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JENSEN TYPE QUADRATIC-QUADRATIC MAPPING IN BANACH SPACES

CHOONKIL PARK, SEONG-KI HONG, AND MYOUNG-JUNG KIM

ABSTRACT. Let X, Y be vector spaces. It is shown that if an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)+f\left(\frac{x-y}{2}+z\right) + f\left(\frac{x-y}{2}-z\right) = f(x)+f(y)+4f(z) \tag{0.1}$$

for all $x, y, z \in X$, then the mapping $f : X \to Y$ is quadratic.
Furthermore, we prove the Cauchy–Rassias stability of the functional equation (0.1) in Banach spaces.

1. Introduction

In 1940, S. M. Ulam [19] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Hyers [5] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \epsilon$$

for all $x \in X$.

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Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0,1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Th. M. Rassias [10] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all $x \in X$. Găvruta [4] generalized the Rassias' result.

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation.* In particular, every solution of the quadratic functional equation is said to be a *quadratic function.* A Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [17] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [3], Czerwik proved the Cauchy–Rassias stability of the quadratic functional equation. Several functional equations have been investigated in [1] and [6]–[18].

In this paper, we solve the functional equation (0.1), and prove the Cauchy–Rassias stability of the functional equation (0.1) in Banach spaces.

2. Jensen type quadratic-quadratic mapping in Banach spaces

LEMMA 2.1. *Let X and Y be vector spaces. If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

(2.1)

$$f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x+y}{2}-z\right) + f\left(\frac{x-y}{2}+z\right) + f\left(\frac{x-y}{2}-z\right)$$
$$= f(x) + f(y) + 4f(z)$$

for all $x, y, z \in X$, then the mapping $f : X \to Y$ is quadratic.

Proof. Letting $x = y$ in (2.1), we get

$$f(x+z)+f(x-z)+f(z)+f(-z)=2f(x)+4f(z)$$

for all $x, z \in X$. Since $f(-z)=f(z)$,

$$f(x+z)+f(x-z)=2f(x)+2f(z)$$

for all $x, z \in X$. So the even mapping $f : X \to Y$ is quadratic. □

The mapping $f : X \to Y$ given in the statement of Lemma 2.1 is called a *Jensen type quadratic-quadratic mapping.* Putting $z = 0$ in (2.1), we get the Jensen type quadratic mapping $2f(\frac{x+y}{2})+2f(\frac{x-y}{2}) = f(x)+f(y)$, and putting $x = y$ in (2.1), we get the quadratic mapping $f(x+z)+f(x-z)=2f(x)+2f(z)$.

From now on, assume that X is a normed vector space with norm $\|\cdot\|$ and that Y is a Banach space with norm $\|\cdot\|$.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y,z) := f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)+f\left(\frac{x-y}{2}+z\right) \\ +f\left(\frac{x-y}{2}-z\right)-f(x)-f(y)-4f(z) \end{aligned}$$

for all $x, y, z \in X$.

THEOREM 2.2. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^3 \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x,y,z) : = \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\right) < \infty, \tag{2.2}$$

$$\|Df(x,y,z)\| \le \varphi(x,y,z) \tag{2.3}$$

for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \le \frac{1}{4}\widetilde{\varphi}(x,x,x) \tag{2.4}$$

for all $x \in X$.

Proof. Letting $x = y = z$ in (2.3), we get

$$\|f(2x) - 4f(x)\| \leq \varphi(x, x, x) \tag{2.5}$$

for all $x \in X$. So

$$\|f(x) - 4f(\frac{x}{2})\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\left\|4^l f(\frac{x}{2^l}) - 4^m f(\frac{x}{2^m})\right\| \leq \sum_{j=l+1}^{m} 4^{j-1}\varphi\left(\frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}\right) \tag{2.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.2) and (2.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$.

By (2.3) and (2.2),

$$\begin{aligned} \|DQ(x, y, z)\| &= \lim_{n\to\infty} 4^n \left\|Df(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n})\right\| \\ &\leq \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \end{aligned}$$

for all $x, y, z \in X$. So $DQ(x, y, z) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is a Jensen type quadratic-quadratic mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get the inequality (2.4).

Now, let $Q' : X \to Y$ be another Jensen type quadratic-quadratic mapping satisfying (2.4). Then we have

$$\begin{aligned} \|Q(x) - Q'(x)\| &= 4^n \left\|Q(\frac{x}{2^n}) - Q'(\frac{x}{2^n})\right\| \\ &\leq 4^n \left(\left\|Q(\frac{x}{2^n}) - f(\frac{x}{2^n})\right\| + \|Q'(\frac{x}{2^n}) - f(\frac{x}{2^n})\|\right) \\ &\leq \frac{2 \cdot 4^n}{4} \widetilde{\varphi}\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right), \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q. $\square$

COROLLARY 2.3. *Let p and θ be positive real numbers with $p > 2$, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\|Df(x,y,z)\| \le \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{3\theta}{2^p - 4}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x,y,z) = \theta(||x||^p + ||y||^p + ||z||^p)$, and apply Theorem 2.2. □

THEOREM 2.4. *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^3 \to [0, \infty)$ satisfying (2.3) such that*

$$\widetilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y, 2^j z) < \infty \tag{2.7}$$

for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{4}\widetilde{\varphi}(x,x,x) \tag{2.8}$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$\|f(x) - \frac{1}{4}f(2x)\| \le \frac{1}{4}\varphi(x,x,x)$$

for all $x \in X$. Hence

$$\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\| \le \sum_{j=l}^{m-1} \frac{1}{4^{j+1}}\varphi(2^j x, 2^j x, 2^j x) \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) and (2.9) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$.

By (2.7) and (2.3),

$$\begin{aligned}\|DQ(x,y,z)\| &= \lim_{n\to\infty} \frac{1}{4^n}\|Df(2^n x, 2^n y, 2^n z)\| \\ &\le \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y, 2^n z) = 0\end{aligned}$$

for all $x, y, z \in X$. So $DQ(x, y, z) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is a Jensen type quadratic-quadratic mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get the inequality (2.8).

The rest of the proof is similar to the proof of Theorem 2.2. □

COROLLARY 2.5. *Let p and θ be positive real numbers with $p < 2$, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\|Df(x,y,z)\| \le \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{3\theta}{4 - 2^p}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y, z) = \theta(||x||^p + ||y||^p + ||z||^p)$, and apply Theorem 2.4. □

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Choonkil Park, Department of Mathematics, Hanyang University, Seoul 133-791, Korea
E-mail: baak@hanyang.ac.kr

Seong-Ki Hong and Myoung-Jung Kim, Department of Mathematics, Chungnam National University, Daejeon 305-764, Korea
E-mail: mrhongsk@hanmail.net
mjkim@math.cnu.ac.kr