ON THE RANGE CLOSURE OF AN ELEMENTARY OPERATOR

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ABSTRACT. Let $A, B \in B(H)$ be Hilbert space contractions, and let $\Delta_{AB}$ be the elementary operator $\Delta_{AB} : X \to AXB - X$. A number of conditions which are equivalent to "$\Delta_{AB}$ has closed range" are proved.

1. Introduction

Given an infinite dimensional complex Hilbert space $H$, let $B(H)$ denote the algebra of operators (equivalently, bounded linear transformations) on $H$. For $A, B \in B(H)$, let $\Delta_{AB} \in B(B(H))$ be the elementary operator defined by

$$\Delta_{AB}(X) = AXB - X.$$ 

Let $\sigma(\Delta_{AB})$ and $\sigma_p(\Delta_{AB})$ denote, respectively, the spectrum and the point spectrum of $\Delta_{AB}$. It was proved in [5, Theorem 2] that if $A_1, A_2 \in B(H)$ are contractions for which $\lambda \in \sigma(A_i)$ and $|\lambda| = 1$ implies $\lambda \in \sigma_p(A_i)$ ($i = 1, 2$), then the range $\Delta_{A_1A_2}(B(H))$ of $\Delta_{A_1A_2}$ is closed. This paper dispenses with the hypothesis on the points in the boundary of the spectrum of the contractions $A_i$ to prove the following theorem. For a Banach space operator $T$, $T \in B(\mathcal{Y})$, let $\text{iso}\sigma(T)$ denote the isolated points of the spectrum of $T$, $\text{asc}(T)$ denote the ascent of $T$, $\text{dsc}(T)$ denote the descent of $T$, and let $H_0(T)$ denote the quasinilpotent part

$$H_0(T) = \{y \in \mathcal{Y} : \lim_{n \to \infty} ||T^n y||^{\frac{1}{n}} = 0\}$$

of $T$. The operator $T$ is semi-regular if $T(\mathcal{Y})$ is closed and $T^{-1}(0) \subseteq \bigcap_{n=1}^{\infty} T^n(\mathcal{Y})$; $T$ admits a generalized Kato decomposition, or GKD, if

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there exists a pair of $T$-invariant closed subspaces $(N, M)$ such that $\mathcal{Y} = N \oplus M$, $T|_M$ is semiregular and $T|_N$ is quasinilpotent. If in this restriction $T|_N$ is nilpotent, then $T$ is said to be *Kato type*. Let $\gamma(T)$, 
\[
\gamma(T) = \inf \{ \frac{||Ty||}{\text{dist}(y, T^{-1}(0))} : y \in \mathcal{Y} \setminus T^{-1}(0) \}
\]
(with the convention that $\gamma(T) = \infty$ if $T = 0$), denote the *reduced minimal modulus* of an operator $T$ [9, p. 203]. Let $\alpha(T) = \dim T^{-1}(0)$ and $\beta(T) = \dim(\mathcal{Y} \setminus T\mathcal{Y})$ denote, respectively, the *null deficiency* and the *image deficiency* of $T$, and let $\text{ind}(T) = \alpha(T) - \beta(T)$ denote the *index* of $T$.

**Theorem 1.1.** If $A, B \in B(H)$ are contractions such that $0 \in \sigma(D_{AB})$ then the following conditions are equivalent.

(i) $B(H) = D_{AB}^{-1}(0) \oplus D_{AB}(B(H))$.

(ii) $0$ is a pole of the resolvent of $D_{AB}$.

(iii) $0 \in \text{iso}(D_{AB})$ and $H_0(D_{AB}) = D_{AB}^{-1}(0)$.

(iv) $D_{AB}(B(H))$ (equivalently, $D_{AB}^*(B(H)^*)$) is closed.

(v) $\text{dsc}(D_{AB}) < \infty$.

(vi) $D_{AB}$ is Kato type.

(vii) $\gamma(D_{AB}) > 0$.

(viii) $\{0\} \neq D_{AB}^{-1}(0)$ is complemented by a $D_{AB}$ invariant closed subspace of $D_{AB}(B(H))$.

Furthermore: (a) If the contractions $A$ (and $B$) are such that the points $\lambda \in \text{iso}(A)$ (resp., $\lambda \in \text{iso}(B)$) with $|\lambda| = 1$ are eigenvalues of $A$ (resp., $B$), then $0$ is a pole of the resolvent of $D_{AB}$ if and only if the set \{$\alpha \in \sigma(A) : \alpha^{-1} \in \sigma(B)$\} is finite.

(b) If $0 < \alpha(D_{AB}) < \infty$, then $0$ is a pole of the resolvent of $D_{AB}$ if and only if $\text{ind}(D_{AB}^*) = -\text{ind}(D_{AB})$.

The equivalent conditions (i)–(viii) of the theorem answer [12, Question 2] and improve [5, Theorem 2]. The main tools that we use in the proof of Theorem 1.1 are the Nirsch-Schneider theorem [3, Theorem 10.10] and elements of "local spectral theory" (for which we refer the reader to the excellent monograph [9]). Our notation and terminology is explained below, and we prove Theorem 1.1 in Section 2.

The *ascent* of $T \in B(\mathcal{Y})$, $\text{asc}(T)$, is the least non-negative integer $n$ such that $T^{-n}(0) = T^{-(n+1)}(0)$ and the *descent* of $T$, $\text{dsc}(T)$, is the least non-negative integer $n$ such that $T^n(\mathcal{Y}) = T^{n+1}(\mathcal{Y})$. Clearly, the operator $T$ is injective if and only if $\text{asc}(T) = 0$ and $T$ is surjective if and only if $\text{dsc}(T) = 0$. In the following, we shall denote the open unit disc, the closed unit disc and the boundary of the closed unit disc in the
complex plane $\mathbb{C}$ by $\mathbf{D}$, $\overline{\mathbf{D}}$ and $\partial \mathbf{D}$, respectively. The numerical range $W(B(\mathcal{Y}), T)$ of $T \in B(\mathcal{Y})$ is the set
\[
\{ f(T) : f \in B(\mathcal{Y})^*, \|f\| = f(T) = 1 \},
\]
where $B(\mathcal{Y})^*$ denotes the (Banach space) dual of $B(\mathcal{Y})$. The numerical range of $T$ is the closed convex hull $\overline{\text{co}V(T)}$ of the spatial numerical range
\[
V(T) = \{ F(Ty) : F \in \mathcal{Y}^*, y \in \mathcal{Y}, \|F\| = \|y\| = F(y) = 1 \},
\]
of $T$ [3, Theorem 9.4]. If we denote the operator conjugate to $T \in B(\mathcal{Y})$ by $T^*$, then $\overline{\text{co}V(T)} = \overline{\text{co}V(T^*)}$ [3, Corollary 9.6(ii)]. Hence:

**Proposition 1.2.** $W(B(\mathcal{Y}), T) = W(B(\mathcal{Y}^*), T^*)$.

We say that $T$ is semi-Fredholm if $T(\mathcal{Y})$ is closed and either $\alpha(T) = \dim(T^{-1}(0))$ or $\beta(T) = \dim(\mathcal{Y}/T(\mathcal{Y}))$ is finite. If $T$ is semi-Fredholm, then the (Fredholm) index of $T$, $\text{ind}(T)$, is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$; $T$ is said to be Fredholm if $\text{ind}(T)$ is finite. The analytic core $K(T)$ of a Banach space operator $T \in B(\mathcal{Y})$ is defined by
\[
K(T) = \{ y \in \mathcal{Y} : \text{there exists a sequence } \{ y_n \} \subset \mathcal{Y} \text{ and } \delta > 0 \\
\text{for which } y = y_0, T(y_{n+1}) = y_n \text{ and } \|y_n\| \leq \delta^n \|y\| \\
\text{for all } n = 1, 2, \ldots \}.
\]
The subspaces $H_0(T)$ and $K(T)$ are generally non-closed, $TK(T) = K(T) \subseteq \bigcap_{n=1}^{\infty} T^n(\mathcal{Y})$, and $T^{-n}(0) \subseteq H_0(T)$ for all $n = 1, 2, \ldots$ (cf. [10]). Observe that if $0 \in \text{iso}(T)$, then $H_0(T)$ and $K(T)$ are closed.

**2. Proof of Theorem 1.1**

$(i) \iff (ii)$. The equivalence $\mathcal{Y} = T^p(0) \oplus T^p(\mathcal{Y}) \iff 0$ is a pole of the resolvent operator holds for every Banach space operator $T \in B(\mathcal{Y})$ [8, Proposition 50.2]. To prove $(ii) \iff (i)$, we prove that $\text{asc}(\triangle_{AB}) = \text{dsc}(\triangle_{AB}) \leq 1$. To achieve this we show that $0$ is in the boundary $\partial W(B(B(H)), \triangle_{AB})$ of the numerical range of $\triangle_{AB}$. An application of the Nirschl-Schneider theorem on the eigenvalues of a Banach space operator in the boundary of the numerical range of the operator [3, Theorem 10.10] will then prove that $\text{asc}(\triangle_{AB}) \leq 1$.

Let $L_A$ and $R_B (\in B(B(H)))$ denote, respectively, the operators of "left multiplication by $A$" and "right multiplication by $B$". Then the operators $A$ and $B$ being Hilbert space contractions it follows that
\[
W(B(B(H)), L_AR_B) \subseteq \overline{\mathbf{D}}
\]
(see [2, Theorem 5.2], [3] and [4]). Since (cf. [2])

\[ W(B(B(H)), \Delta_{AB}) = W(B(B(H)), L_AR_B - 1) = W(B(B(H)), L_AR_B) - 1, \]

it follows that

\[ W(B(B(H)), \Delta_{AB}) \subseteq \{ \lambda \in \mathbb{C} : |\lambda + 1| \leq 1 \}. \]

In particular,

\[ 0 \in \partial W(B(B(H)), \Delta_{AB}) \iff \text{asc}(\Delta_{AB}) \text{ and } \text{asc}(\Delta_{AB}^*) \leq 1. \]

(Observe that if a Banach space operator $T$ has finite ascent and descent, then the conjugate operator $T^*$ satisfies $\text{asc}(T^*) = \text{dsc}(T)$ and $\text{dsc}(T^*) = \text{asc}(T)$. Hence $\text{asc}(\Delta_{AB}^*) = \text{dsc}(\Delta_{AB}^*) \leq 1$, which implies that if (i) holds then 0 is a simple pole of the resolvent of $\Delta_{AB}^*$. Hence 0 is an eigenvalue of both $\Delta_{AB}$ and $\Delta_{AB}^*$.)

(ii) $\iff$ (iii). Evidently, (ii) $\implies$ (iii). For the reverse implication, we observe that if $0 \in \text{iso}(\Delta_{AB})$, then $B(H) = H_0(\Delta_{AB}) \oplus K(\Delta_{AB})$. Thus, if $H_0(\Delta_{AB}) = \Delta_{AB}^{-1}(0)$, then

\[ B(H) = \Delta_{AB}^{-1}(0) \oplus K(\Delta_{AB}) \]
\[ \implies \Delta_{AB}(B(H)) = 0 \oplus \Delta_{AB}(K(\Delta_{AB})) = K(\Delta_{AB}) \]
\[ \implies B(H) = \Delta_{AB}^{-1}(0) \oplus \Delta_{AB}(B(H)). \]

(iv) $\iff$ (i). For an arbitrary subset $M$ of the Banach space $\mathcal{Y}$, let

\[ M^\perp = \{ \phi \in \mathcal{Y}^* : \phi(y) = 0 \text{ for all } y \in M \}, \]

denote the annihilator of $M$ in $\mathcal{Y}^*$. Recall that if $T \in B(\mathcal{Y})$, then $T(\mathcal{Y})^\perp = T^{-1}(0)$, and if $T(\mathcal{Y})$ is closed, then $T^{-1}(0)^\perp = T^*(\mathcal{Y}^*)$.

As we saw above, $\text{asc}(\Delta_{AB})$ and $\text{asc}(\Delta^*_{AB})$ are both less than or equal to 1; hence $\Delta_{AB}^{-1}(0) \cap \Delta_{AB}(B(H)) = \{0\} = \Delta^*_{AB}^{-1}(0) \cap \Delta^*_{AB}(B(H)^*)$ [9, Lemma 4.10.1]. If $\Delta_{AB}(B(H))$ is closed, then $\text{asc}(\Delta_{AB}) \leq 1$ implies $\Delta_{AB}^{-1}(0) + \Delta_{AB}(B(H))^\perp$ is closed [9, Proposition 4.10.4]. We have:

\[ \{\Delta_{AB}^{-1}(0) + \Delta_{AB}(B(H))\}^\perp = \Delta_{AB}(B(H))^\perp \cap \Delta_{AB}^{-1}(0)^\perp = \Delta^*_{AB}^{-1}(0) \cap \Delta^*_{AB}(B(H)^*) = \{0\}. \]

Hence, $\text{dsc}(\Delta_{AB}) \leq 1$, which implies that $\text{asc}(\Delta_{AB}) = \text{dsc}(\Delta_{AB}) \leq 1 \implies (i)$. Evidently, (i) $\implies$ (iv).

(v) $\iff$ (i). Obvious, since $\text{asc}(\Delta_{AB}) \leq 1$.

(vi) $\iff$ (i). The implication (i) $\implies$ (vi) is obvious. For the implication (vi) $\implies$ (i), we observe that both $\text{asc}(\Delta_{AB})$ and $\text{asc}(\Delta^*_{AB})$ are
finite. If $\triangle_{AB}$ is Kato type, then an application of [1, Theorem 2.9] shows that $\text{dsc}(\triangle_{AB}) \leq 1$.

$(iv) \iff (viii)$. Evident (see [9, p. 203]).

$(viii) \iff (iii)$. Observe that if $(viii)$ holds, then $B(H) = \triangle_{AB}^{-1}(0) \oplus M$ for some closed subspace $M$ of $\triangle_{AB}(B(H))$. Consequently,

$$\triangle_{AB}(B(H)) = 0 \oplus \triangle_{AB}(M) \subseteq M \subseteq \triangle_{AB}(B(H)).$$

Hence $B(H) = \triangle_{AB}^{-1}(0) \oplus \triangle_{AB}(B(H)) \implies (iii)$. The reverse implication $(iii) \implies (viii)$ being obvious (from the proof of $(iii) \implies (ii)$), the equivalence follows.

To complete the proof of the theorem we now consider $(a)$ and $(b)$. To prove $(a)$, we observe that if $0$ is a pole of the resolvent of $\triangle_{AB}$, then $0 \in \text{iso}(\triangle_{AB})$. Recall [7] that $\sigma(\triangle_{AB}) = \{\alpha \beta - 1 : \alpha \in \sigma(A), \beta \in \sigma(B)\}$. Hence, if $0 \in \text{iso}(\triangle_{AB})$, then the set $\{\alpha \in \sigma(A) : \alpha^{-1} \in \sigma(B)\}$ is finite. Conversely, if $0 \in \text{iso}(\triangle_{AB})$, and the isolated points of $\sigma(A)$ and $\sigma(B)$ in $\partial D$ are eigenvalues of $A$ and $B$ (respectively), then the argument of the proof of Theorem 2 of [5] implies that $H_0(\triangle_{AB}) = \triangle_{AB}^{-1}(0)$ (i.e., $(iii)$ is satisfied), which implies that $0$ is a pole of the resolvent of $\triangle_{AB}$.

To prove $(b)$, we start by observing that if $0$ is a pole of the resolvent of $\triangle_{AB}$, then $\text{asc}(\triangle_{AB}) = \text{dsc}(\triangle_{AB}) = \text{asc}(\triangle_{AB}^*) = \text{dsc}(\triangle_{AB}^*) \leq 1$. Since $\text{asc}(\triangle_{AB}) < \infty \implies \beta(\triangle_{AB}) \geq \alpha(\triangle_{AB})$ and $\text{dsc}(\triangle_{AB}) < \infty \implies \beta(\triangle_{AB}) \leq \alpha(\triangle_{AB})$ [8, Proposition 38.5], our hypothesis $0 < \alpha(\triangle_{AB}) < \infty$ implies that $\triangle_{AB}$ is Fredholm. Hence $\text{ind}(\triangle_{AB}^*) = -\text{ind}(\triangle_{AB})$.

Conversely, the implications $\text{asc}(\triangle_{AB}) \leq 1 \implies \text{ind}(\triangle_{AB}) \leq 0$ and $\text{asc}(\triangle_{AB}^*) \leq 1 \implies \text{ind}(\triangle_{AB}^*) = -\text{ind}(\triangle_{AB}) \leq 0$ imply that $\alpha(\triangle_{AB}) = \beta(\triangle_{AB}) < \infty$. Hence $\triangle_{AB}$ is Fredholm of index $0$. But then, see [1, Corollary 2.10], $\text{asc}(\triangle_{AB}) = \text{dsc}(\triangle_{AB}) \leq 1 \implies 0$ is a pole of the resolvent of $\triangle_{AB}$.

**Remark.** If $M$ and $N$ are subspaces of a Banach space $\mathcal{V}$, then we say that $M$ is orthogonal to $N$ (in the sense of Garret Birkhoff - see [6, Page 93]), denoted $M \perp N$, if $||m|| \leq ||m + n||$ for all $m \in M$ and $n \in N$. This asymmetric definition of orthogonality agrees with the usual definition of orthogonality in the case in which $\mathcal{V}$ is a Hilbert space. Observe, from the proof of Theorem 1.1, that if $0 \in \sigma_p(\triangle_{AB})$, then $0 \in \partial W(B(B(H)), \triangle_{AB})$, which implies by a result of Sinclair [11, Proposition 1] that $\triangle_{AB}^{-1}(0) \perp \triangle_{AB}(B(H))$. Thus, if $\mu \in \sigma_p(\triangle_{AB}) \cap \partial W(B(B(H)), \triangle_{AB})$, then $(\triangle_{AB} - \mu)^{-1}(0) \perp (\triangle_{AB} - \mu)(B(H))$. For distinct eigenvalues $\mu_1$ and $\mu_2 \in \partial W(B(B(H)), \triangle_{AB})$, $A$ and $B \in B(H)$ are some operators, we have the following.
Proposition 2.1. If \( \mu_i \in \sigma_p(\Delta_{AB}) \cap \partial W(B(B(H)), \Delta_{AB}), i = 1, 2 \) and \( \mu_1 \neq \mu_2 \), then \( (\Delta_{AB} - \mu_1)^{-1}(0) \) and \( (\Delta_{AB} - \mu_2)^{-1}(0) \) are mutually orthogonal.

Proof. The hypothesis \( \mu_1 \in \sigma_p(\Delta_{AB}) \cap \partial W(B(B(H)), \Delta_{AB}) \) implies \( (\Delta_{AB} - \mu_1)^{-1}(0) \perp (\Delta_{AB} - \mu_1)(B(H)) \). Let \( \Delta_{AB}T = \mu_2T; T \in B(H) \).
Since \( (\Delta_{AB} - \mu_1)(T) = (\Delta_{AB} - \mu_2)(T) + (\mu_2 - \mu_1)T \), \( T \in (\Delta_{AB} - \mu_1)(B(H)) \implies (\Delta_{AB} - \mu_1)^{-1}(0) \perp (\Delta_{AB} - \mu_2)^{-1}(0) \).
Reversing the roles of \( \mu_1 \) and \( \mu_2 \), we have \( (\Delta_{AB} - \mu_2)^{-1}(0) \perp (\Delta_{AB} - \mu_1)^{-1}(0) \), and the proof is complete. \( \square \)

References


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