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ITERATIVE ALGORITHMS WITH ERRORS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

Jong Soo Jung

Abstract. The iterative algorithms with errors for nonexpansive mappings are investigated in Banach spaces. Strong convergence theorems for these algorithms are obtained. Our results improve the corresponding results in [5, 13–15, 23, 27–29, 32] as well as those in [1, 16, 19, 26] in framework of a Hilbert space.

1. Introduction

Let E be a real Banach space, C a nonempty closed convex subset of E, and $T_1, \dots, T_N$ nonexpansive mappings from C into itself (recall that a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x,\ y \in C$).

We consider the iterative algorithm: for a positive integer N, nonexpansive mappings $T_1,\ T_2, \dots, T_N$, $u, x_0 \in C$, and $\lambda_n \in (0,1)$,

$$x_{n+1} = \lambda_n u + (1 - \lambda_n) T_{n+1} x_n, \quad n \geq 0, \tag{1}$$

where $T_n := T_{n \bmod N}$, $N > 1$. The convergence of the iterative algorithm (1) has been investigated by many author see, for example, Browder [2], Cho et al. [5], Halpern [12], Lions [16], Reich [20, 21], Shioji and Takahashi [23], Wittmann [26], Xu [27–29] in the case of $N = 1$ and Bauschke [1], Jung [13], Jung and Kim [15], Jung et al. [14], O'Hara et al. [19], Shimizu and Takahashi [22], Zhou et al. [32] in the case of $N > 1$, respectively. The authors above showed that the sequence $\{x_n\}$

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generated by (1) converges strongly to a point in the fixed point set $F = F(T)$ for $N = 1$ and to a point in the common fixed point set $F = \bigcap_{i=1}^{N} F(T_i)$ for $N > 1$ under the following respective conditions in either Hilbert spaces or certain Banach spaces:

(C1) $\lim_{n\to\infty} \lambda_n = 0$; (Halpern [12])

(C2) $\sum_{n=0}^{\infty} \lambda_n = \infty$ or equivalently, $\prod_{n=0}^{\infty}(1-\lambda_n) = 0$; (Halpern [12])

(C3) $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0$; (Lions [16])

(C4) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$; (Wittmann [26])

(C5) $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$; (Bauschke [1])

(C6) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$ or equivalently, $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+N}}{\lambda_{n+N}} = 0$.
(O'Hara et al. [19], Xu [30])

Vert recently, Jung et al. [14] considered the perturbed control condition

$$\text{(C7)} \qquad |\lambda_{n+N} - \lambda_n| \le \circ(\lambda_{n+N}) + \sigma_n, \quad \sum_{n=0}^{\infty} \sigma_n < \infty$$

to obtain the strong convergence of iterative algorithm (1) in uniformly smooth Banach spaces.

On the other hand, using the condition (C1) and (C2), Xu [27] investigated the strong convergence of the iterative algorithm: for nonexpansive mapping T and $u,\ x_1 \in C$,

$$\text{(2)} \qquad x_{n+1} := \lambda_n u + (1-\lambda_n) S_n x_n, \quad n \ge 1,$$

where

$$S_n x := \frac{1}{n} \sum_{k=0}^{n-1} T^k x, \quad n \ge 1, \quad x \in C.$$

In this paper, we consider the iterative algorithms (1) and (2) with errors in the case when $C = E$ in Banach spaces. First, we introduce the iterative algorithm (1) with errors: for $N > 1$, $T_1, \ldots, T_N$ nonexpansive mappings from E into itself, and $u, x_0 \in E$,

$$\text{(IA1)} \qquad x_{n+1} := \lambda_n u + (1-\lambda_n) T_{n+1} x_n + e_n, \quad n \geq 0,$$

where $\{\lambda_n\} \subset (0,1)$ and $\{e_n\} \subset E$, and prove a strong convergence of the iterative algorithm (IA1) under the perturbed control condition (C7) with conditions (C1) and (C2) in a reflexive Banach space having a uniformly Gâteaux differentiable norm and a weakly sequentially continuous duality mapping. Second, we consider the iterative algorithm (2) with errors: for $T : E \to E$ nonexpansive mapping and $u,\ x_1 \in E$,

$$\text{(IA2)} \qquad x_{n+1} := \lambda_n u + (1-\lambda_n) S_n x_n + e_n, \quad n \geq 1,$$

and show that the sequence $\{x_n\}$ generated by (IA2) converges strongly to a fixed point of T under conditions (C1) and (C2) in a uniformly convex Banach space with a Fréchet differentiable norm. By using the perturbed control condition (C7) together with errors $\{e_n\}$ in the case when $C = E$, our results improve the corresponding results in [1, 13–15, 19, 32] for $N > 1$ and [5, 16, 23, 26–29] for $N = 1$ among others.

2. Preliminaries and lemmas

First, as in [14], we mention the relations between conditions (C1)–(C6) and give an example satisfying the perturbed control condition (C7).

In general, the control conditions (C5) and (C6) are not comparable (coupled with the conditions (C1) and (C2)), that is, neither of them implies the others as in the following examples:

EXAMPLE 1. Consider the control sequence $\{\alpha_n\}$ defined by

$$\alpha_n = \begin{cases} \frac{1}{n^s} & \text{if } n \text{ is odd,} \\ \frac{1}{n^s} + \frac{1}{n^t} & \text{if } n \text{ is even,} \end{cases}$$

with $\frac{1}{2} < s < t \leq 1$. Then $\{\alpha_n\}$ satisfies the conditions (C1), (C2) and (C6), but it fails to satisfy the condition (C5), where N is odd.

EXAMPLE 2. Take two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that:

(i) $m_1 = 1,\ \ m_k < n_k$ and $\max\{4n_k, n_k + N\} < m_{k+1}$ for $k \geq 1$,
(ii) $\sum_{i=m_k}^{n_k} \frac{1}{\sqrt{i}} > 1$ for $k \geq 1$.

Define a sequence $\{\mu_n\}$ by

$$\mu_i = \begin{cases} \frac{1}{\sqrt{i}} & \text{if } m_k \le i \le n_k, \quad k \ge 1, \\ \frac{1}{2\sqrt{n_k}} & \text{if } n_k < i < m_{k+1}, \quad k \ge 1. \end{cases}$$

Then $\{\mu_n\}$ is decreasing and $\mu_n \to 0$ as $n \to \infty$. Hence the conditions (C1) and (C5) are satisfied. Noting that

$$\sum_{n=1}^{\infty} \mu_n \ge \sum_{k=1}^{\infty} \sum_{i=m_k}^{n_k} \mu_i = \infty,$$

then we see that the condition (C2) is also satisfied. On the other hand, we have

$$\frac{\mu_{n_k}}{\mu_{n_k+N}} = 2, \quad k \ge 1,$$

which shows that the condition (C6) is not satisfied.

EXAMPLE 3. (Xu [30]) Consider the control sequence $\{\alpha_n\}$ defined by

$$\alpha_n = \begin{cases} \frac{1}{\sqrt{n}} & \text{if } n \text{ is odd}, \\ \frac{1}{\sqrt{n}-1} & \text{if } n \text{ is even}. \end{cases}$$

Then $\{\alpha_n\}$ satisfies (C6), but it fails to satisfy (C5).

EXAMPLE 4. Take $\{\alpha_n\}$ and $\{\mu_n\}$ as in the above Examples 1 and 2. Define a sequence $\{\lambda_n\}$ by

$$\lambda_n = \alpha_n + \mu_n$$

for all $n \ge 1$. Then $\{\lambda_n\}$ satisfies the conditions (C1), (C2) and

$$\text{(C7)} \qquad |\lambda_{n+N} - \lambda_n| \le o(\lambda_{n+N}) + \sigma_n,$$

where $\sum_{n=1}^{\infty} \sigma_n < \infty$, but it fails to satisfy both the conditions (C5) and (C6). For the case $N = 1$, we also refer to [5].

EXAMPLE 5. Let $\{\alpha_n\}$ satisfy (C1), (C2), not (C5), (C6) and let $\{\mu_n\}$ be (C1), (C2), (C5), not (C6). Assume that

$$\lim_{n \to \infty} \frac{\alpha_n}{\mu_n} = 0,$$

and define a sequence $\{\lambda_n\}$ by

$$\lambda_n = \alpha_n + \mu_n$$

for all $n \geq 1$. Then $\{\lambda_n\}$ satisfies the conditions (C1), (C2), not (C5), (C6), (C7).

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f \rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp. $x_n \rightharpoonup x$, $x_n \stackrel{*}{\rightharpoonup} x$) will denote strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to x.

The modulus of convexity of E is defined by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \varepsilon\right\}$$

for every ε with $0 \leq \varepsilon \leq 2$. A Banach space E is said to uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then

$$\left\|\frac{x+y}{2}\right\| \geq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every x, $y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. We also know that if C is a closed convex subset of a uniformly convex Banach space E, then for each $x \in E$, there exist a unique element $u = Px \in C$ with $\|x-u\| = \inf\{\|x-y\| : y \in C\}$. Such a P is called the metric projection of E onto C.

Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. This function φ is called *gauge function.* The *duality mapping* $J_\varphi : E \to E^*$ associated with a gauge function φ is defined by Browder [3] as follows

$$J_\varphi(x) = \{f \in E^* : \langle x, f \rangle = \|x\|\varphi(\|x\|),\ \|f\| = \varphi(\|x\|)\}, \quad x \in E.$$

In the case of $\varphi(t) = t$, we have $J_\varphi = J$, the *normalized duality mapping.* Notice the relation

$$J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x), \quad x \neq 0.$$

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{3}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be *Fréchet differentiable* if for each $x \in U$, this limit is obtained uniformly for $y \in U$. The norm is said to be *uniformly Gâteaux differentiable* if for $y \in U$, the limit is attained uniformly for $x \in U$. The space E is said to have a *uniformly Fréchet differentiable norm* (and E is said to be *uniformly smooth*) if the limit in (3) is attained uniformly for $(x, y) \in U \times U$. It is known that E is smooth if and only if each duality mapping J_φ is single-valued, and that E is uniformly smooth if and only if each duality mapping J_φ is norm to norm uniformly continuous on bounded subsets of E. It is also well-known that if E has a uniformly Gâteaux differentiable norm, J_φ is uniformly norm to weak* continuous on each bounded subsets of E ([7]).

It is relevant to the main theorems of this paper to note that while every uniformly smooth Banach space is a reflexive Banach space with a uniformly Gâteaux differentiable norm, the converse does not hold. Indeed there are reflexive spaces with a uniformly Gâteaux differentiable norm that are not even isomorphic to a uniformly smooth space. To see this consider E to be the direct sum $l^2(l^{p_n})$, the class of all those sequences $x = \{x_n\}$ with $x_n \in l^{p_n}$ and $\|x\| = (\sum_{n<\infty} \|x_n\|^2)^{1/2}$ (see [6]). Now, if $1 < p_n < \infty$ for all $n \geq 1$, where either $\limsup_{n\to\infty} p_n = \infty$ or $\liminf_{n\to\infty} p_n = 1$, then E is a reflexive Banach space with a uniformly Gâteaux differentiable norm, but is not uniformly smooth (see [6, 31]). We also observe that spaces with enjoy the fixed point property for nonexpansive self-mappings are not necessarily spaces with a uniformly Gâteaux differentiable norm. On the other hand, the converse of this fact appears to be unknown as well. For these facts, see also [18].

Following Browder [3], we say that a Banach space E has a *weakly sequentially continuous duality mapping* if there exists a gauge φ such that J_φ is single-valued and weak to weak* sequentially continuous (that is, if for each $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J_\varphi(x_n) \overset{*}{\rightharpoonup} J_\varphi(x)$. It is known that l^p $(1 < p < \infty)$ has a weakly sequentially continuous duality mapping with gauge $\varphi(t) = t^{p-1}$. Setting

$$\Phi(t) := \int_o^t \varphi(\tau) d\tau, \quad t \geq 0,$$

one sees that Φ is a convex function and

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad x \in E,$$

where ∂ denotes the subdifferential in the sense of convex analysis. The subdifferential inequality is such as

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, j_{x+y}\rangle, \quad x,\ y \in E, \quad j_{x+y} \in J_\varphi(x+y).$$

If E is smooth, then we have

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle, \quad x,\ y \in E. \tag{4}$$

or considering the normalized duality mapping J, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle, \quad x,\ y \in E.$$

Recall that a mapping T defined on a subset C of a Banach space E (and taking values in E) is said to be *demiclosed* if for any sequence $\{u_n\}$ in C the following implication holds:

$$u_n \rightharpoonup u \ \text{ and } \ \lim_{n\to\infty} \|Tu_n - w\| = 0$$

imply

$$u \in C \ \text{ and } \ Tu = w.$$

The following lemma can be found in [9, p. 108].

LEMMA 1. *Let E be a reflexive Banach space with a weakly sequentially continuous duality mapping. Let C be a nonempty closed convex subset of E and $T : C \to E$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping.*

Let C be a nonempty closed convex subset of E. C is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C has a fixed point. It is well-known (cf. [10, p.45]) that every weakly compact convex subset of a uniformly smooth Banach space has the fixed point property for nonexpansive mappings.

A mapping Q of C into C is said to be a *retraction* if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is range of Q. Let D be a subset of C and let Q be a mapping of C into D. Then Q is said to be *sunny* if each point on the ray $\{Qx + t(x - Qx) \ : t > 0\}$ is mapped by Q back onto Qx, in other words,

$$Q(Qx + t(x - Qx)) = Qx$$

for all $t \geq 0$ and $x \in C$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D; for more details, see [10]. In a smooth Banach space E, it is known (cf. [10, p.48]) that Q is a sunny nonexpansive retraction of C onto D if and only if the following condition holds:

$$(5) \qquad \langle x - Qx, J_{\varphi}(z - Qx)\rangle \leq 0, \quad x \in C, \quad z \in D.$$

It is also known [20, Theorem 1] (see also [10, pp.49–50] [11, Corollary 3], [24, Theorem 1]) that if E is a reflexive Banach space with a uniformly Gâteaux differentiable norm, every weakly compact convex subset of E has the fixed point property for nonexpansive mappings, and D is the fixed point set of a nonexpansive self-mapping of a closed convex subset C of E, then there exists a (unique) sunny nonexpansive retraction of C onto D. We denote the set of all fixed points of the mapping T by $F(T)$.

Finally, we need the following lemma, which is essentially Lemma 2 of Liu [17] (see also [27, Lemma 2.5]).

LEMMA 2. *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\beta_n + \gamma_n, \quad n \geq 0,$$

where $\{\lambda_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfying the condition:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$ *or equivalently,* $\prod_{n=0}^{\infty}(1-\lambda_n) := \lim_{n\to\infty} \prod_{k=0}^{n}(1 - \lambda_k) = 0$;
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$; *or*
(iii) $\sum_{n=0}^{\infty} \lambda_n\beta_n < \infty$;
(iv) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Main results

First we study the strong convergence of sequence $\{x_n\}$ generated by the following algorithm with errors: for $u, x_0 \in E$,

$$\text{(IA1)} \qquad x_{n+1} := \lambda_n u + (1 - \lambda_n)T_{n+1}x_n + e_n, \quad n \geq 0,$$

where $\{\lambda_n\} \subset [0,1]$ and the computational errors $\{e_n\} \subset E$.

We consider N mappings $T_1, T_2, \ldots, T_N$. For $n > N$, set $T_n := T_{n \mod N}$, where $n \mod N$ is defined as follows: if $n = kN + l$, $0 \leq l < N$, then

$$n \mod N := \begin{cases} l & \text{if } l \neq 0, \\ N & \text{if } l = 0. \end{cases}$$

PROPOSITION 1. *Let E be a reflexive Banach space. Suppose that E has a weakly sequentially continuous duality mapping J_φ with gauge φ. Let $T_1, \ldots, T_N$ be nonexpansive mappings from E into itself with $F := \bigcap_{i=1}^N F(T_i)$ nonempty and*

$$\begin{aligned} F &= F(T_N \cdots T_1) = F(T_1 T_N \cdots T_3 T_2) = \cdots \\ &= F(T_{N-1} T_{N-2} \cdots T_1 T_N). \end{aligned}$$

Assume that the sequence $\{e_n\} \subset E$ satisfies the $\sum_{n=0}^\infty \|e_n\| < \infty$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies the conditions (C1), (C2) *and* (C7). *Let u, $x_0 \in E$ be chosen arbitrarily and let $\{x_n\}$ be generated by*

$$\text{(IA1)} \qquad x_{n+1} = \lambda_n u + (1 - \lambda_n) T_{n+1} x_n + e_n, \quad n \geq 0.$$

If there exists a sunny nonexpansive retraction Q of E onto F, then $\{x_n\}$ converges strongly to $Q_F u$.

Proof. As in [1, 14-16, 21], we divide the proof into several steps.

Step 1: $\{x_n\}$ is bounded. Indeed, let $z \in F$,

$$d = \max\{\|x_0 - z\|, \|u - z\|\} \text{ and } M = d + \sum_{n=0}^\infty \|e_n\|.$$

Then by the nonexpansivity of T_{n+1}, $n \geq 0$

$$\begin{aligned} \|x_1 - z\| &= \|\lambda_0 u + (1 - \lambda_0) T_1 x_0 - z\| + \|e_0\| \\ &\leq \lambda_1 d + (1 - \lambda_1) d + \|e_0\| \\ &= d + \|e_0\|. \end{aligned}$$

Using an induction, we obtain

$$\|x_{n+1} - z\| \leq d + \sum_{k=0}^n \|e_k\| \leq M, \quad n \geq 0.$$

Hence, it follows from $\sum_{n=0}^{\infty} \|e_n\| < \infty$ that $\{x_n\}$ is bounded, and so is $\{T_{n+1}x_n\}$.

Step 2: $\lim_{n\to\infty} \|x_{n+1} - T_{n+1}x_n\| = 0$. Since

$$\begin{aligned}\|x_{n+1} - T_{n+1}x_n\| &\le \lambda_n\|u - T_{n+1}x_n\| + \|e_n\| \\ &\le \lambda_n(\|u\| + \|T_{n+1}x_n\|) + \|e_n\| \\ &\le \lambda_n R + \|e_n\|\end{aligned}$$

for $R = \|u\| + \sup_{n\ge 0} \|T_{n+1}x_n\|$, we have $\lim_{n\to\infty} \|x_{n+1} - T_{n+1}x_n\| = 0$.

Step 3: $\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0$. By Step 2 above, there exists a constant $L > 0$ such that for all $n \ge 1$,

$$\|u - T_{n+1}x_n\| \le L.$$

Since for all $n \ge 1$, $T_{n+N} = T_n$, by (C7) we have

$$\begin{aligned}
&\|x_{n+N} - x_n\| \\
&\le \|(\lambda_{n+N-1} - \lambda_{n-1})(u - T_{n+N}x_{n+N-1})\| \\
&\quad + \|(1 - \lambda_{n+N-1})(T_n x_{n+N-1} - T_n x_{n-1})\| + \|e_{n+N-1} - e_{n-1}\| \\
&\le L|\lambda_{n+N-1} - \lambda_{n-1}| + (1 - \lambda_{n+N-1})\|x_{n+N-1} - x_{n-1}\| \\
&\quad + \|e_{n+N-1}\| + \|e_{n-1}\| \\
&= (1 - \lambda_{n+N-1})\|x_{n+N-1} - x_{n-1}\| + (\circ(\lambda_{n+N-1}) + \sigma_n)L \\
&\quad + \|e_{n+N-1}\| + \|e_{n-1}\|.
\end{aligned} \tag{6}$$

By taking $\alpha_n = \lambda_{n+N-1}$, $s_{n+1} = \|x_{n+N} - x_n\|$, $\alpha_n\beta_n = \circ(\alpha_n)L$ and $\gamma_n = \sigma_n L + \|e_{n+N-1}\| + \|e_{n-1}\|$, from (6) we have

$$s_{n+1} \le (1 - \alpha_n)s_n + \alpha_n\beta_n + \gamma_n,$$

and so by Lemma 2,

$$\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0.$$

Step 4: $\lim_{n\to\infty} \|x_n - T_{n+N}\cdots T_{n+1}x_n\| = 0$. Step 4 coincides with Claim (6) in the proof of [1. Theorem 3.1] (or [15, Lemma 3]), in which Steps 2 and 3 were used.

Step 5: $\limsup_{n\to\infty}\langle u - Q_F u, J_\varphi(x_n - Q_F u)\rangle \le 0$. Let a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ be such that

$$\lim_{j\to\infty} \langle u - Q_F u, J_\varphi(x_{n_j+1} - Q_F u)\rangle = \limsup_{n\to\infty}\langle u - Q_F u, J_\varphi(x_{n+1} - Q_F u)\rangle$$

and $x_{n_j} \rightharpoonup p$ for some $p \in E$. We assume (after passing to another subsequence if necessary) that $n_j+1 \bmod N = i$ for some $i \in \{1, \ldots, N\}$ and that $x_{n_j+1} \rightharpoonup p$. From Step 4, it follows that $\lim_{j\to\infty} \|x_{n_j+1} - T_{i+N} \cdots T_{i+1}x_{n_j+1}\| = 0$. Hence, by Lemma 2, we have $p \in F(T_{i+N} \cdots T_{i+1}) = F$. On the other hand, we know that F is a sunny nonexpansive retract of E ([10, 11, 20, 24]). Thus, by weak continuity of duality mapping J_φ and (5), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle u - Q_F u, J_\varphi(x_{n+1} - Q_F u)\rangle \\
&= \lim_{j\to\infty}\langle u - Q_F u, J_\varphi(x_{n_j+1} - Q_F u)\rangle \\
&= \langle u - Q_F u, J_\varphi(p - Q_F u)\rangle \le 0.
\end{aligned}$$

Step 6: $\lim_{n\to\infty} \|x_n - Q_F u\| = 0$. Since $(x_{n+1} - Q_F u) = (1-\lambda_n)(T_{n+1}x_n - Q_F u) + \lambda_n(u - Q_F u) + e_n$, by the subdifferential inequality (4), we have

$$\begin{aligned}
&\Phi(\|x_{n+1} - Q_F u\|) \\
&= \Phi(\|(1-\lambda_n)(T_{n+1}x_n - Q_F u) + \lambda_n(u - Q_F u) + e_n\|) \\
&\le \Phi((1-\lambda_n)\|T_{n+1}x_n - Q_F u\|) \\
&\quad + \langle \lambda_n(u - Q_F u) + e_n, J_\varphi(x_{n+1} - Q_F u)\rangle \\
&\le (1-\lambda_n)\Phi(\|x_n - Q_F u\|) \\
&\quad + \lambda_n\langle u - Q_F u, J_\varphi(x_{n+1} - Q_F u)\rangle + K\|e_n\|,
\end{aligned}$$

where $K = \sup_{n\ge 0}\{\varphi(\|x_n - Q_F u\|)\}$. An application of Lemma 2 together with Step 5 yields that $\lim_{n\to\infty} \Phi(\|x_n - Q_F u\|) = 0$. This completes the proof. □

From Proposition 1, we have the following result.

THEOREM 1. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings and E has a weakly sequentially continuous duality mapping J_φ with gauge φ. Let $T_1, \ldots, T_N$ be nonexpansive mappings from E into itself with $F := \bigcap_{i=1}^N F(T_i)$ nonempty and*

$$\begin{aligned}
F &= F(T_N \cdots T_1) = F(T_1 T_N \cdots T_3 T_2) = \cdots \\
&= F(T_{N-1} T_{N-2} \cdots T_1 T_N).
\end{aligned}$$

Assume that the sequence $\{e_n\} \subset E$ *satisfies the* $\sum_{n=0}^{\infty} \|e_n\| < \infty$. *Let* $\{\lambda_n\}$ *be a sequence in* $(0,1)$ *which satisfies the conditions* (C1), (C2) *and* (C7). *Let* u, $x_0 \in E$ *be chosen arbitrarily and let* $\{x_n\}$ *be generated by* (IA1). *Then* $\{x_n\}$ *converges strongly to* $Q_F u$, *where* Q *is a sunny nonexpansive retraction of* E *onto* F.

Proof. It follows from [20, Theorem 1] (also [11, Corollary] and [24, Theorem 1]) that there exists a sunny nonexpansive retraction of E onto F. Thus the result follows from Proposition 1. □

As immediate consequences of Theorem 1, we have the following results.

COROLLARY 1. *Let* E *be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping* J_φ *with gauge* φ. *Let* $T_1, \ldots, T_N$ *be nonexpansive mappings from* E *into itself with* $F := \bigcap_{i=1}^{N} F(T_i)$ *nonempty and*

$$\begin{aligned} F = F(T_N \cdots T_1) &= F(T_1 T_N \cdots T_3 T_2) = \cdots \\ &= F(T_{N-1} T_{N-2} \cdots T_1 T_N). \end{aligned}$$

Assume that the sequences $\{\lambda_n\} \subset (0,1)$ *and* $\{e_n\} \subset E$ *are the same as in Theorem 1. Let* u, $x_0 \in E$ *be chosen arbitrarily and let* $\{x_n\}$ *be generated by* (IA1). *Then* $\{x_n\}$ *converges strongly to* $Q_F u$, *where* Q *is a sunny nonexpansive retraction of* E *onto* F.

COROLLARY 2. *Let* H *be a Hilbert space and* $T_1, \ldots, T_N$ *nonexpansive mappings from* E *into itself with* $F := \bigcap_{i=1}^{N} F(T_i)$ *nonempty and*

$$\begin{aligned} F = F(T_N \cdots T_1) &= F(T_1 T_N \cdots T_3 T_2) = \cdots \\ &= F(T_{N-1} T_{N-2} \cdots T_1 T_N). \end{aligned}$$

Assume that the sequences $\{\lambda_n\} \subset (0,1)$ *and* $\{e_n\} \subset H$ *are the same as in Theorem 1. Let* u, $x_0 \in H$ *be chosen arbitrarily and let* $\{x_n\}$ *be generated by* (IA1). *Then* $\{x_n\}$ *converges strongly to* $P_F u$, *where* P *is the nearest point projection of* H *onto* F.

Proof. Note that the metric projection P of H onto $F(T)$ is a sunny nonexpansive retraction. Thus the result follows from Theorem 1. □

As direct consequences of Theorem 1, we also have the following results for $N = 1$.

COROLLARY 3. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings and E has a weakly sequentially continuous duality mapping J_φ with gauge φ. Let $T : E \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Assume that the sequences $\{\lambda_n\} \subset (0,1)$ and $\{e_n\} \subset E$ are the same as in Theorem 1. Let u, $x_0 \in E$ be chosen arbitrarily and let $\{x_n\}$ be generated by*

$$x_{n+1} = \lambda_n u + (1 - \lambda_n)Tx_n + e_n, \quad n \geq 0.$$

Then $\{x_n\}$ converges strongly to Qu, where Q is a sunny nonexpansive retraction of E onto $F(T)$.

COROLLARY 4. *Let H be a Hilbert space and $T : H \to H$ a nonexpansive mapping with $F(T) \neq \emptyset$. Assume that the sequences $\{\lambda_n\} \subset (0,1)$ and $\{e_n\} \subset H$ are the same as in Theorem 1. Let u, $x_0 \in H$ be chosen arbitrarily and let $\{x_n\}$ be generated by*

$$x_{n+1} = \lambda_n u + (1 - \lambda_n)Tx_n + e_n, \quad n \geq 0.$$

Then $\{x_n\}$ converges strongly to $P_F u$, where P is the nearest point projection of H onto F.

REMARK 1. Even though the domain of nonexpansive mappings is whole space E as a closed convex subset of E, our results improves the results of authors mentioned in Introduction 1 as follows:

(1) Theorem 1 extends Theorem 1 of [14] to general Banach space setting together with the computational errors $\{e_n\}$.

(2) By using the condition (C7) together with errors $\{e_n\}$, Theorem 1 (Corollary 1) also improves Theorem 1 of [15] and Theorem 4.1 of [19].

(3) Even for $N = 1$, our proof lines of Theorem 1 are different from those of Cho et al. [5], Xu [27-29], Zhou et al [32], in which, as in [23], Reich's result [19] was utilized to prove their main results.

(4) Corollary 3 generalizes Theorem of [23], Theorem 3.1 of [27], Theorem 2.3 of [28], and Theorem 3.2 of [29].

(5) Corollary 4 is also a complementary one of the result of Wittmann [26] together with the condition (C7) and errors $\{e_n\}$.

Let D be a subset of a Banach space E. Recall that a mapping $T : D \to E$ is said to be firmly nonexpansive if for each x and y in D, the convex function $\phi : [0,1] \to [0,\infty)$ defined by

$$\phi(s) = \|(1-s)x + sTx - ((1-s)y + sTy)\|$$

is nonincreasing. Since ϕ is convex, it is easy to check that a mapping $T : D \to E$ is firmly nonexpansive if and only if

$$\|Tx - Ty\| \leq \|(1-t)(x-y) + t(Tx - Ty)\|$$

for each x and y in D and $t \in [0,1]$. It is clear that every firmly nonexpansive mapping is nonexpansive (cf. [9, 10]).

The following result extends a Lions-type iterative algorithm [16] together with the condition (C7) to a Banach space setting.

COROLLARY 5. *Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping J_φ with gauge φ. Let $T_1, \ldots, T_N$ be firmly nonexpansive mappings from E into itself with $F := \bigcap_{i=1}^N F(T_i)$ nonempty and*

$$\begin{aligned} F &= F(T_N \cdots T_1) = F(T_1 T_N \cdots T_3 T_2) = \cdots \\ &= F(T_{N-1} T_{N-2} \cdots T_1 T_N). \end{aligned}$$

Assume that the sequences $\{\lambda_n\} \subset (0,1)$ and $\{e_n\} \subset E$ are the same as in Theorem 1. Let u, $x_0 \in E$ be chosen arbitrarily and let $\{x_n\}$ be generated by (IA1). *Then $\{x_n\}$ converges strongly to $Q_F u$, where Q is a sunny nonexpansive retraction of E onto F.*

REMARK 2. (1) In Hilbert space, Lions [16, Théorèm 4] had used
(L1) $\lim_{n\to\infty} \lambda_n = 0$,
(L2) $\sum_{k=1}^{\infty} \lambda_{kN+i} = \infty$ for all $i = 0, \ldots, N-1$,
which is more restrictive than (C2), and
(L3)′ $\lim_{k\to\infty} \frac{\sum_{i=1}^N |\lambda_{kN+i} - \lambda_{(k-1)N+i}|}{(\sum_{i=1}^N \lambda_{kN+i})^2} = 0$
in place of (C5).

(2) In general, (C5) and (L3)′ are independent, even when $N = 1$. For examples and more details, see [1].

(3) Corollary 5 also improves Corollary 13 of [14] and Théorèm 4 of [16].

Now, we consider iterative algorithm (IA1) with the mean $S_n x_n$ in place of $T_{n+1} x_n$, where

$$S_n x := \frac{1}{n} \sum_{k=0}^{n-1} T^k x, \quad n \geq 1, \ x \in E.$$

In this case, we do have strong convergence under conditions (C1) and (C2).

We need Bruck's result on the asymptotic behavior of nonexpansive mappings.

LEMMA 3 ([4]). *Let E be a uniformly convex Banach space and let K be a nonempty bounded subset of E. Let $T : E \to E$ be nonexpansive. Then*

$$\lim_{n\to\infty} \sup_{x\in K} \|T(S_n x) - S_n x\| = 0$$

THEOREM 2. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose that E has a weakly continuous duality mapping J_φ with gauge φ. Let $T : E \to E$ be a nonexpansive mapping. Assume that the sequence $\{e_n\} \subset E$ satisfies the $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies the conditions (C1) and (C2). Let $u, x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\text{(IA2)} \qquad x_{n+1} := \lambda_n u + (1-\lambda_n) S_n x_n + e_n, \qquad n \geq 1.$$

If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to Qu, where Q is a sunny nonexpansive retraction of E onto $F(T)$.

Proof. Let $y_n := S_n x_n$. Then Eq. (IA2) can be re-written as

$$x_{n+1} := \lambda_n u + (1-\lambda_n) y_n + e_n, \qquad n \geq 1.$$

Now, we proceed with several steps.

Step 1: $\{x_n\}$ is bounded and so is $\{y_n\}$. Indeed, let $z \in F(T)$ and

$$d = \max\{\|u - z\|, \|x_1 - z\|\}, \quad M = d + \sum_{n=1}^{\infty} \|e_n\|.$$

Then we have

$$\begin{aligned} \|x_2 - z\| &\leq \lambda_1 \|u - z\| + (1-\lambda_1)\|x_1 - z\| + \|e_1\| \\ &\leq \lambda_1 d + (1-\lambda_1) d + \|e_1\| \\ &= d + \|e_1\|. \end{aligned}$$

By induction, we obtain

$$\|x_{n+1} - z\| \leq d + \sum_{k=1}^{n} \|e_k\| \leq M, \quad n \geq 1.$$

Hence, it follows from $\sum_{n=1}^{\infty} \|e_n\| < \infty$ that $\{x_n\}$ is bounded, and so is $\{y_n\}$.

Step 2: $\lim_{n\to\infty} \|Ty_n - y_n\| = 0$ by Lemma 3.

Step 3: $\limsup_{n\to\infty}\langle u - Qu, J_\varphi(y_n - Qu)\rangle \leq 0$. To prove this, we take a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ be such that

$$\lim_{j\to\infty}\langle u - Qu, J_\varphi(y_{n_j} - Qu)\rangle = \limsup_{n\to\infty}\langle u - Qu, J_\varphi(y_n - Qu)\rangle$$

and $x_{y_j} \rightharpoonup p$ for some $p \in E$. It follows from Step 2 and Lemma 1 that $p \in F(T)$. We know that $F(T)$ is sunny nonexpansive retract of E ([10, p. 49]). Thus, by weak continuity of duality mapping J_φ and (5), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u - Qu, J_\varphi(y_n - Qu)\rangle &= \lim_{j\to\infty}\langle u - Qu, J_\varphi(y_{n_j} - Qu)\rangle \\ &= \langle u - Qu, J_\varphi(p - Qu)\rangle \leq 0.\end{aligned}$$

Step 4: $\lim_{n\to\infty}\|x_n - Qu\| = 0$. Indeed, Since $(x_{n+1} - Qu) = (1 - \lambda_n)(y_n - Qu) + \lambda_n(u - Qu) + e_n$, by using the subdifferential inequality (4), we have

$$\begin{aligned}&\Phi(\|x_{n+1} - Qu\|)\\ &\leq \Phi(\|(1-\lambda_n)(y_n - Qu)\|) + \langle \lambda_n(u - Qu) + e_n, J_\varphi(x_{n+1} - Qu)\rangle\\ &\leq \Phi((1-\lambda_n)\|y_n - Qu\|) + \lambda_n\langle u - Qu, J_\varphi(x_{n+1} - Qu)\rangle\\ &\quad + \langle e_n, J_\varphi(x_{n+1} - Qu)\rangle\\ &\leq (1-\lambda_n)\Phi(\|x_n - Qu\|) + \lambda_n\langle u - Qu, J_\varphi(x_{n+1} - Qu)\rangle + K\|e_n\|\\ &= (1-\lambda_n)\Phi(\|x_n - Qu\|) + \lambda_n\beta_n + \gamma_n,\end{aligned}$$

where $\beta_n = \langle u - Qu, J_\varphi(y_n - Qu)\rangle$, $K = \sup_{n\geq 0}\{\varphi(\|x_n - Qu\|)\}$ and $\gamma_n = K\|e_n\|$. Now, applying Lemma 2 with Step 3, we have $\lim_{n\to\infty}\Phi(\|x_n - Qu\|) = 0$. This completes the proof. $\square$

As a direct consequence of Theorem 2, we have the following result.

COROLLARY 6. *Let E be a uniformly convex and uniformly smooth Banach space. Suppose that E has a weakly continuous duality mapping J_φ with gauge φ. Let $T : E \to E$ be a nonexpansive mapping. Assume that the sequences $\{\lambda_n\} \subset [0,1]$ and $\{e_n\} \subset E$ are the same as in Theorem 2. Let $u, x_0 \in E$ and let $\{x_n\}$ be a sequence generated by* (IA2). *If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to Qu, where Q is a sunny nonexpansive retraction of E onto $F(T)$.*

COROLLARY 7. *Let H be a Hilbert space and $T : H \to H$ be a nonexpansive mapping. Assume that the sequences $\{\lambda_n\} \subset [0,1]$ and $\{e_n\} \subset H$ are the same as in Theorem 2. Let $u, x_0 \in H$ and let $\{x_n\}$ be a sequence generated by* (IA2). *If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to Pu, where P is the metric projection of H onto $F(T)$.*

REMARK 3. (1) Theorem 2 (and Corollary 6) improves Theorem 3.2 of [27] with gauge $\varphi(t) = t$ to the case of the iterative algorithm with errors.

(2) Our proof lines of Theorem 2 are different from those of Xu [27], in which he also utilized the Reich's result [20] and the equation $(d/dt)\|x+ty\|^2 = 2\langle y, J(x+ty)\rangle$.

4. Applications to contraction semigroups

As in [27], we consider iterative algorithm for contraction semigroup.

Let E be a Banach space and C be a nonempty closed convex subset of E. Let $\Im = \{S(t) : t \geq 0\}$ be a family of self-mappings of C. Recall that $\Im$ is said to be a *contraction semigroup* on C if the following conditions hold:

(a) $S(0)x = x, \quad x \in C$;
(b) $S(t_1 + t_2)x = S(t_1)S(t_2), \quad t_1,\ t_2 \geq 0,\ x \in C$;
(c) for each $x \in C$, the function $S(t)x$ is continuous in $t \in (0, \infty)$;
(d) for each $t > 0$, $S(t) : C \to C$ is a nonexpansive mapping.

We shall use $F(\Im)$ to denote the set of common fixed points of $\Im$; that is,

$$F(\Im) = \bigcap_{t>0} F(S(t)).$$

LEMMA 5 ([25]). *Let E be a uniformly convex Banach space, C be a closed convex subset of E, and $\Im$ be a contraction semigroup on C. Assume that $F(\Im) \neq \emptyset$. Then there is a family $\{r_t : t \geq 0\}$ of non-negative numbers such that*

$$\lim_{t\to\infty} \sup_{x\in C} \|S(r)\sigma_t(x) - \sigma_t(x)\| = 0, \quad r > 0,$$

where

$$\sigma_t(x) := \frac{1}{t} \int_0^t S(r_t + \tau) d\tau.$$

Now we define iterative algorithm with errors: take $u, x_1 \in E$ arbitrarily and define

$$x_{n+1} := \lambda_n u + (1 - \lambda_n)\sigma_{t_n}(x_n) + e_n, \quad n \geq 1.$$

By employing similar arguments to the discrete cases (Theorem 2 and Corollary 7), we can prove the following results.

THEOREM 3. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose E has a weakly continuous duality mapping J_φ with gauge φ. Let $\{\lambda_n\}$ be a sequence in* $(0, 1)$ *which satisfies the conditions* (C1) *and* (C2). *Let $\Im = \{S(t) : t \geq 0\}$ be a contraction semigroup on E such that $F(\Im) \neq \emptyset$. Assume that the sequences $t_n \in [0, \infty)$ and $\{e_n\} \subset E$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} t_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \|e_n\| < \infty$.

Let $u, x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\text{(IA3)} \qquad x_{n+1} := \lambda_n u + (1 - \lambda_n)\sigma_{t_n}(x_n) + e_n, \quad n \geq 1.$$

Then $\{x_n\}$ converges strongly to a point of $F(\Im)$.

COROLLARY 8. *Let H be a Hilbert space. Let $\{\lambda_n\}$ be a sequence in* $(0, 1)$ *which satisfies the conditions* (C1) *and* (C2). *Let $\Im = \{S(t) : t \geq 0\}$ be a contraction semigroup on E such that $F(\Im) \neq \emptyset$. Assume that the sequences $t_n \in [0, \infty)$ and $\{e_n\} \subset E$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} t_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \|e_n\| < \infty$.

Let $u, x_1 \in H$ and let $\{x_n\}$ be a sequence generated by (IA3). *Then $\{x_n\}$ converges strongly to a point of $F(\Im)$.*

REMARK 6. Even though $C = E$, Theorem 3 and Corollary 8 are new results.

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Department of Mathematics, Dong-A University, Busan 604-714, Korea
E-mail: jungjs@mail.donga.ac.kr