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STRONG LAWS OF LARGE NUMBERS FOR WEIGHTED SUMS OF NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract. For double arrays of constants $\{a_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ and sequences of negatively orthant dependent random variables $\{X_n,\ n \ge 1\}$, the conditions for strong law of large number of $\sum_{i=1}^{k_n} a_{ni}X_i$ are given. Both cases $k_n \uparrow \infty$ and $k_n = \infty$ are treated.

1. Introduction

The history and literature on strong laws of large numbers is vast and rich as this concept is crucial in probability and statistical theory. The literature on concepts of negative dependence is much more limited but still very interesting ([3], [4] and [7]). Negative dependence has been particularly useful in obtaining strong laws of large numbers (cf. [1], [5], [6], [8] and [10]).

Lehmann [6] provided the concept of negative dependence in the bivariate case as follows:

Random variables X and Y are negatively quadrant dependent(NQD) if

$$P\{X \le x, Y \le y\} \le P\{X \le x\}P\{Y \le y\} \tag{1}$$

for all $x, y \in \mathbb{R}$. A collection of random variables $\{X_n, n \ge 1\}$ is said to be pairwise NQD if every pair of random variables in the collection

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satisfies (1). It is important to note that (1) implies

$$P\{X > x, Y > y\} \leq P\{X > x\}P\{Y > y\} \tag{2}$$

for all $x, y \in \mathbb{R}$. Moreover, it follows that (2) implies (1), and hence, they are equivalent for pairwise NQD.

Ebrahimi and Ghosh [2] showed that (1) and (2) are not equivalent for $n \geq 3$(i.e., (3) and (4) below are not equivalent). Consequently, the following definition is needed to define sequences of negatively dependent random variables:

The random variables $X_1, X_2, \ldots, X_n$ are said to be

(a) lower negatively orthant dependent(LNOD) if for each n

$$P\{X_1 \leq x_1, \ldots, X_n \leq x_n\} \leq \prod_{i=1}^{n} P\{X_i \leq x_i\} \tag{3}$$

for all $x_1, \ldots, x_n \in \mathbb{R}$,

(b) upper negatively orthant dependent(UNOD) if for each n

$$P\{X_1 > x_1, \ldots, X_n > x_n\} \leq \prod_{i=1}^{n} P\{X_i > x_i\} \tag{4}$$

for all $x_1, \ldots, x_n \in \mathbb{R}$,

(c) negatively orthant dependent (NOD) if both (3) and (4) hold.

This notion was introduced by Ebrahimi and Ghosh [2]. In this paper we will give some results of almost sure convergence of weighted sums of NOD sequences, which have not appeared before. In section 2 we consider the case of triangular-type weight arrays and in section 3 we consider the case of infinite double arrays.

We will use the following concept in this paper. Let $\{X_n, n \geq 1\}$ be a sequence of random variables and let X be a nonnegative random variable. If there exists a constant $C(0 < C < \infty)$ satisfying $\sup_{n\geq 1} P\{|X_n| \geq t\} \leq CP\{X \geq t\}$ for any $t \geq 0$, then $\{X_n, n \geq 1\}$ is said to be stochastically dominated by X (briefly $\{X_n, n \geq 1\} \prec X$).

Throughout the remainder of this paper, C will stand for a constant whose value may vary from line to line.

2. Triangular arrays

LEMMA 2.1. ([8]) *If $\{X_n, n \geq 1\}$ is a sequence of NOD random variables and $\{f_n\}$ is a sequence of Borel functions all of which are monotone increasing (or all monotone decreasing), then $\{f_n(X_n)\}$ is a sequence of NOD random variables.*

THEOREM 2.2. ([8]) *Let* $\{X_i,\ 1 \le i \le n\}$ *be a sequence of nonnegative random variables which are upper negatively orthant dependent. Then*

$$E(\Pi_{i=1}^{n} X_i) \le \Pi_{i=1}^{n} E(X_i). \tag{5}$$

THEOREM 2.3. *Let* $0 < r \le 2$. *Assume that* $\{X_n, n \ge 1\}$ *is a sequence of NOD random variables and stochastically dominated by a nonnegative random variable* X, *i.e.,* $X_n \prec X$. *Let* $\{a_{ni}, 1 \le i < k_n < \infty,\ 1 \le k_n \uparrow,\ n \ge 1\}$ *be an array of constants and* $\{d_n, n \ge 1\}$ *a sequence of constants such that* $0 < d_n \uparrow$ *and* $d_n = O(n^{\frac{1}{r}})$. *If the following conditions are satisfied:*

(i) *there exists a positive number p such that*

$$\sum_{n=1}^{\infty} k_n^{-p} < \infty, \tag{6}$$

$$\text{(ii)}\ \sum_{n=1}^{\infty} P(X \ge d_n) < \infty, \tag{7}$$

$$\text{(iii)}\ \max_{1 \le i \le k_n} |a_{ni}|\ d_i = O(\frac{1}{\log k_n}) \tag{8}$$

and

(iv) *one of the following statements is satisfied*

a) *if* $1 \le r \le 2$, $\frac{d_n}{n} \downarrow$, $EX_n = 0$,

$$\sum_{i=1}^{k_n} a_{ni}^2\ d_i^{2-r} = o(\frac{1}{\log k_n}) \tag{9}$$

b) *if* $0 < r < 1$,

$$\sum_{i=1}^{k_n} |a_{ni}|\ d_i^{1-r} = o(1). \tag{10}$$

Then

$$\sum_{i=1}^{k_n} a_{ni}\ X_i \to 0 \ \text{ a.s.} \tag{11}$$

Proof. In view of $d_n = O(n^{\frac{1}{r}})$, (7) yields $EX^r < \infty$. From (7) and $\{X_n\} \prec X$, $\Sigma_{n=1}^{\infty} P\{|X_n| > d_n\} < \infty$, i.e., $P\{|X_n| > d_n, \text{i.o.}\} = 0$ follows. In the following statement, without loss of generality we assume

$a_{ni} \geq 0,\ n \geq 1, i \geq 1$ and $EX^r = 1$. If $q_n = \max_{1 \leq i \leq k_n} |a_{ni}| d_i$. Then $q_n = O(\frac{1}{\log k_n})$ by (8).

(a) If $1 \leq r \leq 2$, let

$$S_n^{'} = \sum_{i=1}^{k_n} a_{ni}(X_i^{'} - EX_i^{'}) \text{ and } S_n^{''} = \sum_{i=1}^{k_n} a_{ni}(X_i^{''} - EX_i^{''}),$$

where $X_i^{'} = (-d_i) \vee (X_i \wedge d_i)$ and $X_i^{''} = X_i - X_i^{'}$. Obviously,

$$S_n = \sum_{n=1}^{k_n} a_{ni} X_i = S_n^{'} + S_n^{''}.$$

Now if $g(x) = x^{-2}(e^x - 1 - x)$, then $g(x)$ is nonnegative and increasing ([9]). For $t > 0$ and $1 \leq i \leq k_n$, noting that$|X_i^{'} - EX_i^{'}| \leq 2d_i$ and $E|X_i^{'} - EX_i^{'}|^r \leq 2^{r-1}(2E|X_i|^r) \leq C2^r$, where C comes from $\sup_n P(|X_n| \geq t) \leq CP(X \geq t)$, we have

$$\begin{aligned}
& E\ \exp\{t a_{ni}(X_i^{'} - EX_i^{'})\} \\
= \ & 1 + E\{\exp[t a_{ni}(X_i^{'} - EX_i^{'})] - 1 - t a_{ni}(X_i^{'} - EX_i^{'})\} \\
\leq \ & 1 + t^2 a_{ni}^2 g(2tq_n) E\{|X_i^{'} - EX_i^{'}|^r (2d_i)^{2-r}\} \\
\leq \ & \exp\{4Ct^2 a_{ni}^2 d_i^{2-r} g(2tq_n)\}. \qquad (12)
\end{aligned}$$

From (12) and Theorem 2.2 , it follows that

$$E(e^{tS_n^{'}}) \leq \exp\{4Ct^2 g(2tq_n) \sum_{i=1}^{k_n} a_{ni}^2 d_i^{2-r}\}. \qquad (13)$$

For all $\epsilon > 0$, set $t = (p+1)\varepsilon^{-1} \log k_n$. From (8) and (9), there exists a positive constant K_ε such that

$$\begin{aligned}
& P(S_n^{'} > \varepsilon) \\
\leq \ & e^{-\varepsilon t} E e^{tS_n^{'}} \\
\leq \ & \exp\{-(p+1)\log k_n \\
& +4C(p+1)^2 \varepsilon^{-2} g(K_\varepsilon)(\log k_n)^2 (\sum_{i=1}^{k_n} a_{ni}^2 d_i^{2-r})\} \\
\leq \ & \exp\{-p \log k_n\} = k_n^{-p}. \qquad (14)
\end{aligned}$$

From (6) and (13), it follows that

$$\overline{\lim}_{n \to \infty} S_n^{'} \leq 0 \quad \text{a.s.} \qquad (15)$$

Replacing X_i by $-X_i$, we get

$$\overline{\lim}_{n\to\infty}(-S_n^{'}) \leq 0. \quad \text{a.s.} \tag{16}$$

Consequently, by (15) and (16)

$$S_n^{'} \to 0 \quad \text{a.s.} \tag{17}$$

Now we will show that

$$S_n^{''} \to 0 \quad \text{a.s.} \tag{18}$$

If there exists $0 < K < \infty$ such that $d_n < K$, $n \geq 1$, then from (7) and $\{X_n\} \prec X$, we have $X \leq K$ a.s. for all $n \geq 1$. Replacing d_n by K we get $S_n^{''} = 0$ a.s. and, of course, $S_n^{''} \to 0$ a.s.

Now we assume that $d_n \uparrow \infty$. On the other hand, from the conditions about d_n there exists a function $d(x)$ on $(0, \infty)$ which is positive, continuous, increasing, $d(x)/x \downarrow$ and $d(n) = d_n$. Set $d^-(x) = \inf\{y > 0, d(y) \geq x\}$. It is easy to know that $d^-(x)/x \uparrow$ on $(0, \infty)$ and $d^-(X)$ is integrable. Thus we have

$$\begin{aligned} \frac{n}{d_n}\int_{\{|X_n| \geq d_n\}} |X_n| dp &< C\frac{n}{d_n}\int_{\{X \geq d_n\}} X dp \\ &\leq C\int_{\{X \geq d_n\}} d^-(X) dp = o(1). \end{aligned}$$

So there exist a sequence of positive constant $o(1)$ and $C > 0$ such that

$$\begin{aligned} (\sum_{i=1}^{k_n} a_{ni} EX_i^{''})^2 &\leq (\sum_{i=1}^{k_n} a_{ni}\epsilon_i d_i i^{-1})^2 \\ &\leq C(\sum_{i=1}^{k_n} a_{ni}^2 d_i^{2-r})(\sum_{i=1}^{k_n} \epsilon_i^2 i^{-1}) \\ &= o(1). \end{aligned} \tag{19}$$

From (9), noting that $d_n \uparrow$, there exists $C > 0$ such that

$$\sum_{i=1}^{k_n} a_{ni}^2 \leq C(\sum_{i=1}^{k_n} a_{ni}^2 d_i^{2-r}) = o(\frac{1}{\log k_n}). \tag{20}$$

From (20), we obtain

$$(\sum_{i=1}^{k_n} a_{ni} X_i^{''})^2 \leq (\sum_{i=1}^{k_n} a_{ni}^2)(\sum_{i=1}^{k_n} |X_i|^2 I(|X_i| \geq d_i)) \tag{21}$$

From (19) and (21), $S_n^{''} \to 0$ a.s., i.e., (11) holds.

(b) If $0 < r < 1$, let $S_n' = \sum_{i=1}^{k_n} a_{ni} X_i'$ and $S_n'' = \sum_{i=1}^{k_n} a_{ni} X_i''$, where $X_i' = (-d_i) \vee (X_i \wedge d_i)$ and $X_i'' = X_i - X_i'$. Obviously, $\sum_{i=1}^{k_n} a_{ni} X_i = S_n' + S_n''$. Set $g_1(x) = x^{-1}(e^x - 1)$, then $g_1(x)$ is an increasing function. Since $E|X_i|^r \leq CEX^r = C$ (where C comes from $\sup_n P(|X_n| \geq t) \leq CP(|X| \geq t)$), we have

$$\begin{aligned} E\exp(ta_{ni}X_i') &\leq 1 + Cta_{ni}d_i^{1-r} g_1(tq_n) \\ &\leq \exp\{Cta_{ni}d_i^{1-r} g_1(tq_n)\} \end{aligned}$$

and

$$E\exp(tS_n') \leq \exp\{Ctg_1(tq_n) \sum_{i=1}^{k_n} a_{ni} d_i^{1-r}\} \tag{22}$$

by Theorem 2.2. Let $t = (p+1)\varepsilon^{-1} \log\ k_n$. It follows as earlier method that

$$S_n' \to 0 \ \text{ a.s.}$$

On the other hand, via (10), there exists $C > 0$ such that

$$\sum_{i=1}^{k_n} a_{ni}^2 \leq C(\sum_{i=1}^{k_n} a_{ni} d_i^{1-r})^2 = o(1).$$

Consequently,

$$\begin{aligned} S_n''^2 &= |\sum_{i=1}^{k_n} a_{ni} X_i''|^2 \\ &\leq (\sum_{i=1}^{k_n} a_{ni}^2)(\sum_{i=1}^{k_n} |X_i|^2 I(|X_i| \geq d_i)) \to 0 \ \text{ a.s.} \end{aligned}$$

So (11) holds. The proof is complete. □

According to next theorem a slight strengthening of (8) permits a weakening of (9):

THEOREM 2.4. *Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables and stochastically dominated by a nonnegative random variable X, i.e., $X_n < X$, and let $\{a_{ni}, 1 \leq i < k_n < \infty,\ 1 \leq k_n \uparrow,\ n \geq 1\}$ be an array of constants. If $E|X|^r < \infty$, $1 \leq r \leq 2$, $EX = 0$ and (6) holds, then*

$$\max_{1 \leq i \leq k_n} |a_{ni}| = O(k_n^{-\frac{1}{r}} (\log k_n)^{-1}) \tag{23}$$

implies (11).

Proof. Choosing $d_i = i$ and letting

$$X_i' = (-i^{\frac{1}{r}}) \vee (X \wedge i^{\frac{1}{r}}) \text{ and } X_i'' = X_i - X_i',$$

the proof follcws the same pattern as that of Theorem 2.3 noting that (23)

$$\sum_{i=1}^{k_n} a_{ni}^2 \leq k_n^{\frac{r-2}{r}} (\log k_n)^{-2}.$$

$\square$

REMARK. If $\sum_{i=1}^{k_n} a_{ni}^2 i^{\frac{2}{r}-1} = o(\frac{1}{\log k_n})$, then (23) can be weakened to

$$\max_{1 \leq i \leq k_n} |a_{ni}| i^{\frac{1}{r}} = O(\frac{1}{\log k_n}).$$

Moreover, if $\sum_{i=1}^{k_n} a_{ni}^2 d_i^{2-r} = o(\frac{1}{\log k_n})$, where $d_n = O(n^{\frac{1}{r}})$ then (23) can be weakened to (8).

LEMMA 2.5. *Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$, $\sup_n |X_n| \leq C$ a.s. $(0 < C < \infty)$. If $\{a_{ni}, 1 \leq i \leq k_n \uparrow \infty, n \geq 1\}$ is a sequence of constants satisfying*

$$\text{(24)} \qquad \text{(i)} \max_{1 \leq i \leq k_n} |a_{ni}| = O(\frac{1}{k_n}),$$

(ii) *there exists $p > 0$ such that*

$$\text{(25)} \qquad \sum_{n=1}^{\infty} \frac{1}{k_n^p} < \infty,$$

then

$$\sum_{i=1}^{k_n} a_{ni} X_i \to 0 \text{ a.s.}$$

Proof. Assume $a_{ni} \geq 0$. From the inequality $e^x \leq 1 + x + \frac{1}{2}x^2 e^{|x|}$ and (24) it follows that, for all $t > 0$

$$\begin{aligned} E\exp\{t a_{ni} X_i\} &\leq 1 + E\{\frac{1}{2} t^2 a_{ni}^2 X_i^2 \exp(t a_{ni} |X_i|)\} \\ &\leq \exp\{C \frac{t^2}{k_n^2} \exp(C \frac{t}{k_n})\}. \end{aligned}$$

By Theorem 2.2 we get

$$E\exp\{t \sum_{i=1}^{k_n} a_{ni} X_i\} \leq \exp\{C \frac{t^2}{k_n} \exp(C \frac{t}{k_n})\}.$$

For all $\epsilon > 0$, set $t = p(\frac{\log k_n}{\epsilon})$. Noticing that $C\frac{t^2}{k_n}\exp(C\frac{t}{k_n})$ is bounded, it follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\{\sum_{i=1}^{k_n} a_{ni}X_i > \epsilon\} \\
\leq\ &\sum_{n=1}^{\infty} \exp\{-t\epsilon + C\frac{t^2}{k_n}\exp(C\frac{t}{k_n})\} \\
\leq\ &C\sum_{n=1}^{\infty} \exp(-p\ \log\ k_n) < \infty.
\end{aligned}$$

Consequently, we have

$$\overline{\lim_{n\to\infty}} \sum_{i=1}^{k_n} a_{ni}X_i \leq 0 \quad \text{a.s.}$$

Replacing X_i by $-X_i$, we obtain

$$\overline{\lim_{n\to\infty}} \sum_{i=1}^{k_n} a_{ni}(-X_i) \leq 0 \quad \text{a.s.}$$

So

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} a_{ni}X_i = 0 \quad \text{a.s.}$$

The proof is complete. □

THEOREM 2.6. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NOD random variables with $EX = 0$ and $\{a_{ni}, 1 \leq i \leq k_n, \uparrow \infty, n \geq 1\}$ an array of constants satisfying $\max_{1\leq i\leq k_n} |a_{ni}| = O(\frac{1}{k_n})$. If there exists a positive p such that $\sum_{n=1}^{\infty} \frac{1}{k_n^p} < \infty$, then $\sum_{i=1}^{k_n} a_{ni}X_i \to 0$ a.s.*

Proof. Assume $a_{ni} \geq 0$ and let $X_i^{'} = (-M)\bigvee(X_i \bigwedge M)$ and $X_i^{''} = X_i - X_i^{'}$. Then $\{X_i^{'} - EX_i^{'}\}$ satisfies the conditions in Lemma 2.5 and thus

$$\sum_{i=1}^{k_n} a_{ni}(X_i^{'} - EX_i^{'}) \to 0 \quad \text{a.s.} \tag{26}$$

On the other hand, since $\max_{1\le i\le k_n}|a_{ni}|=O(\frac{1}{k_n})$

$$\begin{aligned}&|\sum_{i=1}^{k_n}a_{ni}(X_i^{''}-EX_i^{''})|\\ \le\ &\frac{C}{k_n}\sum_{i=1}^{k_n}|X_i|I(|X_i|>M)+CE|X|I(|X|>M).\end{aligned}$$

Since NOD sequence is pairwise NQD, we get

$$\frac{1}{k_n}\sum_{i=1}^{k_n}|X_i|I(|X_i|>M)\to E|X|I(|X|>M)\ \text{ a.s.}$$

by Theorem 1 in [8]. Noticing that $E|X|I(|X|>M)\to 0$ *as* $M\to\infty$, we obtain

$$\sum_{i=1}^{k_n}a_{ni}(X_i^{''}-EX_i^{''})\to 0\ \text{ a.s.}\tag{27}$$

From (26) and (27), $\sum_{i=1}^{k_n}a_{ni}X_i\to 0$ a.s.

Theorem 5 in [1] is a special case of the above theorem. □

COROLLARY 2.7. *Suppose that $\{X,X_n,n\ge 1\}$ is a sequence of identically distributed and independent random variables with $EX=0$ and $\{a_{ni},1\le i\le n,n\ge 1\}$ is a triangular array of constants satisfying $\max_{1\le i\le n}|a_{ni}=O(\frac{1}{n})$. Then $\sum_{i=1}^{n}a_{ni}X_i\to 0$ a.s.*

3. Infinite double arrays

LEMMA 3.1. *Assume that $\{X_n,n\ge 1\}$ is a sequence of NOD random variables with $EX_n=0$ and $|X_n|\le d_n$, $n\ge 1$ and $\{a_{ni},i\ge 1,n\ge 1\}$ is an array of constants satisfying*

$$\sup_i d_i|a_{ni}|=o(\frac{1}{\log n}).\tag{28}$$

If one of the following conditions holds

$$\text{(i) } EX_n^2\le\sigma^2<\infty,\ n\ge 1\quad\text{and}\quad\sum_{i=1}^{\infty}a_{ni}^2=o(\frac{1}{\log n}),\tag{29}$$

$$\text{(ii) } E|X_n|\le\alpha<\infty,\ n\ge 1\quad\text{and}\quad\sum_{i=1}^{\infty}|a_{ni}|=o(1).\tag{30}$$

Then

$$S_n = \sum_{i=1}^{\infty} a_{ni} X_i \to 0 \text{ a.s..} \tag{31}$$

Proof. Assume $a_{ni} \geq 0$. Obviously. $E\sum_{i=1}^{\infty} a_{ni}|X_i| < \infty$ thus $S_n, n \geq 1$ are almost surely defined. From (28), there exist $0 < \gamma_n = o(1)$ such that $|a_{ni}X_i| \leq d_i a_{ni} \leq \frac{\gamma_n}{\log n}, n \geq 1, i \geq 1$.

If (i) holds, for all $t > 0$, we can show similarly to the display (12) that

$$\begin{aligned} E\exp\{ta_{ni}X_i\} &= 1 + t^2 a_{ni}^2 EX_i^2 g(ta_{ni}X_i) \\ &\leq \exp\{\sigma^2 t^2 a_{ni}^2 g(\frac{t\gamma_n}{\log n})\}. \end{aligned}$$

Since $\{X, X_n, n \geq 1\}$ are NOD, by applying Fatou's lemma we obtain

$$E\exp(t\sum_{i=1}^{\infty} a_{ni}X_i) \leq \exp\{\sigma^2 t^2 g(\frac{t\gamma_n}{\log n})\sum_{i=1}^{\infty} a_{ni}^2\}.$$

From (29), there exists $0 < \delta_n = o(1)$ such that $\sum_{i=1}^{\infty} a_{ni}^2 = \frac{\delta_n}{\log n}$. Set $t_n = \rho_n^{-1}\log n$, where $\rho_n = \max(\delta_n^{\frac{1}{2}}, \gamma_n^{\frac{1}{2}})$, necessarily $\rho_n = o(1)$. For all $\epsilon > 0$ and enough large n, we have

$$\begin{aligned} & P\{\sum_{i=1}^{\infty} a_{ni}X_i > \epsilon\} \\ \leq\ & \exp\{-t_n\epsilon + \sigma^2 t_n^2 g(\frac{t_n\gamma_n}{\log n})\sum_{i=1}^{\infty} a_{ni}^2\} \\ =\ & \exp\{-\frac{\epsilon \log n}{\rho_n}(1 + o(1))\} \leq \frac{1}{n^2}. \end{aligned}$$

So $\sum_{n=1}^{\infty} P\{\sum_{i=1}^{\infty} a_{ni}X_i > \epsilon\} < \infty$, i.e., $\overline{\lim}_{n\to\infty}\sum_{i=1}^{\infty} a_{ni}X_i \leq 0$ a.s.. In the same way we can show that $\overline{\lim}_{n\to\infty}\sum_{i=1}^{\infty} a_{ni}(-X_i) \leq 0$ a.s.. Thus (31) holds.

If (ii) holds, for all $t > 0$, we can show similarly to the display preceding (22) that

$$\begin{aligned} E\exp\{ta_{ni}X_i\} &= 1 + ta_{ni}EX_i g_1(ta_{ni}X_i) \\ &\leq \exp\{ta_{ni}EX_i g_1(ta_{ni}X_i)\} \\ &\leq \exp\{\alpha t g_1(\frac{t\gamma_n}{\log n})a_{ni}\}. \end{aligned}$$

Consequently,

$$E\exp(t\sum_{i=1}^{\infty} a_{ni}X_i) \le \exp\{\alpha t g_1(\frac{t\gamma_n}{\log n})\sum_{i=1}^{\infty} a_{ni}\}.$$

Set $t_n = \rho_n^{-1}\log n$ with $\rho_n = \gamma_n^{\frac{1}{2}}$, then for all $\epsilon > 0$,

$$P\{\sum_{i=1}^{\infty} a_{ni}X_i > \epsilon\} \le \frac{1}{n^2}$$

From this, (31) holds too. □

THEOREM 3.2. *Suppose that $\{X_n, n \ge 1\}$ is a sequence of NOD random variables with $EX_n = 0$, X is a nonnegative random variable such that $\{X_n\} \prec X$ and $\{a_{ni}, i \ge 1, n \ge 1\}$ is an array constants satisfying*

$$\sum_{i=1}^{\infty} a_{ni}^2 = o(\frac{1}{\log n}) \text{ and } \sup_{i\ge 1}|a_{ni}|i^{\frac{1}{2}} = o(\frac{1}{\log n}). \tag{32}$$

If for some $\epsilon > 0$,

$$EX^2(\log^+ X)^{\frac{1}{2}}[\log^+(\log^+ X)]^{\frac{1}{2}+\varepsilon} < \infty \tag{33}$$

then

$$\sum_{i=1}^{\infty} a_{ni}X_i \to 0 \text{ a.s..}$$

Proof. Assume $a_{ni} \ge 0$. Let $X_i^{'} = (-i^{\frac{1}{2}}) \vee (X_i \wedge i^{\frac{1}{2}})$ and $X_i^{''} = X_i - X_i^{'}$, then $|X_i^{'} - EX_i^{'}| \le 2i^{\frac{1}{2}}$ and $E(X_i^{'} - EX_i^{'})^2 \le EX_i^{'2} \le CEX^2 < \infty$. From Lemma 3.1,

$$\sum_{i=1}^{\infty} a_{ni}(X_i^{'} - EX_i^{'}) \to 0 \text{ a.s.} \tag{34}$$

From (33), $E^2XI(X \ge i^{\frac{1}{2}}) = O(\frac{1}{i\log i(\log\log i)^{1+2\epsilon}}), (i \to \infty)$. And so

$$\sum_{i=1}^{\infty} E^2XI(X \ge i^{\frac{1}{2}}) < \infty. \tag{35}$$

Obviously,

$$\sum_{i=1}^{\infty} P(|X_i| \ge i^{\frac{1}{2}}) \le C\sum_{i=1}^{\infty} P(X \ge i^{\frac{1}{2}}) \tag{36}$$

and

$$P(|X_i| \geq i^{\frac{1}{2}}, \text{i.o.}) = 0. \tag{37}$$

From(32), (35) and (37),

$$\begin{aligned} &[\sum_{i=1}^{\infty} a_{ni}(X_i^{''} - EX_i^{''})]^2 \\ \leq\ & 4(\sum_{i=1}^{\infty} a_{ni}^2) \sum_{i=1}^{\infty} (X_i{}^2 I(|X_i| \geq i^{\frac{1}{2}}) \\ & + CE^2 X I(|X_i| \geq i^{\frac{1}{2}}) \to 0 \text{ a.s.} \end{aligned} \tag{38}$$

From (34) and (38), $\sum_{i=1}^{\infty} a_{ni} X_i \to 0$ a.s.. □

THEOREM 3.3. *Suppose that* $\{X_n, n \geq 1\}$ *is a sequence of NOD random variables with* $EX_n = 0$, X *is a nonnegative random variable such that* $\{X_n\} \prec X$ *with* $EX < \infty$ *and* $\{a_{ni},\ i \geq 1, n \geq 1\}$ *is an array of constants satisfying*

$$\sum_{i=1}^{\infty} |a_{ni}| = o(1) \text{ and } \sup_{i \geq 1} i|a_{ni}| = o(\frac{1}{\log n}). \tag{39}$$

Then

$$\sum_{i=1}^{\infty} a_{ni} X_i \to 0 \text{ a.s..}$$

Proof. Assume $a_{ni} \geq 0$. Let $X_i^{'} = (-i) \vee (X_i \wedge i)$ and $X_i^{''} = X_i - X_i^{'}$. From Lemma 3.1,

$$\sum_{i=1}^{\infty} a_{ni}(X_i^{'} - EX_i^{'}) \to 0 \text{ a.s.} \tag{40}$$

Since $EX < \infty$ and $\{X_n\} \prec X$, we have

$$P\{|X_i| \geq i, \text{ i.o.}\} = 0. \tag{41}$$

From (39) and (41), we obtain

$$|\sum_{i=1}^{\infty} a_{ni} X_i^{''}| \leq \sum_{i=1}^{\infty} a_{ni} \sum_{i=1}^{\infty} |X_i| I(|X_i| \geq i) \to 0 \text{ a.s.} \tag{42}$$

Again from (39), we get

$$
\begin{aligned}
|\sum_{i=1}^{\infty} a_{ni} E X_i^{''}| &\leq (\sup_i E|X_i^{''}|) \sum_{i=1}^{\infty} a_{ni} \\
&\leq C(EX) \sum_{i=1}^{\infty} a_{ni} = o(1). \qquad (43)
\end{aligned}
$$

Finally, we obtain $\sum_{i=1}^{\infty} a_{ni} X_i \to 0$ a.s. from (40), (42) and (43). □

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