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FUGLEDE-PUTNAM THEOREM FOR p-HYPONORMAL OR CLASS $\mathcal{Y}$ OPERATORS

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ABSTRACT. We say operators A, B on Hilbert space satisfy Fuglede-Putnam theorem if $AX = XB$ for some X implies $A^*X = XB^*$. We show that if either (1) A is p-hyponormal and B^* is a class $\mathcal{Y}$ operator or (2) A is a class $\mathcal{Y}$ operator and B^* is p-hyponormal, then A, B satisfy Fuglede-Putnam theorem.

1. Introduction

Our aim is to extend the Fuglede-Putnam theorem ([4], [7]). Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces and $B(\mathcal{H}), B(\mathcal{K})$ the algebras of all bounded linear operators on $\mathcal{H}, \mathcal{K}$. The familiar Fuglede-Putnam theorem is as follows:

THEOREM 1 (Fuglede-Putnam [4], [7]). *If $A \in B(\mathcal{H}), B \in B(\mathcal{K})$ be normal and $AX = XB$ for some $X \in B(\mathcal{K}, \mathcal{H})$, then $A^*X = XB^*$.*

Many authors have extented this theorem for several classes of operators, for examples [3], [5], [6], [10], [13], [15], [17]. We say operators A, B satisfy Fuglede-Putnam theorem if $AX = XB$ implies $A^*X = XB^*$. The aim of this paper is to show that if either (1) A is p-hyponormal and B^* is a class $\mathcal{Y}$ operator or (2) A is a class $\mathcal{Y}$ operator and B^* is p-hyponormal, then A, B satisfy Fuglede-Putnam theorem. We remark that B. P. Duggal [3] proved if A, B^* are p-hyponormal operators, then A, B satisfy Fugled-Putnam theorem, and A. Uchiyama and T. Yoshino [15] proved if A, B^* are class $\mathcal{Y}$ operators, then A, B satisfy Fugled-Putnam theorem.

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An operator $A \in B(\mathcal{H})$ is said to be p-hyponormal if $(A^*A)^p \geq (AA^*)^p$, where $p > 0$. This definition is due to Aluthge [1] and many authors studied interesting properties of p-hyponormal operators by using Aluthge transform (see [1], [6]). A is said to be a class $\mathcal{Y}_\alpha$ operator for $\alpha \geq 1$ (or $A \in \mathcal{Y}_\alpha$) if there exists a positive number k_α such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2 (A-\lambda)^*(A-\lambda) \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \leq \alpha \leq \beta$. Let $\mathcal{Y} = \cup_{1\leq\alpha}\mathcal{Y}_\alpha$. We remark that a class $\mathcal{Y}_1$ operator A is M-hyponormal, i.e., there exists a positive number M such that

$$(A-\lambda)(A-\lambda)^* \leq M^2(A-\lambda)^*(A-\lambda) \quad \text{for all} \quad \lambda \in \mathbb{C},$$

and M-hyponormal operators are class $\mathcal{Y}_2$ operators (see [15]). A is said to be dominant if for any $\lambda \in \mathbb{C}$ there exists a positive number M_λ such that

$$(A-\lambda)(A-\lambda)^* \leq M_\lambda^2(A-\lambda)^*(A-\lambda).$$

It is obvious that M-hyponormal operators are dominant, but the converse does not hold. Let $\{f_n\}_{n=-\infty}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Define $Tf_n = 2^{-|n|}f_{n+1}$. It is known that T is a dominant operator which is not a class $\mathcal{Y}$ operator. (Hence T is not M-hyponormal.) We remark T is not p-hyponormal, as $\langle (T^*T)^p f_1, f_1\rangle = 4^{-p} < 1 = \langle (TT^*)^p f_1, f_1\rangle$ (see [11], [15]). Let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Define $Sf_1 = f_2, Sf_2 = 2f_3, Sf_n = f_{n+1}$ for $n = 3, 4, \ldots$. Wadhwa [16] proved S is M-hyponormal, hence S is a class $\mathcal{Y}$ operator. But S is not p-hyponormal for any $0 < p$, as $\langle (S^*S)^p f_3, f_3\rangle = 1 < 2^p = \langle (SS^*)^p f_3, f_3\rangle$. However it is not known that there exists a p-hyponormal operator which is not a class $\mathcal{Y}$ operator. Also, it is not known that there exists a class $\mathcal{Y}$ operator which is not dominant.

2. Results

We will recall some known results which will be used in the sequel.

LEMMA 2. (Uchiyama and Yoshino [15]) *Let $A \in B(\mathcal{H})$ be a class $\mathcal{Y}$ operator and $\mathcal{M} \subset \mathcal{H}$ invariant under A. If $A|_\mathcal{M}$ is normal, then $\mathcal{M}$ reduces A.*

LEMMA 3 (Uchiyama [14]). *Let $A \in B(\mathcal{H})$ be p-hyponormal and $\mathcal{M} \subset \mathcal{H}$ be invariant under A. If $A|_\mathcal{M}$ is normal, then $\mathcal{M}$ reduces A.*

LEMMA 4 (Stampfli and Wadhwa [11]). *Let $A \in B(\mathcal{H})$ be dominant. Let $\delta \subset \mathbb{C}$ be closed. If there exists a bounded function $f : \mathbb{C} \setminus \delta \to \mathcal{H}$ such that $(A-\lambda)f(\lambda) = x \neq 0$ for some $x \in \mathcal{H}$, then there exists an analytic function $g : \mathbb{C} \setminus \delta \to \mathcal{H}$ such that $(A-\lambda)g(\lambda) = x$.*

REMARK. In [11], the authors assert f is analytic. But they use Putnam's result [9], i.e., if $A = \int \lambda dE(\lambda)$ is normal, then

$$\bigcap\{(A-\lambda)\mathcal{H} | \lambda \in \mathbb{C} \setminus \delta\} = E(\delta)\mathcal{H}$$
$$= \{x \in \mathcal{H} | \exists \text{ analytic } g : \mathbb{C} \setminus \delta \to \mathcal{H} \text{ such that } (A-\lambda)g(\lambda) = x\}.$$

Hence we must substitute a bounded function f by an analytic function g. If A is pure, i.e., A has no-nonzero reducing subspace $\mathcal{M}$ such that $A|_{\mathcal{M}}$ is normal, then $\ker A = \{0\}$ as $\ker A \subset \ker A^*$. Hence $f = g$. This is pointed by Professor F. Hiai.

The following result is due to Takahashi [12]. We denote by [ran A] the closure of the range of A.

LEMMA 5 (Takahashi [12]). *Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$. Then the following assertions are equivalent.*

(1) A, B satisfy Fugled-Putnam theorem.

(2) If $AC = CB$ for some operator $C \in B(\mathcal{K}, \mathcal{H})$, then [ran C] *reduces A, $(\ker C)^{\perp}$ reduces B, and $A|_{[\text{ran } C]}, B|_{(\ker C)^{\perp}}$ are normal.*

REMARK. In (2), $C_1 : (\ker C)^{\perp} \ni x \to Cx \in$ [ran C] is a quasi-affinity (i.e., C_1 is injective and has dense range) such that $A|_{[\text{ran } C]}C_1 = C_1 B|_{(\ker C)^{\perp}}$. Then $A|_{[\text{ran } C]}, B|_{(\ker C)^{\perp}}$ are unitarily equivalent normal operators by a corollary of the Fuglede-Putnam theorem (see Theorem 1.6.4 of [8] and its proof).

LEMMA 6. *Let $A \in B(\mathcal{H})$ be an injective p-hyponormal operator and $B^* \in B(\mathcal{K})$ be a class $\mathcal{Y}$ operator. If $AC = CB$ for some operator $C \in B(\mathcal{K}, \mathcal{H})$, then $A^*C = CB^*$. Moreover,* [ran C] *reduces A, $(\ker C)^{\perp}$ reduces B, and $A|_{[\text{ran } C]}, B|_{(\ker C)^{\perp}}$ are unitarily equivalent normal operators.*

Proof. (Case $1/2 \leq p \leq 1$) Since B^* is class $\mathcal{Y}$, there exist positive numbers α and k_α such that

$$|BB^* - B^*B|^{\alpha} \leq k_\alpha^2 (B-\lambda)(B-\lambda)^* \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

Hence for $x \in |BB^* - B^*B|^{\alpha/2}\mathcal{K}$ there exists a bounded function $f : \mathbb{C} \to \mathcal{K}$ such that

$$(B-\lambda)f(\lambda) = x \quad \text{for all} \quad \lambda \in \mathbb{C}$$

by [2]. Let $A = U|A|$ be the polar decomposition of A and define its Aluthge transform by $\tilde{A} = |A|^{1/2}U|A|^{1/2}$. Then $\tilde{A}$ is hyponormal by [1] (the author assumed U is unitary, however this assumption is not necessary.) Then

$$\begin{aligned}(\tilde{A} - \lambda)|A|^{1/2}Cf(\lambda) &= |A|^{1/2}(A - \lambda)Cf(\lambda) \\ &= |A|^{1/2}C(B - \lambda)f(\lambda) = |A|^{1/2}Cx\end{aligned}$$

for all $\lambda \in \mathbb{C}$.

We assert $|A|^{1/2}Cx = 0$. Because if $|A|^{1/2}Cx \neq 0$, there exists an analytic function $g : \mathbb{C} \to \mathcal{H}$ such that $(\tilde{A} - \lambda)g(\lambda) = |A|^{1/2}Cx$ by Lemma 4. Since

$$g(\lambda) = (\tilde{A} - \lambda)^{-1}|A|^{1/2}Cx \to 0 \text{ as } \lambda \to \infty,$$

we have $g(\lambda) = 0$, and hence $|A|^{1/2}Cx = 0$. This is a contradiction.

Then

$$|A|^{1/2}C|BB^* - B^*B|^{\alpha/2}\mathcal{K} = \{0\}.$$

Since $\ker A = \ker |A| = \{0\}$, we have

$$C(BB^* - B^*B) = 0.$$

Since $[\text{ran } C]$ is invariant under A and $(\ker C)^{\perp}$ is invariant under B^*, we can write

$$A = \begin{pmatrix} A_1 & S \\ 0 & A_2 \end{pmatrix} \text{ on } \mathcal{H} = [\text{ran } C] \oplus [\text{ran } C]^{\perp},$$

$$B = \begin{pmatrix} B_1 & 0 \\ T & B_2 \end{pmatrix} \text{ on } \mathcal{K} = (\ker C)^{\perp} \oplus (\ker C),$$

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} : (\ker C)^{\perp} \oplus (\ker C) \to [\text{ran } C] \oplus [\text{ran } C]^{\perp}.$$

Then

$$\begin{aligned}0 &= C(BB^* - B^*B) \\ &= \begin{pmatrix} C_1(B_1B_1^* - B_1^*B_1 - T^*T) & C_1(B_1T^* - T^*B_2) \\ 0 & 0 \end{pmatrix}\end{aligned}$$

and

$$C_1(B_1B_1^* - B_1^*B_1 - T^*T) = 0.$$

Since C_1 is injective and has dense range,

$$B_1B_1^* - B_1^*B_1 - T^*T = 0$$

and

$$B_1B_1^* = B_1^*B_1 + T^*T \geq B_1^*B_1.$$

This implies B_1^* is hyponormal. Since $AC = CB$, we have

$$A_1C_1 = C_1B_1$$

where A_1 is p-hyponormal by [14]. Hence A_1, B_1 are normal and

$$A_1^*C_1 = C_1B_1^*$$

by [3]. Then $S = 0$ by Lemma 3 and $T = 0$ by Lemma 2. Hence

$$A^*C = \begin{pmatrix} A_1^*C_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1B_1^* & 0 \\ 0 & 0 \end{pmatrix} = CB^*.$$

Hence $A|_{[\text{ran } C]}, B|_{(\ker C)^\perp}$ are normal by Lemma 5 and unitarily equivalent by its remark.

(Case $0 < p < 1/2$) Let $A = U|A|$ be the polar decomposition of A and define its Aluthge transform by $\tilde{A} = |A|^{1/2}U|A|^{1/2}$. Then $\tilde{A}$ is $(p + 1/2)$-hyponormal by [1] and

$$\tilde{A}|A|^{1/2}C = |A|^{1/2}AC = |A|^{1/2}CB.$$

Let $\tilde{A} = V|\tilde{A}|$ be the polar decomposition and $\hat{A} = |\tilde{A}|^{1/2}V|\tilde{A}|^{1/2}$. Then $\hat{A}$ is hyponormal and

$$\hat{A}|\tilde{A}|^{1/2}|A|^{1/2}C = |\tilde{A}|^{1/2}|A|^{1/2}CB.$$

Since $\sigma_p(\tilde{A}) = \sigma_p(A) = \emptyset$, we have $C(BB^* - B^*B) = 0$ by an similar arguments in the case $1/2 \leq p \leq 1$. The rest is the same to the case $1/2 \leq p \leq 1$. □

THEOREM 7. *Let $A \in B(\mathcal{H})$ and $B^* \in B(\mathcal{K})$. If either* (1) *A is p-hyponormal and B^* is a class $\mathcal{Y}$ operator or* (2) *A is a class $\mathcal{Y}$ operator and B^* is p-hyponormal, then $AC = CB$ for some operator $C \in B(\mathcal{K}, \mathcal{H})$ implies $A^*C = CB^*$. Moreover,* [ran C] *reduces A,* $(\ker C)^\perp$ *reduces B, and $A|_{[\text{ran } C]}, B|_{(\ker C)^\perp}$ are unitarily equivalent normal operators.*

Proof. (1). Decompose A into normal part A_1 and pure part A_2 as

$$A = A_1 \oplus A_2 \quad \text{on } \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2,$$

and write

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} : \mathcal{K} \to \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Since $\ker A_2 \subset \ker A_2^*$ and A_2 is pure, A_2 is injective. $AC = CB$ implies

$$\begin{pmatrix} A_1C_1 \\ A_2C_2 \end{pmatrix} = \begin{pmatrix} C_1B \\ C_2B \end{pmatrix}.$$

Hence

$$A^*C = \begin{pmatrix} A_1^*C_1 \\ A_2^*C_2 \end{pmatrix} = \begin{pmatrix} C_1B^* \\ C_2B^* \end{pmatrix} = CB^*$$

by [3] and Lemma 6. The rest follows from Lemma 5 and its remark.

(2). Since $AC = CB$, we have $B^*C^* = C^*A^*$. Hence $BC^* = B^{**}C^* = C^*A^{**} = C^*A$ by (1) and $A^*C = CB^*$. The rest follows from Lemma 5 and its remark. □

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