the nondegenerate hypersurface obtained from the LSA $\mathcal{A}_{n3}$ is the 3-dimensional paraboloid, that is, the graph of the function $F(a_1, a_2, a_3) = \frac{1}{2}(a_1^2 + a_2^2 + a_3^2)$. Hence the Lie group $\mathbb{A}_{n3}$ obtained from the LSA $\mathcal{A}_{n3}$ acts simply transitively on the paraboloid. But there does not exist any simply transitive affine group acting on the domain over the paraboloid, which contain this Lie group $\mathbb{A}_{n3}$. On the paraboloid, we know, the abelian Lie group obtained from the trivial LSA also acts simply transitively. This gives the example mentioned in Remark 2.7.

The following table shows the compatible H's and B's which will give a nondegenerate graph extension, and the equation of the nondegenerate hypersurfaces which will give the graph domains : for $a = a_1e_1 + a_2e_2 + a_3e_3 \in \mathcal{A}$,

	H	B	$F(a_1, a_2, a_3)$
$\mathcal{A}_{n1}$ $t \neq -1$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$	$a_1a_3 - a_2a_3^2 + \frac{1}{2}a_2^2 + \frac{(1+t)}{4}a_3^4$
$\mathcal{A}_{n1}$ $t = -1$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \pm1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$	$a_1a_3 - a_2a_3^2 \pm \frac{1}{2}a_2^2$
$\mathcal{A}_{n2}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{3}{4} & b_{12} & 0 \\ 0 & \frac{1}{2} & b_{12} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$	$a_1a_3 - \frac{1}{2}a_2^2$
$\mathcal{A}_{n4}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \pm1 \end{pmatrix}$	$\begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & b_{32} & \frac{1}{2} \end{pmatrix}$	$a_1a_2 - \frac{1}{3}a_2^3 \pm \frac{1}{2}a_3^2$
$\mathcal{A}_{n7}$ $t = -1$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \pm1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$	$a_1a_3 + a_2a_3^2 \pm \frac{1}{2}a_2^2$
$\mathcal{A}_{n7}$ $t = 2$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \pm1 \end{pmatrix}$	$\begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$	$a_1a_2 - a_2^2a_3 \pm \frac{1}{2}a_3^2$
$\mathcal{A}_{n9}$ (3,0)	$\begin{pmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & b_{12} & b_{13} \\ -b_{12} & \frac{1}{2} & b_{23} \\ -b_{13} & -b_{23} & \frac{1}{2} \end{pmatrix}$	$\pm\frac{1}{2}(a_1^2 + a_2^2 + a_3^2)$
$\mathcal{A}_{n9}$ (2,1)	$\begin{pmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \mp1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & b_{12} & b_{13} \\ -b_{12} & \frac{1}{2} & b_{23} \\ b_{13} & b_{23} & \frac{1}{2} \end{pmatrix}$	$\pm\frac{1}{2}(a_1^2 + a_2^2 - a_3^2)$

Note that the maximal degree of the polynomial occurs only on the LSA $\mathcal{A}_{n1}(t \neq -1)$. In this case, if we put $t = 0$, then the LSA is abelian filiform, so the associated hypersurface is the 3-dimensional Cayley hypersurface [3]. Moreover, if we put $t = -\dfrac{1}{3}$ then the hypersurface is equal to the example in [4], on which the difference tensor K satisfies $\nabla K = 0$, $K^2 \not\equiv 0$. In fact, if $\nabla K = 0$, the t must be equal to $-\frac{1}{3}$. In this case, the LSA $\mathcal{A}_{n1}(t = \frac{1}{3})$ is non-abelian filiform.

For the LSA $\mathcal{A}_{n2}$, the degree of the associated polynomial is 2. In this case, the difference tensor $K = 0$, equivalently, the cubic form C vanishes identically (cf. [14]). Note that, for an abelian LSA, the associated polynomial is quadratic if and only if the LSA is trivial (cf. [3]).

REMARK 5.2. From the low dimensional classification of graph extension, it seems that the graph extension does not have any ideal. Moreover, from [11], 2-dimensional and 3-dimensional incomplete simple LSA's are all the nondegenerate graph extension of a complete unimodular LSA. Therefore it would be interesting to ask whether the nondegenerate graph extension is the incomplete simple LSA.

Low dimensional homogeneous hypersurface was classified using preferred normal form in [5, 6]. By comparing their results with ours, we guess that which homogeneous hypersurface does not admit simply transitive unimodular Lie group action or which graph domain does not admit simply transitive Lie group action. Furthermore, some homogeneous hypersurfaces in [5, 6] may be associated to the graph extensions of incomplete or degenerate unimodular LSA's.

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