COMMUTATOR LENGTH OF SOLVABLE GROUPS SATISFYING MAX-N

MEHRI AKHAVAN-MALAYERI

ABSTRACT. In this paper we find a suitable bound for the number of commutators which is required to express every element of the derived group of a solvable group satisfying the maximal condition for normal subgroups. The precise formulas for expressing every element of the derived group to the minimal number of commutators are given.

1. Introduction

Let G be a group and G' be its commutator subgroup. Denote by $c(G)$ the minimal number such that every element of G' can be expressed as a product of at most $c(G)$ commutators. A group G is called a c-group if $c(G)$ is finite.

Let $F_{n,t} = \langle x_1, \ldots, x_n \rangle$ and $M_{n,t} = \langle x_1, \ldots, x_n \rangle$ be respectively the free nilpotent group of rank n and nilpotency class t and the free metabelian nilpotent group of rank n and nilpotency class t. P. W. Stroud, in his Ph. D. thesis [9] in 1966, proved that for all t, every element of the commutator subgroup $F'_{n,t}$ can be expressed as a product of n commutators. In 1985 H. Allambergenov and V. A. Romankov [4] proved that $c(M_{n,t})$ is precisely n provided $n \geq 2$, $t \geq 4$, or $n \geq 3$, $t \geq 3$. In [5] C. Bavard and G. Meigniez considered the same problem for the n-generator free metabelian group M_n. They showed that the minimum number $c(M_n)$ of commutators required to express an arbitrary element of the derived subgroup M'_n satisfies the inequality

$$[n/2] \leq c(M_n) \leq n,$$

where $[n/2]$ is the greatest integer part of $n/2$.

Since $F_{n,3}$ groups are metabelian, the result of Allambergenov and Romankov [4] shows that $c(M_n) \geq n$ for $n \geq 3$, and in [3] we considered the remaining case $n = 2$. We have $c(M_n) = n$, for all $n \geq 2$. These results were extended in [3] to the larger class of abelian by nilpotent groups and it was shown that $c(G) = n$ if G is a (non-abelian) free abelian by nilpotent group of rank n.

In [2] we proved that $2 \leq c(W) \leq 3$, where $W = G \wr C_\infty$ is the wreath product of a non-trivial group G with the infinite cyclic group.

In the case of a finite d-generator solvable group G of solvability length r Hartley [6] proved that $c(G) \leq d + (2d - 1)(r - 1)$. And in a recent paper D. Segal [8] has proved that in a finite d-generator solvable group G, every element of G' can be expressed as a product of $72d^2 + 46d$ commutators.

The problem remains open for the d-generator solvable group in general. In the section of open problems in the site of Magnus project (http://www.grouptheory.org), M. I. Kargapolov asks the question ($S4$) as follows:

"Is there a number $N = N(k, d)$ so that every element of the commutator subgroup of a free solvable group of rank k and solvability length d, is a product of N commutators?"

The answer is "yes" for free metabelian groups, see [4] and for free solvable groups of solvability length 3, see [7].

In [1] we found lower and upper bound for the commutator length of a finitely generated nilpotent by abelian group. We also considered a $n-$generator solvable group G such that G has a nilpotent by abelian normal subgroup K of finite index. If K is a $s-$generator group then $c(G) \leq s(s + 1)/2 + 72n^2 + 47n$. We considered the class of solvable group of finite Prüfer rank s and we proved that every element of its commutator subgroup is equal to a product of at most $s(s + 1)/2 + 72s^2 + 47s$. And as a consequence of the above results we proved that if A is a normal subgroup of a solvable group G such that G/A is a $d-$generator finite group. And A has finite Prüfer rank s. Then $c(G) \leq s(s + 1)/2 + 72(s^2 + n^2) + 47(s + n)$. These bounds depend only on the number of generators of the groups.

In the present paper we consider a solvable group satisfying the maximal condition for normal subgroups. We find an upper bound for the commutator length of this class of groups. The bound depends on the number of generators of the group G, the solvability length of the group and the number of generators of the group $G^{(k)}$ as a G-subgroup. In

particular, if in a finitely generated solvable group G each term of the derived series is finitely generated as a G-subgroup, then G is a c-group.

We also give the precise formulas for expressing every element of the derived group to the product of commutators.

2. Main results

NOTATION. Let N be a subgroup of a group G, and $x, y \in G$. Then $x^y = y^{-1}xy$, $[x, y] = x^{-1}y^{-1}xy$ and $[N, x] = \{[n, x] : n \in N\}$.

The main results of this paper are as follows.

LEMMA 1. Let $G = \langle x_1, \ldots, x_n \rangle$ be a finitely generated solvable group of solvability length r and $G^{(r-2)} = \langle y_1^G, \ldots, y_s^G \rangle$. Then

$$G^{(r-1)} = [G^{(r-1)}, G]\left(\prod_{i=1}^{s}\left[G^{(r-2)}, y_i\right]\right).$$

THEOREM 1. Let $G = \langle x_1, \ldots, x_n \rangle$ be a solvable group of solvability length r satisfying the maximal condition for normal subgroups. Then $c(G) \leq n[r/2] + \sum_{k=1}^{r-2} s_k$, where $G^{(k)} = \langle y_{k1}^G, \ldots, y_{ks_k}^G \rangle$ for suitable elements $y_{k_i} \in G$, and $[r/2]$ is the greatest integer part of $r/2$. Moreover we may have the following three expressions.

(0) For $r = 2$, every element of G' is a product of commutators:

$$\prod_{i=1}^{n}[g_i a_{1i}^{-1}, x_i].$$

(1) For $r = 2m \geq 4$, an even number, every element of G' is a product of commutators:

$$\prod_{i=1}^{n}\left[a_{(2m-1)i}, x_i\right]$$

$$\prod_{i=1}^{s_{2m-2}}\left[h_{(2m-2)i}, y_{(2m-2)i}\right] \prod_{i=1}^{s_{2m-3}}\left[h_{(2m-3)i}, y_{(2m-3)i}\right]$$

$$\prod_{i=1}^{n}\left[a_{(2m-3)i}a_{(2m-2)i}^{-1}, x_i\right]^{d_{(2m-3)i}} \cdots \prod_{i=1}^{s_4}\left[h_{4i}, y_{4i}\right] \prod_{i=1}^{s_3}\left[h_{3i}, y_{3i}\right]$$

$$(1) \quad \prod_{i=1}^{n}\left[a_{3i}a_{4i}^{-1}, x_i\right]^{d_{3i}} \prod_{i=1}^{s_2}\left[h_{2i}, y_{2i}\right] \prod_{i=1}^{s_1}\left[h_{1i}, y_{1i}\right] \prod_{i=1}^{n}\left[g_i a_{1i}^{-1} a_{2i}^{-1}, x_i\right]^{d_{1i}}.$$

(2) *For* $r = 2m + 1$, *an odd number, every element of* G' *is a product of commutators:*

$$\prod_{i=1}^{s_{2m-1}} \left[h_{(2m-1)i} y_{(2m-1)i} \right] \prod_{i=1}^{n} \left[a_{(2m-1)i} a_{(2m)i}^{-1}, x_i \right]^{d_{(2m-1)i}}$$

$$\prod_{i=1}^{s_{2m-2}} \left[h_{(2m-2)i}, y_{(2m-2)i} \right] \prod_{i=1}^{s_{2m-3}} \left[h_{(2m-3)i}, y_{(2m-3)i} \right]$$

$$\prod_{i=1}^{n} \left[a_{(2m-3)i} a_{(2m-2)i}^{-1}, x_i \right]^{d_{(2m-3)i}} \cdots$$

$$\prod_{i=1}^{s_4} \left[h_{4i}, y_{4i} \right] \prod_{i=1}^{s_3} \left[h_{3i}, y_{3i} \right] \prod_{i=1}^{n} \left[a_{3i} a_{4i}^{-1}, x_i \right]^{d_{3i}}$$

$$(2) \qquad \prod_{i=1}^{s_2} \left[h_{2i}, y_{2i} \right] \prod_{i=1}^{s_1} \left[h_{1i}, y_{1i} \right] \prod_{i=1}^{n} \left[g_i a_{1i}^{-1} a_{2i}^{-1}, x_i \right]^{d_{1i}}$$

where $g_i \in G$, a_{ki}, $d_{ki} \in G^{(k)}$ *for* $i = 1, \ldots, n$ *and* $h_{ki} \in G^{(k)}$ *for* $i = 1, \ldots, n$.

In particular, we have the following consequence of Theorem 1.

COROLLARY 1. *If in a finitely generated solvable group* G *each term of the derived series is finitely generated as a* G-*subgroup, then* G *is a c-group.*

3. Proofs

The proof of lemma 1 requires the following lemma proved by Rhemtulla in [7].

LEMMA (RHEMTULLA). *Let* $G = \langle A, x_1, \ldots, x_n \rangle$, *where* A *is an abelian normal subgroup of* G. *Then the set*

$$S = \{ [a_1, x_1][a_2, x_2] \cdots [a_n, x_n]; a_i \in A \}$$

is precisely the subgroup $[A, G]$.

And we also need the following lemma proved by Stroud [9];

LEMMA (P. STROUD). *Let* $G = \langle x_1, \ldots, x_n \rangle$ *be a nilpotent group. Then every element of* G' *is a product of* n *commutators* $[x_1, g_1] \cdots [x_n, g_n]$ *for suitable* g_i *in* G.

Now we turn to the proof of Lemma 1.

Proof of Lemma 1. Put $A = [G^{(r-1)}, G]$. Since $G^{(r-1)}$ is an abelian normal subgroup of G, every element of A can be written in the form $[x_1, a_1] \cdots [x_n, a_n]$, where $a_i \in G^{(r-1)}$, by Rhemtulla's Lemma. We may assume $A = 1$. Since $G^{(r-1)}$ is a central subgroup of G, the map $a \to [a, y_i]$ is a homomorphism from $G^{(r-2)}$ to $G^{(r-1)}$. And it follows that $A_i = [G^{(r-2)}, y_i] \leq G^{(r-1)} \leq Z(G)$ is normal in G, and $B = \prod_{i=1}^{s} A_i$ is a normal subgroup of G.

It will suffice now to prove that $B = G^{(r-1)}$ modulo A. Now for any x, $y \in G$, it is clear that $[y_j^x, y_i^y] = [y_j^{xy^{-1}}, y_i]^y \in A_i^y = A_i$. Hence $G^{(r-1)} \leq B$, and this completes the proof.

Now we continue by proving Theorem 1:
We will use the following easily checked commutator identity;

$$(3) \qquad\qquad [x, y][z, x] = [zy^{-1}, x]^{y^x}$$

Proof of Theorem 1. The proof is by induction on r. If $r = 2$, by Stroud's Lemma every element $g \in G'$ modulo $[G', G]$ has the form $\prod_{i=1}^{n} [g_i, x_i]$. By Rhemtulla's Lemma, every element of $[G, G']$ has the form $\prod_{i=1}^{n} [x_i, a_{1i}]$, where $a_{1i} \in G^{(1)}$. Hence

$$
\begin{aligned}
g &= \prod_{i=1}^{n} [x_i, a_{1i}] \prod_{i=1}^{n} [g_i, x_i] \\
&= \prod_{i=1}^{n} [g_i a_{1i}^{-1}, x_i]^{a_{1i}^{x_i}} \\
&= \prod_{i=1}^{n} [g_i a_{1i}^{-1}, x_i].
\end{aligned}
$$

Suppose that $r = 3$, and $G^{(1)} = \langle y_{11}^G, \ldots, y_{1s_1}^G \rangle$. By using the case $r = 2$, any $g \in G'$ can be expressed in the form $g = vu$, where u is in the form $\prod_{i=1}^{n} [g_i a_{1i}^{-1}, x_i]$, (where $a_{1i} \in G^{(1)}$, and $g_i \in G$). And by Lemma 1 $v \in G^{(2)}$ is in the form $\prod_{i=1}^{s_1} [h_{1i}, y_{1i}] \prod_{i=1}^{n} [x_i, a_{2i}]$, where $a_{2i} \in G^{(2)}$ and $h_{1i} \in G^{(1)}$. Hence

$$g = \prod_{i=1}^{s_1} [h_{1i}, y_{1i}] \prod_{i=1}^{n} [x_i, a_{2i}] \prod_{i=1}^{n} [g_i a_{1i}^{-1}, x_i].$$

Now by using equation (3) we have

$$g = \prod_{i=1}^{s_1} [h_{1i}, y_{1i}] \prod_{i=1}^{n} [g_i a_{1i}^{-1} a_{2i}^{-1}, x_i]^{d_{1i}},$$

where $d_{1i} \in G^{(1)}$.

We continue by induction on r. Suppose $r \geq 3$ and G is of class $r+1$. Let $g \in G'$ and $G^{(r-1)} = \langle y_{(r-1)1}^G, y_{(r-1)2}^G, \ldots, y_{(r-1)s_{r-1}}^G \rangle$. By induction g can be expressed in the form vu, where v belongs to $G^{(r)}$ and hence $v = \prod_{i=1}^{s_{r-1}} \left[h_{(r-1)i}, y_{(r-1)i} \right] \prod_{i=1}^{n} \left[x_i, a_{ri} \right]$ for some elements $h_{(r-1)i} \in G^{(r-1)}$ and $a_{ri} \in G^{(r)}$. Now by induction if $r = 2m$ is an even number u can be expressed in form (1), and if $r = 2m + 1$ is an odd number u can be expressed in form (2).

Finally if $r = 2m$, then

$$
g = v \prod_{i=1}^{n} \left[a_{(2m-1)i}, x_i \right]
$$
$$
\prod_{i=1}^{s_{2m-2}} \left[h_{(2m-2)i}, y_{(2m-2)i} \right] \prod_{i=1}^{s_{2m-3}} \left[h_{(2m-3)i}, y_{(2m-3)i} \right]
$$
$$
\prod_{i=1}^{n} \left[a_{(2m-3)i} a_{(2m-2)i}^{-1}, x_i \right]^{d_{(2m-3)i}} \cdots
$$
$$
\prod_{i=1}^{s_4} \left[h_{4i}, y_{4i} \right] \prod_{i=1}^{s_3} \left[h_{3i}, y_{3i} \right] \prod_{i=1}^{n} \left[a_{3i} a_{4i}^{-1}, x_i \right]^{d_{3i}}
$$
$$
\prod_{i=1}^{s_2} \left[h_{2i}, y_{2i} \right] \prod_{i=1}^{s_1} \left[h_{1i}, y_{1i} \right] \prod_{i=1}^{n} \left[g_i a_{1i}^{-1} a_{2i}^{-1}, x_i \right]^{d_{1i}}
$$

Therefore ;

$$
g = \prod_{i=1}^{s_{2m-1}} \left[h_{2m-1i}, y_{2m-1i} \right] \prod_{i=1}^{n} \left[x_i, a_{ri} \right]
$$
$$
\prod_{i=1}^{s_{2m-2}} \left[h_{2m-2i}, y_{2m-2i} \right] \prod_{i=1}^{s_{2m-3}} \left[h_{2m-3i}, y_{2m-3i} \right]
$$
$$
\prod_{i=1}^{n} \left[a_{2m-3i} a_{2m-2i}^{-1}, x_i \right]^{d_{2m-3i}} \cdots
$$
$$
\prod_{i=1}^{s_4} \left[h_{4i}, y_{4i} \right] \prod_{i=1}^{s_3} \left[h_{3i}, y_{3i} \right] \prod_{i=1}^{n} \left[a_{3i} a_{4i}^{-1}, x_i \right]^{d_{3i}}
$$
$$
\prod_{i=1}^{s_2} \left[h_{2i}, y_{2i} \right] \prod_{i=1}^{s_1} \left[h_{1i}, y_{1i} \right] \prod_{i=1}^{n} \left[g_i a_{1i}^{-1} a_{2i}^{-1}, x_i \right]^{d_{1i}}
$$

And if $r = 2m + 1$ is an odd number then :

$$g = \prod_{i=1}^{n} [x_i, a_{ri}] \prod_{i=1}^{s_{2m}} [h_{(2m)i}, y_{(2m)i}]$$

$$\prod_{i=1}^{s_{2m-1}} [h_{2m-1i} y_{2m-1i}], \prod_{i=1}^{n} [a_{2m-1i} a_{(2m)i}^{-1}, x_i]^{d_{2m-1i}}$$

$$\prod_{i=1}^{s_{2m-2}} [h_{2m-2i}, y_{2m-2i}] \prod_{i=1}^{s_{2m-3}} [h_{2m-3i}, y_{2m-3i}]$$

$$\prod_{i=1}^{n} [a_{2m-3i} a_{2m-2i}^{-1}, x_i]^{d_{2m-3i}} \cdots$$

$$\prod_{i=1}^{s_4} [h_{4i}, y_{4i}] \prod_{i=1}^{s_3} [h_{3i}, y_{3i}] \prod_{i=1}^{n} [a_{3i} a_{4i}^{-1}, x_i]^{d_{3i}}$$

$$\prod_{i=1}^{s_2} [h_{2i}, y_{2i}] \prod_{i=1}^{s_1} [h_{1i}, y_{1i}] \prod_{i=1}^{n} [g_i a_{1i}^{-1} a_{2i}^{-1}, x_i]^{d_{1i}}$$

Thus g is a product of $[r+1/2]n + \sum_{i=1}^{r-1} s_i$ commutators, as required. $\square$

Finally we have the following corollary:

COROLLARY 1. *If in a finitely generated solvable group G each term of the derived series is finitely generated as a G-subgroup, then G is a c-group.*

We note that, for example if in Theorem 1 we let G be the n-generator free metabelian group M_n. Then the maximal upper bound is obtained.

References

[1] M. Akhavan-Malayeri, *On commutator length of certain classes of solvable groups*, Internat. J. Algebra Comput. **15** (2005), no. 1, 143–147.

[2] _____, *Commutator length and square length of the wreath product of a free group by the infinite cyclic group*, Houston J. Math. **27** (2001), no. 4, 753–756.

[3] M. Akhavan-Malayeri and A, Rhemtulla, *Commutator length of abelian-by-nilpotent groups*, Glasg. Math. J. **40** (1998), no. 1, 117–121.

[4] Kh. S. Allambergenov and V.A. Romankov, *Products of commutators in groups*, Dokl. Akad. Nauk. UzSSR. (1984), no. 4, 14–15(Russian).

[5] C. Bavard and G. Meigniez, *Commutateurs dans les groupes métabéliens*, Indag. Mathem. (N.S.) **3** (1992), no. 2, 129–135.

[6] B. Hartley, *Subgroup of finite index in profinite groups*, Math. Z. **168** (1979), no. 1, 71–76.

[7] A. Rhemtulla, *Commutators of certain finitely generated soluble groups*, Canad. J. Math. **21** (1969), 1160–1164.

[8] D. Segal, *Closed subgroups of profinite groups*, Proc. London Math. Soc. (3) **81** (2000), no. 1, 29-54.

[9] P. Stroud, *Ph. D. Thesis*, Cambridge, 1966.

ALZAHRA UNIVERSITY, VANAK, TEHRAN 19834, IRAN
E-mail: makhavanm@yahoo.com, mmalayer@alzahra.ac.ir