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STATIONARITY AND β-MIXING PROPERTY OF A MIXTURE AR-ARCH MODELS

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ABSTRACT. We consider a MAR model with ARCH type conditional heteroscedasticity. MAR-ARCH model can be derived as a smoothed version of the double threshold AR-ARCH model by adding a random error to the threshold parameters. Easy to check sufficient conditions for strict stationarity, β-mixing property and existence of moments of the model are given via Markovian representation technique.

1. Introduction

During the past two decades there has been a growing interest in nonlinear time series models. The threshold autoregressive model (TAR) Tong [9] and the autoregressive conditionally heteroscedastic model (ARCH) Engle [3] have been among the most widely used models. Recently, Le *et al* [6] introduced the mixture transition distribution models to capture the flat stretches, bursts and outliers in time series. This model is generalized to the mixture autoregressive (MAR) models which consist of a mixture of several AR components Wong and Li [10].

Adopting MAR model, it is possible to model multimodal conditional distributions and capture conditional heteroscedasticity. Unlike the TAR models, the MAR model implies conditional heteroscedasticity even though the innovation is homoscedastic. But many financial time series exhibit conditional heteroscedasticity which is not of the type inherent in the MAR model.

As an extension of MAR model, a mixture autoregressive conditional heteroscedasticity (MAR-ARCH) model is proposed for modeling nonlinear time series that is the conditional mean follows MAR process,

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whereas the conditional variance of the process follows a mixture ARCH process. The MAR-ARCH model retains all of the nice properties of the MAR model and flexible squared autocorrelation structure. It is shown that the MAR-ARCH model can be successfully applied in finance and macroeconomics (see, e.g., Zeevi *et al* [13], Wong and Li [11, 12] Lanne and Saikkonen [4]).

In this paper, we consider the MAR model with ARCH errors defined by

$$(1.1)\quad y_t = \sum_{i=1}^{m}(b_{i0} + b_{i1}y_{t-1} + b_{i2}y_{t-2} + \cdots + b_{ip}y_{t-p} + \sigma_{it}\epsilon_t) \times I(c_{i-1} + \eta_t \le y_{t-d} < c_i + \eta_t),$$

where $\sigma_{it}^2 = \alpha_{i0} + \alpha_{i1}u_{it-1}^2 + \cdots + \alpha_{iq}u_{it-q}^2$ with $u_{it} = y_t - b_{i0} - b_{i1}y_{t-1} - \cdots - b_{ip}y_{t-p}$ and $I(\cdot)$ is the indicator function. Here $d, 1 \le d \le p$ is a delayed parameter; $-\infty = c_0 < c_1 < \cdots < c_{m-1} < c_m = \infty (m \ge 2)$ are threshold parameters ; $b_{ij}, \alpha_{il} (i = 1, 2, \ldots, m, j = 0, \ldots, p, l = 0, \ldots, q)$ are unknown parameters with $\alpha_{i0} > 0, \alpha_{il} \ge 0$; $p \ge 1, q \ge 0$. Assume that $\{\eta_t\}$ and $\{\epsilon_t\}$ are independent processes such that η_t is normal with mean 0 and variance σ_η and ϵ_t is normal with mean 0 and unit variance.

(1.1) shows that MAR-ARCH model can be considered as a flexible alternative of the double threshold AR-ARCH model. The second order stationarity and autocorrelation structure of the MAR-ARCH process were examined in Wong and Li [11] and Lanne and Saikkonen [4]. Estimation, testing hypothesis and examples are discussed in Zeevi *et al* [13], Lanne and Saikkonen [4, 5] etc.

Our aim is to give sufficient conditions for strict stationarity and β-mixing property of the process y_t given in (1.1). It is one way to prove the geometric ergodicity of the properly defined associated Markov process Y_t on R^{p+q},

$$(1.2)\quad Y_t = (y_t, y_{t-1}, \ldots, y_{t-p-q+1}),$$

by using Markov chain technique and from which we get the desired results of y_t. Existence of moments is also considered.

Section 2 presents the results. All proofs are in section 3.

2. Main results

In this section, we first consider the irreducibility of the Markov chain Y_t defined in (1.1) and (1.2).

Let ϕ and Φ denote the density function and the cumulative distribution function of the standard normal distribution, respectively.

A conditional density function $f_{t-1}(y_t)$ of y_t in (1.1), given $y_{t-j}, j \geq 1$ is known as (see, Lennan and Saikkonen [4])

$$f_{t-1}(y_t) = \sum_{i=1}^{m} \frac{1}{\sigma_{it}} \phi((y_t - b_{i0} - b_{i1}y_{t-1} - \cdots - b_{ip}y_{t-p})/\sigma_{it})\pi_{i,t-d},$$

where the mixing proportions $\pi_{i,t-d}(i = 1, 2, \ldots, m)$ are defined as

$$\pi_{i,t-d} = \begin{cases} 1 - \Phi((y_{t-d} - c_1)/\sigma_\eta), i = 1 \\ \Phi((y_{t-d} - c_{i-1})/\sigma_\eta) - \Phi((y_{t-d} - c_i)/\sigma_\eta), i = 2, \ldots, m-1 \\ \Phi((y_{t-d} - c_{m-1})/\sigma_\eta), i = m. \end{cases}$$

Therefore the $(p+q)$-step transition probability density function $g(\cdot)$ of $\{Y_t, t \geq 0\}$, that is, it is the joint density of Y_{t+p+q} conditional on $y_{t-j}, j \geq 0$ is given by

$$g((y_{t+p+q}, y_{t+p+q-1}, \ldots, y_{t+1})) = \Pi_{j=0}^{p+q-1} f_{t+j}(y_{t+j+1}).$$

LEMMA 2.1. *$Y_t, t \geq 0$ in (1.2) is an aperiodic λ-irreducible Markov chain with Lebesgue measure λ on R^{p+q} and every compact set is a small set.*

THEOREM 2.1. (*Meyn and Tweedie* [8]) *Suppose that the Markov process $\{Y_t : t \geq 0\}$ is aperiodic λ-irreducible and B is a small set. Suppose there are constants $\rho < 1, \epsilon > 0$ and a measurable function $V \geq 1$ such that*

$$E[V(Y_t)|Y_{t-1} = z] \leq \rho V(z) - \epsilon, \quad z \in B^c \tag{2.3}$$

and

$$\sup_{z \in B} E[V(Y_t)|Y_{t-1} = z] < \infty. \tag{2.4}$$

Then the Markov process Y_t is geometrically ergodic.

LEMMA 2.2. *Let $v(z) = \sum_{i=1}^{n} \gamma_i |z_i|^r$, $z = (z_1, \ldots, z_n)$, $n \in Z^+, r > 0$. If $\sum_{i=1}^{n} \xi_i < 1$ with $\xi_i \geq 0$, we may choose $\gamma_i > 0$, $i = 1, \ldots, n$ so that for some positive constant $\rho < 1$,*

$$\gamma_1 (\sum_{i=1}^{n} \xi_i |z_i|^r) + \sum_{i=2}^{n} \gamma_i |z_{i-1}|^r \leq \rho v(z). \tag{2.5}$$

Define that

$$b_j = \max_{1\le i\le m} |b_{ij}| \quad (0 \le j \le p),$$
$$\alpha_l = \max_{1\le i\le m} \alpha_{il} \quad (0 \le l \le q).$$

Next theorem is our main result.

THEOREM 2.2. (1) *If* b_j, α_l $(1 \le j \le p,\ 1 \le l \le q)$ *and* ϵ_t *hold one of the following;*

(*a*) *for* $0 < r \le 1$, $\sum_{j=1}^{p} b_j^r + (1+\sum_{j=1}^{p} b_j^r)(\sum_{l=1}^{q} \alpha_l^{r/2})E|\epsilon_t|^r < 1$,

(*b*) *for* $1 < r \le 2$, $(\sum_{j=1}^{p} b_j)^r + (1+\sum_{j=1}^{p} b_j)^r(\sum_{l=1}^{q} \alpha_l^{r/2})E|\epsilon_t|^r < 1$,
then y_t *in (1.1) is strictly stationary and* β*-mixing with* $E_\pi|y_t|^r < \infty$, *where* π *is the stationary distribution of* y_t.

(2) *If* $4(\sum_{j=1}^{p} b_j)^4 + 6(1+\sum_{j=1}^{p} b_j)^4(\sum_{l=1}^{q}\alpha_l)^2 < 1$, *then* y_t *in* (1.1) *is strictly stationary and* β*-mixing with* $E_\pi(y_t^4) < \infty$.

3. Proofs

Proof of Lemma 2.1. Let $g(y|z)$ denote the joint density of Y_{t+p+q} given $Y_t = z$. For any $z \in R^{p+q}$ and $A \in \mathcal{B}(R^{p+q})$ with $\lambda(A) > 0$,

$$\sum_{t=1}^{\infty} p^{(t)}(z,A) \ge p^{(p+q)}(z,A) = \int_A g(y|z)dy > 0,$$

since g is continuous and positive. If C is a compact set, then we have that

$$\min_{(z,y)\in C\times C} g(y|z) > \delta$$

for some $\delta > 0$. For any $z \in C$ and any $A \in \mathcal{B}(R^{(p+q)})$, we have that

$$p^{(p+q)}(z,A) \ge p^{(p+q)}(z, A\cap C) = \int_{A\cap C} g(y|z)dy \ge \delta\lambda(A\cap C)$$

and hence every compact subset is small. □

Proof of Lemma 2.2. Let $\delta > 0$ be such that $\sum_{i=1}^{n} \xi_i + \delta = 1$. Choose $\gamma_1 > 0$ arbitrary and define

$$\gamma_{i+1} = \gamma_1(1 - \xi_1 - \cdots - \xi_i - \frac{i\delta}{n}), \quad i = 1,2,\ldots,n-1. \tag{3.6}$$

Then following two inequalities hold:

$$\gamma_1\xi_i + \gamma_{i+1} \le \gamma_i(1-\delta/n), \quad 1 \le i \le n-1, \tag{3.7}$$
$$\gamma_1\xi_n \le \gamma_n(1-\delta/n). \tag{3.8}$$

Using (3.3) and (3.4), we obtain that

$$\sum_{i=1}^{n-1}(\gamma_1\xi_i+\gamma_{i+1})|z_i|^r+\gamma_1\xi_n|z_n|^r \le (1-\delta/n)\sum_{i=1}^{n}\gamma_i|z_i|^r.$$

Thus (2.3) holds with $\rho=(1-\delta/n)$. □

Proof of Theorem 2.2. The process y_t in (1.1) can be rewritten as

$$y_t=g_1(y_{t-1},\dots,y_{t-p},\eta_t)+g_2(y_{t-1},\dots,y_{t-p-q},\eta_t)\epsilon_t.$$

Here

$$g_1(y_{t-1},\dots,y_{t-p},\eta_t)=\sum_{i=1}^{m}(b_{i0}+b_{i1}y_{t-1}+\cdots+b_{ip}y_{t-p})I_{itd},$$

$$g_2^2(y_{t-1},\dots,y_{t-p-q},\eta_t)=\sum_{i=1}^{m}(\alpha_{i0}+\alpha_{i1}u_{it-1}^2+\cdots+\alpha_{iq}u_{it-q}^2)I_{itd},$$

and $I_{itd}=I(c_{i-1}+\eta_t\le y_{t-d}<c_i+\eta_t)$.

(1) We first define a test function $V: R^{p+q}\to R$ by

$$V(z_1,\dots,z_{p+q})=\sum_{i=1}^{p+q}\gamma_i|z_i|^r+1,$$

where γ_i $(1\le i\le p+q)$ are to be defined later.

We note that if $0<r\le 1$, then

$$|a_0+\sum a_iz_i|^r\le|a_0|^r+\sum|a_iz_i|^r. \tag{3.9}$$

If $r>1$, then by convexity of $|z|^r$, we obtain that for any fixed $\epsilon>0$, there exists $M=M(\epsilon)>0$ such that if $\|z\|>M$,

$$|a_0+\sum a_iz_i|^r\le(1+\epsilon)^r(\sum|a_i|)^{r-1}(\sum|a_i||z_i|^r). \tag{3.10}$$

Applying the inequalities (3.6) and (3.7) yields that

(3.11)

$$|g_1(z_1,\dots,z_p,\eta_t)|^r\le\begin{cases}b_0^r+\sum_{j=1}^{p}b_j^r|z_j|^r, & 0<r\le 1\\ (1+\epsilon)^r(\sum_{j=1}^{p}b_j)^{r-1}(\sum_{j=1}^{p}b_j|z_j|^r), & 1<r\le 2,\end{cases}$$

and

$$\begin{aligned}|g_2(z_1,\dots,z_{p+q},\eta_t)|^r&\le(\alpha_0+\textstyle\sum_{l=1}^{q}\alpha_lu_l^2)^{r/2}\\&\le\alpha_0^{r/2}+\textstyle\sum_{l=1}^{q}\alpha_l^{r/2}u_l^r\end{aligned} \tag{3.12}$$

with $u_l = |z_l| + b_0 + b_1|z_{l+1}| + \cdots + b_p|z_{l+p}|, \quad l = 1, 2, \ldots, q$. Moreover,

(3.13)

$$\begin{aligned} &|g_2(z_1, \ldots, z_{p+q}, \eta_t)|^r \\ \leq\ & \begin{cases} \sum_{l=1}^q \alpha_l^{r/2}(|z_l|^r + \sum_{j=1}^p b_j^r|z_{l+j}|^r) + C_1, & 0 < r \leq 1 \\ \sum_{l=1}^q \alpha_l^{r/2}(1+\epsilon)^r(1 + \sum_{j=1}^p b_j)^{r-1} \\ \qquad \times(|z_l|^r + \sum_{j=1}^p b_j|z_{l+j}|^r) + C_2, & 1 < r \leq 2 \end{cases} \\ =\ & \sum_{i=1}^{p+q} \beta_i|z_i|^r + C_3, \end{aligned}$$

where $\sum_{i=1}^{p+q} \beta_i = (1+\sum_{j=1}^p b_j^r)(\sum_{l=1}^q \alpha_l^{r/2})$ for $0 < r \leq 1$, and $\sum_{i=1}^{p+q} \beta_i = (1+\epsilon)^r(1+\sum_{j=1}^p b_j)^r(\sum_{l=1}^q \alpha_l^{r/2})$ for $1 < r \leq 2$. Let, in this section, C_j be a generic notation for positive constants.

Now since ϵ_t is symmetric, the fact that the inequality $(1+x)^r + (1-x)^r \leq 2(|x|^r + 1)$ holds for $1 < r \leq 2$, $-1 \leq x \leq 1$ implies that for $0 < r \leq 2$,

$$\begin{aligned} E_{\epsilon_t}|g_1 + g_2\epsilon_t|^r &= 1/2E_{\epsilon_t}[|g_1 + g_2\epsilon_t|^r + |g_1 - g_2\epsilon_t|^r] \\ &\leq |g_1|^r + |g_2|^r E|\epsilon_t|^r. \end{aligned} \tag{3.14}$$

Combining (3.8)-(3.11), we have that for $0 < r \leq 2$ and $z = (z_1, \ldots, z_{p+q})$ with $\|z\| > M$,

$$\begin{aligned} E[V(Y_t) \mid Y_{t-1} = z] &\leq \gamma_1(|g_1^*|^r + |g_2^*|^r E|\epsilon_t|^r) + \sum_{i=2}^{p+q} \gamma_i|z_{i-1}|^r + 1 \\ &\leq \gamma_1(\sum_{i=1}^{p+q} \xi_i|z_i|^r) + \sum_{i=2}^{p+q} \gamma_i|z_{i-1}|^r + 1 + C_4, \end{aligned} \tag{3.15}$$

where $g_1^*(z_1, \ldots, z_p) = b_0 + \sum_{i=1}^p b_i|z_i|$, $g_2^{*2}(z_1, \ldots, z_{p+q}) = \alpha_0 + \sum_{l=1}^q \alpha_l u_l^2$, and

(3.16)

$$\begin{aligned} &\sum_{i=1}^{p+q} \xi_i \\ =\ & \begin{cases} \sum_{j=1}^p b_j^r + (1 + \sum_{j=1}^p b_j^r)(\sum_{l=1}^q \alpha_l^{r/2})E|\epsilon_t|^r < 1, & 0 < r \leq 1 \\ (1+\epsilon)^r((\sum_{j=1}^p b_j)^r + (1 + \sum_{j=1}^p b_j)^r(\sum_{l=1}^q \alpha_l^{r/2})E|\epsilon_t|^r) < 1, \\ & 1 < r \leq 2. \end{cases} \end{aligned}$$

Define γ_i $(i = 1, 2, \ldots, p+q)$ by the same manner as given in (3.2). Then it follows from (3.12), (3.13), assumption (a) or (b) and Lemma 2.2

that there exists some constant $\rho < 1$ such that for $\|z\| > M$,

$$E[V(Y_t)|Y_{t-1} = z] \leq \rho V(z) + C_5.$$

Thus inequalities (2.1) and (2.2) in Theorem 2.1 hold with some $\epsilon > 0$ and compact set $B = \{\|z\| \leq M\}$ for sufficiently large $M < \infty$, since $V(z)$ increases as $\|z\|$ increases. Therefore the geometric ergodicity and hence the strict stationarity and β-mixing property of y_t are obtained. Existence of the r-th moment of y_t is also derived.

(2) To prove part (2), we define $V : R^{p+q} \to R$ by

$$V(z_1, \ldots, z_{p+q}) = \sum_{i=1}^{p+q} \gamma_i z_i^4 + 1. \tag{3.17}$$

Note that from $E(\epsilon_t) = E(\epsilon_t^3) = 0$ and (3.7), we get that for $\|z\| > M$,

$$E(g_1 + g_2\epsilon_t)^4 \leq (1 + 3E\epsilon_t^2)g_1^{*4} + (E\epsilon_t^4 + 3E\epsilon_t^2)g_2^{*4}, \tag{3.18}$$

with

$$g_1^{*4}(z_1, \ldots, z_p) \leq (1+\epsilon)^4(\sum_{j=1}^{p} b_j z_j^4)(\sum_{j=1}^{p} b_j)^3 = (1+\epsilon)^4 \sum_{i=1}^{p} \eta_i z_i^4, \tag{3.19}$$

and

$$\begin{aligned} g_2^{*4}(z_1, \ldots, z_{p+q}) \quad &\leq \quad (1+\epsilon)^6(\sum_{l=1}^{q} \alpha_l)(1 + \sum_{j=1}^{p} b_j)^3 \\ &\quad \times \sum_{l=1}^{q} \alpha_l (z_l^4 + \sum_{j=1}^{p} b_j z_{l+j}^4) \\ &= \quad (1+\epsilon)^6 \sum_{i=1}^{p+q} \delta_i z_i^4, \end{aligned} \tag{3.20}$$

where g_1^* and g_2^* are given in (3.12), $\sum_{i=1}^{p} \eta_i = (1+\epsilon)^4(\sum b_i)^4$, and $\sum_{i=1}^{p+q} \delta_i = (1+\epsilon)^6(\sum_{l=1}^{q} \alpha_l)^2(1 + \sum_{j=1}^{p} b_j)^4$.

From (3.14)–(3.17), $E(\epsilon_t^2) = 1$ and $E(\epsilon_t^4) = 3$, we have that if $\|Z\| > M$,

$$\begin{aligned} E[V(Y_t)|Y_{t-1} = (z_1, \ldots, z_{p+q})] \quad &\leq \quad \gamma_1[4g_1^{*4} + 6g_2^{*4}] + \sum_{i=2}^{p+q} \gamma_i z_{i-1}^4 + 1 \\ &\leq \quad \gamma_1 \sum_{i=1}^{p+q} (4\eta_i + 6\delta_i) z_i^4 + \sum_{i=2}^{p+q} \gamma_i z_{i-1}^4 \end{aligned} \tag{3.21}$$

where $\eta_{p+1} = \cdots = \eta_{p+q} = 0$. By assumption, we may choose $\epsilon > 0$ so small that $4(1+\epsilon)^4 \sum_{i=1}^{p+q} \eta_i + 6(1+\epsilon)^6 \sum_{i=1}^{p+q} \delta_i < 1$, and hence deduce the conclusion from (3.18), Lemma 2.1 and Theorem 2.1. □

REMARK 1. Recall that the geometric ergodicity of a Markov chain implies the β-mixing (absolute regularity) and strongly mixing of the process.

REMARK 2. Instead of normal distribution assumption of ϵ_t, other symmetric mean zero distribution may be used without difficulty.

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