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GEOMETRY OF CONTACT STRONGLY PSEUDO-CONVEX CR-MANIFOLDS

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ABSTRACT. As a natural generalization of a Sasakian space form, we define a contact strongly pseudo-convex CR-space form (of constant pseudo-holomorphic sectional curvature) by using the Tanaka-Webster connection, which is a canonical affine connection on a contact strongly pseudo-convex CR-manifold. In particular, we classify a contact strongly pseudo-convex CR-space form (M, η, φ) with the pseudo-parallel structure operator $h(= 1/2L_\xi\varphi)$, and then we obtain the nice form of their curvature tensors in proving Schur-type theorem, where L_ξ denote the Lie derivative in the characteristic direction ξ.

1. Introduction

A *contact manifold* (M, η) is a smooth manifold M^{2n+1} together with a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It means that $d\eta$ has a maximal rank $2n$ on the contact distribution (or sub-bundle) $D(=$ kernel of $\eta)$. This fact arises naturally the *characteristic vector field* ξ on M, and then leads to the decomposition $TM = D \oplus \{\xi\}$. Given a contact structure η, we have two associated structures. One is a Riemannian structure (or metric) g, and then we call $(M; \eta, g)$ a *contact Riemannian manifold.* The other is an *almost CR-structure* (η, L), where L is the *Levi form* associated with an endomorphism J on D such that $J^2 = -I$. In particular, if J is integrable, then we call it the (integrable) CR-structure. The associated almost CR-structure is said to be *pseudohermitian, strongly pseudo-convex* if the Levi form is hermitian and positive definite. We call such a manifold a *contact strongly*

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pseudo-convex almost CR-manifold. There is a one-to-one correspondence between the two associated structures by the relation

$$g = L + \eta \otimes \eta,$$

where we denote by the same letter L the natural extension of the Levi form to a (0,2)-tensor field on M, that is, $i_\xi L = 0$, where i_ξ denotes the interior product by ξ. We also denote by φ the natural extension of J, which means that $\varphi|_D = J$ and $\varphi\xi = 0$. Then the above correspondence may be rephrased by the relation between (η, g) and (η, φ). From this point of view, we have two geometries for a given contact manifold, that is, one is formed by the Levi-Civita connection ∇, the other is derived by the *Tanaka-Webster connection* $\hat{\nabla}$, which is a canonical affine connection on a strongly pseudo-convex CR-manifold.

The normality of a contact Riemannian structure is defined in [13] (see, section 2). A normal contact Riemannian manifold is called a Sasakian manifold. A Sasakian structure has another picture, namely, a contact strongly pseudo-convex CR-structure whose characteristic vector field is a Killing vector field with respect to its associated Riemannian structure. In this context, we have two sides for a Sasakian space form: one is defined by a Sasakian manifold with constant φ-holomorphic sectional curvatures with respect to ∇ and the other is of constant pseudo-holomorphic sectional curvature with respect to $\hat{\nabla}$. Indeed, in [8] we defined a contact Riemannian space form which extends a Sasakian space form in the Riemannian view point. Corresponding to that, in this paper we introduce a notion, say, a *contact strongly pseudo-convex CR-space form*, which is a contact strongly pseudo-convex CR-manifold M of constant pseudo-holomorphic sectional curvature c (with respect to $\hat{\nabla}$), that is, M satisfies for any unit vector field X orthogonal to ξ

$$L(\hat{R}(X, \varphi X)\varphi X, X) = c \ (constant).$$

The main purpose of this paper is to find a proper class of contact strongly pseudo-convex CR-space forms (containing Sasakian space forms) and to study their geometric properties.

In particular, a contact strongly pseudo-convex CR-manifold satisfies CR-integrability or the condition of η-parallel φ (that is, $g((\nabla_X\varphi)Y, Z) = 0$ for all vector fields X, Y, Z orthogonal to ξ). We note that it is also equivalent to the condition of *pseudo-parallel* φ which is defined by

$$L((\hat{\nabla}_X\varphi)Y, Z) = 0$$

for all vector fields X, Y, Z orthogonal to ξ. Here, it is remarkable that the normality of a contact Riemannian structure implies the integrability of the associated CR-structure. But the converse does not always hold. In fact, there are some examples of contact Riemannian manifolds which have integrable CR-structures but are not Sasakian. Other than all 3-dimensional contact Riemannian manifolds ([17]), we see that their associated CR-structures are integrable for (non-Sasakian) contact (k,μ)-spaces (cf. [3], [8]). This class was introduced in [3] and their spaces are studied intensively in [4], [5] and [9]. In particular, their local classification is given in [5].

We restrict our attention to a more suitable class of contact strongly pseudo-convex CR-manifolds endowed with an additional property, namely, it is imposed by the condition of *pseudo-parallel h*:

$$L((\hat{\nabla}_X h)Y, Z) = 0$$

for all vector fields X, Y, Z orthogonal to ξ, where h denotes, up to a scaling factor, the Lie derivative of φ in the direction of ξ. As concerns this condition, we note that it is also equivalent to η-parallel h (with respect to ∇), i.e., $g((\nabla_X h)Y, Z) = 0$ for all vector fields X, Y, Z orthogonal to ξ. Recently, E. Boeckx and the present author [6] proved that a contact Riemannian space with η-parallel h is either a K-contact space (in which case, h vanishes identically) or a (k,μ)-space. A contact strongly pseudo-convex CR-manifold with pseudo-parallel h is called a *pseudo-parallel contact strongly pseudo-convex CR-manifold*, or shortly, a *pseudo-parallel contact CR-space.*

In Section 2, we collect preliminary notions and results which are needed in later sections. In Section 3, we study the Tanaka-Webster curvature tensor $\hat{R}$ of a contact strongly pseudo-convex CR-manifold. In Section 4, we classify a pseudo-parallel contact strongly pseudo-convex CR-space form. In more detail, a pseudo-parallel contact strongly pseudo-convex CR-space of constant pseudo-holomorphic sectional curvature c is pseudo-homothetic to one of the following: (1) the (normalized) model spaces of Sasakian space forms, (2) the unit tangent sphere bundle of a space of constant curvature -1, or (3) a non-Sasakian Lie group with a special left-invariant contact metric, $SU(2)$, $SL(2,R)$, the group $E(2)$ of rigid motions of Euclidean 2-space, the group $E(1,1)$ of rigid motions of the Minkowski 2-space (Corollary 4.3). It is remarkable that the case (2) above is neither Sasakian nor a space of constant φ-holomorphic sectional curvature.

In Section 5, we obtain the curvature form of a contact strongly pseudo-convex CR-manifold with constant pseudo-holomorphic sectional curvature. Finally, in Section 6, for the class of pseudo-parallel contact strongly pseudo-convex CR-manifolds, we prove a Schur-type theorem. Then we have the nice form of the curvature tensor of a pseudo-parallel contact strongly pseudo-convex CR-space form.

2. Preliminaries

We start by collecting some fundamental materials about contact Riemannian geometry and contact strongly pseudo-convex CR-manifold. We refer to [2] for further details. All manifolds in the present paper are assumed to be connected and of class C^∞.

A $(2n+1)$-dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, there exists a unique vector field ξ, called the *characteristic vector field*, satisfying $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$ for any vector field X. It is well-known that there also exists a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$
\begin{aligned}
&g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),\\
&d\eta(X,Y) = g(X, \varphi Y),\\
&\varphi^2 X = -X + \eta(X)\xi,
\end{aligned} \tag{2.1}
$$

where X and Y are vector fields on M. From (2.1), it follows that

$$
\varphi\xi = 0, \ \ \eta \circ \varphi = 0, \ \ \eta(X) = g(X, \xi). \tag{2.2}
$$

A Riemannian manifold M equipped with structure tensors (η, g) satisfying (2.1) is said to be a *contact Riemannian manifold* or *contact metric manifold* and it is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold M, we define an operator h by $h = \frac{1}{2} L_\xi \varphi$, where L denotes Lie differentiation. Then we may observe that the *structural operator* h is symmetric and satisfies

$$
h\xi = 0, \quad h\varphi = -\varphi h, \tag{2.3}
$$

$$
\nabla_X \xi = -\varphi X - \varphi h X, \tag{2.4}
$$

where ∇ is Levi-Civita connection. We denote by R the Riemannian curvature tensor defined by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for all vector fields X, Y, Z on M. Along a trajectory of ξ, the *characteristic Jacobi operator* $l = R(\cdot, \xi)\xi$ is also symmetric. Moreover, we have

$$\nabla_\xi h = \varphi - \varphi l - \varphi h^2, \tag{2.5}$$

$$\begin{aligned} g(R(X,Y)\xi, Z) = {} & g((\nabla_Y\varphi)X - (\nabla_X\varphi)Y, Z) \\ & + g((\nabla_Y\varphi h)X - (\nabla_X\varphi h)Y, Z) \end{aligned} \tag{2.6}$$

for all vector fields X, Y, Z on M. A contact Riemannian manifold for which ξ is a Killing vector field, is called a K-contact manifold. It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. For a contact Riemannian manifold M, one may define naturally an almost complex structure $\mathfrak{J}$ on $M \times \mathbb{R}$ by

$$\mathfrak{J}(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure $\mathfrak{J}$ is integrable, M is said to be normal or Sasakian. It is known that M is normal if and only if M satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A Sasakian manifold is also characterized by the condition

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X \tag{2.7}$$

for all vector fields X and Y on the manifold and this is equivalent to

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.8}$$

for all vector fields X and Y.

For a contact Riemannian manifold $M = (M; \eta, g)$, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus$

$\{\xi\}_p$(direct sum), where we denote $D_p = \{v \in T_pM | \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution orthogonal to ξ. The $2n$-dimensional distribution (or subbundle) D is called the *contact distribution* (*or contact subbundle*). Its associated almost CR-structure is given by the holomorphic subbundle

$$\mathcal{H} = \{X - iJX : X \in D\}$$

of the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM, where $J = \varphi|D$, the restriction of φ to D. Then we see that each fiber $\mathcal{H}_x$ $(x \in M)$ is of complex dimension n and $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$. Furthermore, we have $\mathbb{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}$. We say that *the associated CR-structure is integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. For $\mathcal{H}$ we define the Levi form by

$$L : D \times D \to \mathcal{F}(M), \quad L(X,Y) = -d\eta(X, JY),$$

where $\mathcal{F}(M)$ denotes the algebra of differential functions on M. Then we see that the Levi form is hermitian and positive definite, that is, the CR-structure is *strongly pseudo-convex, pseudo-hermitian CR-structure.* We call the pair (η, L) a *strongly pseudo-convex, pseudo-hermitian structure* on M. Since $d\eta(\varphi X, \varphi Y) = d\eta(X,Y)$, we see that $[JX, JY] - [X, Y] \in D$ and $[JX, Y] + [X, JY] \in D$ for $X, Y \in D$. Furthermore, if M satisfies the condition

$$[J, J](X, Y) = 0$$

for $X, Y \in D$, then the pair (η, L) is called a *strongly pseudo-convex* (*integrable*) *CR-structure* and $(M; \eta, L)$ is called a *strongly pseudo-convex pseudohermitian CR-manifold.* It may be easily proved that the almost CR-structure is integrable if and only if M satisfies the integrability condition $Q = 0$, where Q is a (1,2)-tensor field on M defined by

$$Q(X,Y) = (\nabla_X \varphi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX) \tag{2.9}$$

for all vector fields X, Y on M (see [17, Proposition 2.1]). Taking account of (2.7) we see that for a Sasakian manifold the associated CR-structure is integrable (cf. [12]).

Now, we review the *generalized Tanaka-Webster connection* ([17]) on a contact strongly pseudo-convex almost CR-manifold $M = (M; \eta, L)$. The generalized Tanaka-Webster connection $\hat{\nabla}$ is defined by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X, Y on M. Together with (2.4), $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + A(X,Y), \tag{2.10}$$

where we have put

$$A(X,Y) = \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi hX) - g(\varphi X + \varphi hX, Y)\xi. \tag{2.11}$$

We see that the generalized Tanaka-Webster connection $\hat{\nabla}$ has the torsion

$$\hat{T}(X,Y) = 2g(X,\varphi Y)\xi + \eta(Y)\varphi hX - \eta(X)\varphi hY.$$

In particular, for a K-contact manifold (2.11) reduces as follows:

$$A(X,Y) = \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi. \tag{2.12}$$

Furthermore, it was proved that

PROPOSITION 2.1 ([17]). *The generalized Tanaka-Webster connection $\hat{\nabla}$ on a contact Riemannian manifold $M = (M;\eta, g)$ is the unique linear connection satisfying the following conditions:*
(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g = 0$;
(iii-1) $\hat{T}(X,Y) = 2g(X,\varphi Y)\xi$, $X,\ Y \in D$;
(iii-2) $\hat{T}(\xi,\varphi Y) = -\varphi\hat{T}(\xi, Y)$, $Y \in D$;
(iv) $(\hat{\nabla}_X\varphi)Y = Q(X,Y)$, $X,\ Y \in TM$.

The Tanaka-Webster connection ([14], [20]) on a nondegenerate (integrable) CR-manifold is defined as the unique linear connection satisfying (i), (ii), (iii-1), (iii-2) and $Q = 0$ (CR-integrability). The metric affine connection $\hat{\nabla}$ is a natural generalization of the Tanaka-Webster connection. In fact, in [1] the authors treat the use of $\hat{\nabla}$ in the non-integrable case.

From Proposition 2.1 we immediately see that the CR-integrability condition $Q = 0$ is equivalent to the condition of *pseudo-parallel* φ (*with respect to* $\hat{\nabla}$)

$$L((\hat{\nabla}_X\varphi)Y, Z) = 0$$

for all vector fields X, Y, Z orthogonal to ξ. Since we know that $\nabla_\xi\varphi = 0$ holds (cf. [2] p. 67) in a contact Riemannian manifold, we see further that CR-integrability is also equivalent to the condition of *η-parallel φ*, i.e., $g((\nabla_X\varphi)Y,Z)=0$ for all vector fields X,Y,Z orthogonal to ξ.

From (2.3), (2.10) and (2.11) we have

$$
\begin{aligned}
(\hat{\nabla}_X h)Y &= (\nabla_X h)Y + A(X,hY) - hA(X,Y) \\
&= (\nabla_X h)Y + 2\eta(X)\varphi hY + g((\varphi h+\varphi h^2)X,Y)\xi \\
&\quad + \eta(Y)(\varphi hX + \varphi h^2X).
\end{aligned}
\tag{2.13}
$$

In [6] we studied a contact Riemannian manifold which satisfies the condition that *h is η-parallel* (*with respect to* ∇), i.e., $g((\nabla_X h)Y,Z)=0$ for any vector fields X,Y,Z orthogonal to ξ. Also from (2.13) we see that this is equivalent to the condition that

$$L((\hat{\nabla}_X h)Y,Z)=0$$

for any vector fields X,Y,Z orthogonal to ξ, i.e., *h is pseudo-parallel* (*with respect to* $\hat{\nabla}$). We call a contact strongly pseudo-convex CR-manifold with pseudo-parallel h, a *pseudo-parallel contact strongly pseudo-convex CR-manifold*, or in short, a *pseudo-parallel contact CR-space.*

Here, we recall the notion of a *pseudo-homothetic transformation* (or *D-homothetic transformation*) of a contact metric manifold ([15]). This transformation means a change of structure tensors of the form

$$\bar{\eta}=a\eta,\ \bar{\xi}=1/a\xi,\ \bar{\varphi}=\varphi,\ \bar{g}=ag+a(a-1)\eta\otimes\eta, \tag{2.14}$$

where a is a positive constant. From (2.14), we have $\bar{h}=(1/a)h$. By using the well-known formula

$$
\begin{aligned}
2g(\nabla_XY,Z) =& Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z]) \\
&-g(Y,[X,Z])+g(Z,[X,Y])
\end{aligned}
$$

we have

$$\bar{\nabla}_XY=\nabla_XY+E(X,Y), \tag{2.15}$$

where E is the (1,2)-type tensor defined by

$$E(X,Y)=-(a-1)[\eta(Y)\varphi X+\eta(X)\varphi Y]-\frac{a-1}{a}g(\varphi hX,Y)\xi.$$

REMARK 1. (1) From (2.14) and (2.15), we see that the condition of pseudo-parallel φ (or η-parallel φ) is invariant under a pseudo-homothetic transformation. Indeed by direct computations we have

$$(\bar{\nabla}_X\bar{\varphi})Y = (\nabla_X\varphi)Y + (a-1)\eta(Y)\varphi^2X - (a-1)/ag(X,hY)\xi.$$

(2) The condition of pseudo-parallel h (or η-parallel h) is also invariant under a pseudo-homothetic transformation. Namely, for a pseudo-homothetic transformation we have

$$\begin{aligned}(\bar{\nabla}_X\bar{h})Y = 1/a\Big((\nabla_Xh)Y + (a-1)\eta(Y)h\varphi X + 2(a-1)\eta(X)h\varphi Y \\ - (a-1)/ag(\varphi hX, hY)\xi\Big).\end{aligned}$$

3. The Tanaka-Webster curvature tensor of a contact CR-manifold

Let $(M;\eta,L)$ be a contact strongly pseudo-convex CR-manifold. In this section, we define the Tanaka-Webster curvature tensor of $\hat{R}$ (with respect to $\hat{\nabla}$) (in the extended meaning) by

$$\hat{R}(X,Y)Z = \hat{\nabla}_X(\hat{\nabla}_YZ) - \hat{\nabla}_Y(\hat{\nabla}_XZ) - \hat{\nabla}_{[X,Y]}Z \tag{3.1}$$

for all vector fields X, Y, Z in M. Then we have

PROPOSITION 3.1.

$$\hat{R}(X,Y)Z = -\hat{R}(Y,X)Z,$$
$$g(\hat{R}(X,Y)Z,W) = -g(\hat{R}(X,Y)W,Z).$$

The first identity follows trivially from the definition of $\hat{R}$. Since the connection parallelizes the metric form, (i.e., $\hat{\nabla}g = 0$), we have also the second one by a similar way as the case of Riemanian curvature tensor. We remark that the Tanaka-Webster connection is not torsion-free, the Jacobi- or Bianchi-type identities do not hold, in general. From (3.1), together with $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$, $\hat{\nabla}g = 0$, $\hat{\nabla}\varphi = 0$, the straightforward computations yield

$$\begin{aligned}&\hat{R}(X,Y)Z \\ &= R(X,Y)Z + \eta(Z)\Big(\varphi P(X,Y) + \varphi(A(X,hY) - A(Y,hX)) \\ &\quad - \varphi h(A(X,Y) - A(Y,X))\Big)\end{aligned}$$

$$
\begin{aligned}
&- g\Big(\varphi P(X,Y) + \varphi(A(X,hY) - A(Y,hX)) \\
&- \varphi h(A(X,Y) - A(Y,X)), Z\Big)\xi \\
&- 2g(\varphi X,Y)A(\xi,Z) - \eta(X)A(\varphi hY,Z) + \eta(Y)A(\varphi hX,Z) \\
&- \eta(X)\varphi A(Y,Z) + \eta(Y)\varphi A(X,Z) + \eta(A(X,Z))(\varphi Y + \varphi hY) \\
&- \eta(A(Y,Z))(\varphi X + \varphi hX) + g(\varphi X + \varphi hX, A(Y,Z))\xi \\
&- g(\varphi Y + \varphi hY, A(X,Z))\xi.
\end{aligned}
$$

We put $P(X,Y) = (\nabla_X h)Y - (\nabla_Y h)X$, then we see that P is a $(1,2)$-type tensor field on M. By using (2.1), (2.2), (2.3) and (2.11) we have

$$\hat{R}(X,Y)Z = R(X,Y)Z + B(X,Y)Z, \tag{3.2}$$

where

$$
\begin{aligned}
&B(X,Y)Z \\
&= \eta(Z)\varphi P(X,Y) - g(\varphi P(X,Y),Z)\xi - \eta(Z)\Big(\eta(Y)(X+hX) \\
&\quad - \eta(X)(Y+hY)\Big) + \eta(Y)g(X+hX,Z)\xi - \eta(X)g(Y+hY,Z)\xi \\
&\quad + g(\varphi Y + \varphi hY, Z)(\varphi X + \varphi hX) - g(\varphi X + \varphi hX, Z)(\varphi Y + \varphi hY) \\
&\quad - 2g(\varphi X,Y)\varphi Z
\end{aligned}
$$

for all vector fields X, Y, Z in M. From (3.2), by making use of (2.2) and (2.3), we obtain

$$
\begin{aligned}
&\hat{R}(X,Y)\xi \\
&= R(X,Y)\xi + \varphi P(X,Y) + \eta(X)(Y+hY) - \eta(Y)(X+hX).
\end{aligned} \tag{3.3}
$$

Now, we give

DEFINITION 3.2. Let $(M;\eta,L)$ be a contact strongly pseudo-convex CR-manifold with the associated Levi form L. Then M is said to be of *constant pseudo-holomorphic sectional curvature* c (*with respect to the Tanaka-Webster connection*) if M satisfies

$$L(\hat{R}(X,\varphi X)\varphi X, X) = c$$

for any unit vector field $X \perp \xi$. A complete and simply connected contact strongly pseudo-convex CR-manifold of constant pseudo-holomorphic sectional curvature is called a *contact strongly pseudo-convex CR-space form.*

Here, we recall

DEFINITION 3.3 ([16]). Let $(M;\eta,g)$ be a Sasakian manifold. Then M is called a space of constant φ-holomorphic sectional curvature c_0 if M satisfies

$$g(R(X,\varphi X)\varphi X,X)=c_0$$

for any unit vector field $X\perp\xi$. A complete and simply connected Sasakian space of constant φ-holomorphic sectional curvature is called a Sasakian space form.

Now, we prove that

PROPOSITION 3.4. *The contact strongly pseudo-convex CR-space form is a pseudo-homothetic-invariant.*

Proof. From (2.15), by long but tedious computations, we get

$$\begin{aligned} &g(\bar R(X,\varphi X)\varphi X,X)\\ &=g(R(X,\varphi X)\varphi X,X)-(a-1)[3+g(hX,X)]\\ &\quad-\frac{a-1}{a}[g(\varphi hX,X)^2+g(hX,X)(g(hX,X)-1)]\\ &\quad+\frac{(a-1)^2}{a}g(hX,X)\end{aligned}\tag{3.4}$$

for any unit horizontal vector $X\in D(p)$ (with respect to g), $p\in M$. For any unit horizontal vector X, from (3.2), we get

$$\begin{aligned}&\bar L(\hat{\bar R}(X,\bar\varphi X)\bar\varphi X,X)\\ &=3+\bar g(\bar R(X,\bar\varphi X)\bar\varphi X,X)-\bar g(\bar\varphi\bar hX,X)^2-\bar g(\bar hX,X)^2.\end{aligned}$$

Along with (2.14) and (3.4), we have

$$\bar L(\hat{\bar R}(X,\bar\varphi X)\bar\varphi X,X)=aL(\hat R(X,\varphi X)\varphi X,X).$$

If we denote by $\hat K(X,\varphi X)$ the pseudo-holomorphic sectional curvature $L(\hat R(X,\varphi X)\varphi X,X)$ for a unit horizontal vector X, then this is rewritten by

$$\hat{\bar K}(X,\bar\varphi X)=a\hat K(X,\varphi X).$$

Thus, we have proved. □

REMARK 2. Making use of (2.14) and (3.4) we can see that the contact space of constant φ-holomorphic sectional curvature is not a pseudo-homothetic invariant, in general.

4. Pseudo-parallel contact strongly pseudo-convex CR-space form

We start this section by reviewing in brief a (k, μ)-space. In [3], the (k, μ)-nullity distribution of a contact Riemannian manifold M, for the pair $(k, \mu) \in \mathbb{R}^2$, is defined by

$$\begin{aligned} &N(k, \mu) : p \to N_p(k, \mu) \\ &= \{z \in T_pM | R(x, y)z = (kI + \mu h)(g(y, z)x - g(x, z)y) \\ &\quad \text{for any } x, y \in T_pM\}. \end{aligned}$$

A (k, μ)*-space* is a contact Riemannian manifold with ξ belonging to the (k, μ)-nullity distribution, that is,

$$R(X, Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y), \tag{4.1}$$

where I denotes the identity transformation. It is proved in [3] that the (k, μ)-spaces are invariant under a pseudo-homothetic transformation in the range of (k, μ). More precisely, a pseudo-homothetic transformation with constant a change (k, μ) into $(\bar{k}, \bar{\mu})$, where

$$\bar{k} = \frac{k + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}. \tag{4.2}$$

Also, the associated CR-structures of the (k, μ)-spaces are integrable, that is, they are contact strongly pseudo-convex CR-manifolds. This class contains Sasakian manifolds with $k = 1$ and $h = 0$. The unit tangent sphere bundle is a (k, μ)-space if and only if the base manifold is of constant curvature c with $k = c(2 - c)$ and $\mu = -2c$ ([3]). (By virtue of the result of Y. Tashiro [19], we know that for $c \neq 1$, the unit tangent sphere bundle is non-Sasakian.) In [4], [5] the curvature tensor R of contact (k, μ)-space is determined completely for $k < 1$. Furthermore, E. Boeckx [5] classified non-Sasakian (k, μ)-spaces up to a pseudo-homothetic transformation.

In [3], the authors proved following useful formulas:

$$\begin{aligned} &(\nabla_X h)Y \\ &= [(1 - k)g(X, \varphi Y) - g(X, \varphi hY)]\xi \\ &\quad - \eta(Y)[(1 - k)\varphi X + \varphi hX] - \mu\eta(X)\varphi hY \end{aligned} \tag{4.3}$$

and

$$P(X,Y) = (\nabla_X h)Y - (\nabla_Y h)X \\ = (1-k)[2g(X,\varphi Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X] \\ + (1-\mu)[\eta(X)\varphi hY - \eta(Y)\varphi hX]. \tag{4.4}$$

Then from (4.3), we immediately see that a (k,μ)-space has a pseudo-parallel structure. Moreover, together with the result in [6] we have

THEOREM 4.1. *A pseudo-parallel contact strongly pseudo-convex CR-manifold is Sasakian or a (k,μ)-space.*

Now, from (3.2), we have for unit vector field $X \perp \xi$

$$L(\hat{R}(X,\varphi X)\varphi X, X) \\ = 3 + g(R(X,\varphi X)\varphi X, X) - g(\varphi hX, X)^2 - g(hX,X)^2. \tag{4.5}$$

Hence, we see that M is of constant pseudo-holomorphic sectional curvature c if and only if

$$K(X,\varphi X)(= g(R(X,\varphi X)\varphi X, X)) \\ = (c-3) + g(\varphi hX, X)^2 + g(hX,X)^2. \tag{4.6}$$

We prove

THEOREM 4.2. *Let M be a contact (k,μ)-space. Then M is of constant pseudo-holomorphic sectional curvature c if and only if* (1) *M is Sasakian space of constant φ-holomorphic sectional curvature $c_0 = (c-3)$,* (2) *$\mu = 2$ and $c = 0$, or* (3) *dim M=3 and $\mu = (2-c)$.*

Proof. We let M be a non-Sasakian contact (k,μ)-space ($k \neq 1$). Then we already know that (cf. [3] or [8])

$$K(X,\varphi X) = (1-2\mu) + \frac{k+1-\mu}{k-1}[g(\varphi hX, X)^2 + g(hX,X)^2]. \tag{4.7}$$

Thus, from (4.6) and (4.7), we can deduce the following three cases: (1) $k=1$ ($h=0$) and M is a Sasakian space form, (2) $k<1$, $\mu=2$ and $c=0$, (3) If dim $M=3$, then we see that $g(\varphi hX,X)^2 + g(hX,X)^2 = 1/2(\text{trace of } h^2)$. But, since $h^2 = (k-1)\varphi^2$ (cf. [3]), we have $\mu = (2-c)$. □

In the proof of Proposition 3.4, we see that for a Sasakian space (whose structure is invariant under a pseudo-homothetic transformation) the constancy of pseudo-holomorphic sectional curvature is invariant under pseudo-homothetic transformations (indeed, $\bar{c} = c/a$, $a > 0$). From (4.2), we also see that a $(k, 2)$-space is invariant under a pseudo-homothetic transformation and $I_M = \frac{1-\mu/2}{\sqrt{1-k}} = 0 > -1$. Thus, due to the classification theorem of a (k,μ)-space in [5], we see that $(k, 2)$-space is pseudo-homothetic to $T_1M(-1)$. For the three-dimensional non-Sasakian (k,μ)-space, the local classification is given in [3]. Further in [5] E. Boeckx showed up there picture up to a pseudo-homothetic invariant I_M, indeed there are $SL(2,R)$ with $I_{SL(2,R)} < -1$ or $-1 < I_{SL(2,R)} < 1$, $E(1,1)$ with $I_{E(1,1)} = -1$, $E(2)$ with $I_{E(2)} = 1$, $SU(2)$ with $I_{SU(2)} > 1$. Thus, From Theorems 4.1 and 4.2, we have

COROLLARY 4.3. *Let M be a pseudo-parallel contact strongly pseudo-convex CR-space. Then M is of constant pseudo-holomorphic sectional curvature c if and only if M is pseudo-homothetic to one of the following:*

(1) the unit sphere S^{2n+1} with the natural Sasakian structure with $c_0 = 1$ for $c > 0$, R^{2n+1} with Sasakian φ-holomorphic sectional curvature $c_0 = -3$ for $c = 0$, or $B^n \times R$ with Sasakian φ-holomorphic sectional curvature $c_0 = -7$ for $c < 0$, where B^n is a simply connected bounded domain in C^n with constant holomorphic sectional curvature -4,

(2) the unit tangent sphere bundle of a space of constant curvature -1, or

(3) a non-Sasakian Lie group with a special left-invariant contact metric, $SU(2)$, $SL(2,R)$, the group $E(2)$ of rigid motions of Euclidean 2-space, the group $E(1,1)$ of rigid motions of the Minkowski 2-space.

The Sasakian structure (η, g) on $R^{2n+1}(x^i, y^i, z)$ $(i = 1, \ldots, n)$ is given by the canonical contact structure

$$\eta = \frac{1}{2}(dz - \sum_i^n y^i dx^i)$$

and the Riemannian metric g given by the quadratic form

$$ds^2 = \frac{1}{4}(\eta \otimes \eta + \sum_i^n ((dx^i)^2 + (dy^i)^2)).$$

We know that the standard contact metric structure of the unit tangent sphere bundle $T_1M(1)$ of a space of constant curvature 1 is

Sasakian. However, we can check that it has neither constant φ-holomorphic sectional curvature nor constant pseudo-holomorphic sectional curvature. As stated already, unit tangent sphere bundles are (k,μ)-spaces if and only if the base manifold is of constant curvature b with $k = b(2-b)$ and $\mu = -2b$. Thus, we have

COROLLARY 4.4. *The standard contact strongly pseudo-convex CR-structure of a unit tangent sphere bundle $T_1M(b)$ of $(n+1)$-dimensional space of constant curvature b has constant pseudo-holomorphic sectional curvature c if and only if*

(1) $b = -1$ *and* $c = 0$, *or*

(2) $n = 1$ *and* $b = 1/2(c-2)$.

For a regular (i.e., the foliation defined by the vector field ξ is regular) *Sasakian space form* $M^{2n+1}(c_0)$ of constant φ-holomorphic sectional curvature c_0, the quotient $M^{2n+1}(c_0)/\xi$ with the induced metric and the complex structure J given by $J\pi_*X = \pi_*\varphi X$ is a complex space form $\widetilde{M}^n((c_0+3)/4)$, where $\pi : M \to M/\xi$ is the Riemannian submersion. Closing this section we state

REMARK 4. (1) Three-dimensional non-Sasakian contact (k,μ)-spaces have constant φ-holomorphic sectional curvature (cf. [3], [8]) and at the same time constant pseudo-holomorphic sectional curvatures $c = (2-\mu)$.

(2) A contact $(k,2)$-space $(k \neq 1)$ is non-Sasakian and of non-constant φ-holomorphic sectional curvature (see (4.7)), but has constant "pseudo-holomorphic sectional curvature (with respect to the Tanaka-Webster connection)".

5. The curvature tensor of a contact strongly pseudo-convex CR-space form

In this section, we study the curvature of a contact strongly pseudo-convex CR-space form. Let M be a contact strongly pseudo-convex CR-manifold. We put

$$C(X,Y)Z = R(X,Y)Z + g(hY,Z)hX - g(hX,Z)hY$$

for all vector fields X, Y, Z on M. Then we see that C is a (1,3)-type tensor field on M. From this, by using the symmetries of the curvature tensor R and the symmetry of structure tensor h, we easily see that C also satisfies the symmetries, that is,

$$
\begin{aligned}
&(1)\ C(X,Y)Z = -C(Y,X)Z,\\
&(2)\ g(C(X,Y)Z,W) = -g(C(X,Y)W,Z),\\
&(3)\ g(C(X,Y)Z,W) = g(C(Z,W)X,Y),\\
&(4)\ C(X,Y)Z + C(Y,Z)X + C(Z,X)Y = 0.
\end{aligned}
$$

Further, we see, together with (4.6), that M has pointwise constant pseudo-holomorphic sectional curvature $H(p)$, $p \in M$, if and only if

$$g(C(X,\varphi X)\varphi X, X) = H_1(p)$$

for a unit horizontal vector X, where we have put $H_1(p) = H(p) - 3$.

First of all, from (2.9), we see that M satisfies

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \tag{5.1}$$

for all vector fields X and Y. Comparing with (2.7), it follows that a contact strongly pseudo-convex CR-manifold is normal (or Sasakian) if and only if $h = 0$.

Then we have the following

PROPOSITION 5.1. *For all vector fields X, Y, Z on M,*

$$C(X,Y)\xi = \eta(Y)(X + hX) - \eta(X)(Y + hY) - \varphi P(X,Y), \tag{5.2}$$

$$
\begin{aligned}
&g(C(\xi,X)Y,Z)\\
&= \eta(Z)g(Y + hY, X) - \eta(Y)g(Z + hZ, X) + g(\varphi P(Z,Y), X),
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
&R(X,Y)\varphi Z\\
&= \varphi R(X,Y)Z - g(Y + hY, Z)(\varphi X + \varphi hX)\\
&\quad + g(X + hX, Z)(\varphi Y + \varphi hY)\\
&\quad + g(\varphi X + \varphi hX, Z)(Y + hY) - g(\varphi Y + \varphi hY, Z)(X + hX)\\
&\quad + g(P(X,Y),Z)\xi - \eta(Z)P(X,Y),
\end{aligned} \tag{5.4}
$$

and

$$\begin{aligned}(5.5)\qquad & C(X,Y)\varphi Z - \varphi C(X,Y)Z \\ &= R(X,Y)\varphi Z - \varphi R(X,Y)Z \\ &\quad + g(hY,\varphi Z)hX - g(hX,\varphi Z)hY \\ &\quad - g(hY,Z)\varphi hX + g(hX,Z)\varphi hY.\end{aligned}$$

Proof. First, together with (2.3) we see that $C(X,Y)\xi = R(X,Y)\xi$ and $g(C(\xi,X)Y,Z) = g(R(\xi,X)Y,Z)$. But, from (2.6), (5.1) and the fundamental symmetries of the curvature tensor, we compute $R(X,Y)\xi$ and $g(R(\xi,X)Y,Z)$. So, we obtain (5.2) and (5.3). The Ricci identity for φ is given as

$$(5.6)\qquad R(X,Y)\varphi Z - \varphi R(X,Y)Z = (\nabla^2_{X,Y}\varphi)Z - (\nabla^2_{Y,X}\varphi)Z,$$

where $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla_X Y}$. From (5.1) we have

$$\begin{aligned}(\nabla^2_{X,Y}\varphi)Z =& - g(Y+hY,Z)(\varphi X + \varphi hX) + g(\varphi X + \varphi hX, Z)(Y+hY) \\ &+ g((\nabla_X h)Y,Z)\xi - \eta(Z)(\nabla_X h)Y,\end{aligned}$$

and thus (5.4) and (5.5) follow easily from this, (5.6) and the definition of the tensor C. □

Now, we prove

PROPOSITION 5.2. *Let M be a contact strongly pseudo-convex CR-manifold. Then the necessary and sufficient condition for M to have pointwise constant pseudo-holomorphic sectional curvature $H = H(p)$, $p \in M$, is*

$$\begin{aligned}(5.7)\qquad & g(R(X,Y)Z,W) \\ &= \frac{1}{4}\Big\{H\Big[\big(g(Y,Z)-\eta(Y)\eta(Z)\big)\big(g(X,W)-\eta(X)\eta(W)\big) \\ &\quad - \big(g(X,Z)-\eta(X)\eta(Z)\big)\big(g(Y,W)-\eta(Y)\eta(W)\big)\Big] \\ &\quad + (H-4)\Big[g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) \\ &\quad - 2g(\varphi X,Y)g(\varphi Z,W)\Big]\Big\} \\ &\quad + g(hY,Z)\big(g(X,W)-\eta(X)\eta(W)\big) \\ &\quad - g(hX,Z)\big(g(Y,W)-\eta(Y)\eta(W)\big)\end{aligned}$$

$$\begin{aligned}
&+ g(hX,W)\big(g(Y,Z)-\eta(Y)\eta(Z)\big)\\
&- g(hY,W)\big(g(X,Z)-\eta(X)\eta(Z)\big)\\
&- g(\varphi hY,Z)g(\varphi hX,W)+g(\varphi hX,Z)g(\varphi hY,W)\\
&+\eta(X)\Big[g\big(\varphi P(W,Z)-\eta(W)\varphi P(\xi,Z)-\eta(Z)\varphi P(W,\xi),Y\big)\Big]\\
&-\eta(Y)\Big[g\big(\varphi P(W,Z)-\eta(W)\varphi P(\xi,Z)-\eta(Z)\varphi P(W,\xi),X\big)\Big]\\
&+\eta(Z)\Big[g\big(\varphi P(Y,X)-\eta(Y)\varphi P(\xi,X)-\eta(X)\varphi P(Y,\xi),W\big)\Big]\\
&-\eta(W)\Big[g\big(\varphi P(Y,X)-\eta(Y)\varphi P(\xi,X)-\eta(X)\varphi P(Y,\xi),Z\big)\Big]\\
&-\eta(X)\eta(Z)\Big[g(Y+hY,W)-\eta(Y)\eta(W)+g(\varphi P(\xi,Y),W)\Big]\\
&+\eta(X)\eta(W)\Big[g(Y+hY,Z)-\eta(Y)\eta(Z)+g(\varphi P(\xi,Y),Z)\Big]\\
&+\eta(Y)\eta(Z)\Big[g(X+hX,W)-\eta(X)\eta(W)+g(\varphi P(\xi,X),W)\Big]\\
&-\eta(Y)\eta(W)\Big[g(X+hX,Z)-\eta(X)\eta(Z)+g(\varphi P(\xi,X),Z)\Big]
\end{aligned}$$

for all vector fields X, Y, Z, W in M.

Proof. For $X, Y \in D$, using the fundamental properties of the tensor C and the curvature tensor R, (2.1), (2.2) and (2.3), we obtain from (5.4) and (5.5)

$$g(C(X,\varphi X)Y,\varphi Y)=g(C(X,\varphi Y)Y,\varphi X)+g(C(X,Y)\varphi X,\varphi Y) \tag{5.8}$$

and

(5.9)

$$\begin{aligned}
&g(C(X,Y)\varphi X,\varphi Y)\\
&= g(C(X,Y)X,Y)\\
&\quad - g(X,Y)^2-2g(hX,Y)^2-2g(X,Y)g(hX,Y)+g(X,X)g(Y,Y)\\
&\quad + g(X,X)g(hY,Y)+g(Y,Y)g(hX,X)\\
&\quad + 2g(hX,X)g(hY,Y)-g(\varphi X,Y)^2\\
&\quad + 2g(\varphi hX,Y)^2-2g(\varphi hX,X)g(\varphi hY,Y).
\end{aligned}$$

Similarly, from (5.4) and (5.5) we get

$$\begin{aligned}
&g(C(X,\varphi Y)X,\varphi Y)\\
&= g(C(X,\varphi Y)Y,\varphi X)\\
&\quad + g(X,Y)^2-2g(hX,Y)^2-2g(\varphi hX,X)g(\varphi hY,Y)-g(X,X)g(Y,Y)
\end{aligned}$$

$$(5.10)\qquad \begin{aligned} &- g(Y,Y)g(hX,X) + g(X,X)g(hY,Y) \\ &+ 2g(hX,X)g(hY,Y) + g(\varphi X,Y)^2 \\ &+ 2g(\varphi hX,Y)^2 + 2g(\varphi X,Y)g(\varphi hX,Y) \end{aligned}$$

and

$$(5.11)\qquad \begin{aligned} &g(C(Y,\varphi X)Y,\varphi X) \\ &= g(C(X,\varphi Y)Y,\varphi X) \\ &\quad + g(X,Y)^2 - 2g(hX,Y)^2 - 2g(\varphi hX,X)g(\varphi hY,Y) - g(X,X)g(Y,Y) \\ &\quad + g(Y,Y)g(hX,X) - g(X,X)g(hY,Y) \\ &\quad + 2g(hX,X)g(hY,Y) + g(\varphi X,Y)^2 \\ &\quad + 2g(\varphi hX,Y)^2 - 2g(\varphi X,Y)g(\varphi hX,Y). \end{aligned}$$

We now suppose that M has a pointwise constant pseudo-holomorphic sectional curvature $H(p)$, i.e., for any $X \in D(p)$,

$$L(\hat{R}(X,\varphi X)\varphi X,X) = H(p)g(X,X)^2.$$

Then together with (4.5) we immediately get

$$(5.12)\qquad g(C(X,\varphi X)\varphi X,X) = H_1(p)g(X,X)^2,$$

where we have put $H_1(p) = H(p) - 3$. Substituting X by $X+Y$ and $X-Y$ for $X, Y \in D$ in (5.12) respectively, and summing them, we get

$$(5.13)\qquad \begin{aligned} &2g(C(X,\varphi X)\varphi Y,Y) + C(R(X,\varphi Y)\varphi Y,X) \\ &\quad + 2g(C(X,\varphi Y)\varphi X,Y) + g(C(Y,\varphi X)\varphi X,Y) \\ &= 2H_1\{2g(X,Y)^2 + g(X,X)g(Y,Y)\}. \end{aligned}$$

From (5.8), (5.9), (5.10), (5.11) and (5.13), we get

$$(5.14)\qquad \begin{aligned} &3g(C(X,\varphi Y)Y,\varphi X) + g(C(X,Y)X,Y) \\ &+ 2g(hX,Y)^2 + 2g(X,Y)g(hX,Y) \\ &- g(X,X)g(hY,Y) - g(Y,Y)g(hX,X) \\ &- 4g(hX,X)g(hY,Y) - 4g(\varphi hX,Y)^2 + 4g(\varphi hX,X)g(\varphi hY,Y) \\ &= H_1\{2g(X,Y)^2 + g(X,X)g(Y,Y)\}. \end{aligned}$$

Replacing Y by φY in (5.14) and using (2.1), (2.2) and (2.3), we have

(5.15)

$$\begin{aligned} &3g(C(X,Y)\varphi Y,\varphi X) - g(C(X,\varphi Y)X,\varphi Y) \\ &+ 4g(\varphi hX,Y)^2 - 2g(X,\varphi Y)g(hX,\varphi Y) \\ &+ g(X,X)g(hY,Y) - g(Y,Y)g(hX,X) \\ &+ 4g(hX,X)g(hY,Y) - 4g(hX,Y)^2 - 4g(\varphi hX,X)g(\varphi hY,Y) \\ &= H_1\{2g(X,\varphi Y)^2 + g(X,X)g(Y,Y)\}. \end{aligned}$$

From (5.15), together with (5.9) and (5.10), we get

(5.16)

$$\begin{aligned} &3g(C(X,Y)Y,X) + g(C(X,\varphi Y)\varphi X,Y) \\ &+ 2g(X,Y)^2 + 4g(hX,Y)^2 + 6g(X,Y)g(hX,Y) - 2g(X,X)g(Y,Y) \\ &- 3g(X,X)g(hY,Y) - 3g(Y,Y)g(hX,X) - 4g(hX,X)g(hY,Y) \\ &+ 2g(X,\varphi Y)^2 - 4g(\varphi hX,Y)^2 + 4g(\varphi hX,Y)g(\varphi hY,Y) \\ &= H_1\{2g(X,\varphi Y)^2 + g(X,X)g(Y,Y)\}. \end{aligned}$$

From (5.14) and (5.16), we have

(5.17)

$$\begin{aligned} &4g(C(X,Y)Y,X) \\ &= (H_1+3)\{g(X,X)g(Y,Y) - g(X,Y)^2\} + 3(H_1-1)g(X,\varphi Y)^2 \\ &\quad - 2\{2g(hX,Y)^2 + 4g(X,Y)g(hX,Y) - 2g(X,X)g(hY,Y) \\ &\quad - 2g(Y,Y)g(hX,X) \\ &\quad - 2g(hX,X)g(hY,Y) - 2g(\varphi hX,Y)^2 + 2g(\varphi hX,X)g(\varphi hY,Y)\} \end{aligned}$$

for all $X, Y \in D$. Substituting $X = X + Z$ in (5.17), we obtain

(5.18)

$$\begin{aligned} &4g(C(X,Y)Y,Z) \\ &= (H_1+3)\{g(X,Z)g(Y,Y) - g(X,Y)g(Y,Z)\} \\ &\quad + 3(H_1-1)g(X,\varphi Y)g(Z,\varphi Y) - 4\{g(hX,Y)g(hY,Z) \\ &\quad + g(X,Y)g(hY,Z) \\ &\quad + g(Y,Z)g(hX,Y) - g(X,Z)g(hY,Y) - g(Y,Y)g(hX,Z) \\ &\quad - g(hX,Z)g(hY,Y) \\ &\quad - g(\varphi hX,Y)g(\varphi hZ,Y) + g(\varphi hX,Z)g(\varphi hY,Y)\}. \end{aligned}$$

If we substitute $Y = Y + W$ in (5.18) again and use (2.3), then we obtain

(5.19)
$$\begin{aligned}
&4\{g(C(X,Y)W,Z) + g(C(X,W)Y,Z)\}\\
&= (H_1+3)\{2g(X,Z)g(Y,W) - g(X,Y)g(W,Z) - g(X,W)g(Y,Z)\}\\
&\quad + 3(H_1-1)\{g(X,\varphi Y)g(Z,\varphi W) + g(X,\varphi W)g(Z,\varphi Y)\}\\
&\quad - 4\{g(hX,Y)g(hZ,W) + g(hX,W)g(hZ,Y) + g(X,Y)g(hZ,W)\\
&\quad + g(X,W)g(hZ,Y) + g(Z,Y)g(hX,W) + g(Z,W)g(hX,Y)\\
&\quad - 2g(X,Z)g(hY,W) - 2g(Y,W)g(hX,Z) - 2g(hX,Z)g(hY,W)\\
&\quad - g(\varphi hX,Y)g(\varphi hZ,W) - g(\varphi hX,W)g(\varphi hZ,Y)\\
&\quad + 2g(\varphi hX,Z)g(\varphi hY,W)\}
\end{aligned}$$

and we have

(5.20)
$$\begin{aligned}
&4\{g(C(X,Z)W,Y) + g(C(X,W)Z,Y)\}\\
&= (H_1+3)\{2g(X,Y)g(Z,W)\\
&\quad - g(X,Z)g(W,Y) - g(X,W)g(Z,Y)\}\\
&\quad + 3(H_1-1)\{g(X,\varphi Z)g(Y,\varphi W)\\
&\quad + g(X,\varphi W)g(Y,\varphi Z)\} - 4\{g(hX,Z)g(hY,W) + g(hX,W)g(hY,Z)\\
&\quad + g(X,Z)g(hY,W) + g(X,W)g(hY,Z) + g(Y,Z)g(hX,W)\\
&\quad + g(Y,W)g(hX,Z) - 2g(X,Y)g(hZ,W) - 2g(Z,W)g(hX,Y)\\
&\quad - 2g(hX,Y)g(hZ,W) - g(\varphi hX,Z)g(\varphi hY,W)\\
&\quad - g(\varphi hX,W)g(\varphi hY,Z) + 2g(\varphi hX,Y)g(\varphi hZ,W)\}.
\end{aligned}$$

We subtract (5.20) from (5.19). Then by using the Bianchi-type identity for the curvature-like tensor field C and (2.3), we get

(5.21)
$$\begin{aligned}
&4g(C(X,Y)Z,W)\\
&= (H_1+3)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}\\
&\quad + (H_1-1)\{g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W)\\
&\quad - 2g(\varphi X,Y)g(\varphi Z,W)\}\\
&\quad + 4\{g(hY,Z)g(X,W) - g(hX,Z)g(Y,W) + g(Y,Z)g(hX,W)\\
&\quad - g(X,Z)g(hY,W) + g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W)\\
&\quad - g(\varphi hY,Z)g(\varphi hX,W) + g(\varphi hX,Z)g(\varphi hY,W)\},
\end{aligned}$$

where $X, Y, Z, W \in D(p)$. We now let X be an arbitrary vector field on M. Then we may write

$$X = X^T + \eta(X)\xi,$$

where X^T denotes *the horizontal part* of X. Then we have for all vector fields X, Y, Z, W in M:

(5.22)

$$\begin{aligned}
&g(C(X,Y)Z,W) \\
&= g(C(X^T,Y^T)Z^T,W^T) + \eta(X)g(C(\xi,Y^T)Z^T,W^T) \\
&\quad + \eta(Y)g(C(X^T,\xi)Z^T,W^T) + \eta(Z)g(C(X^T,Y^T)\xi,W^T) \\
&\quad + \eta(W)g(C(X^T,Y^T)Z^T,\xi) + \eta(X)\eta(Z)g(C(\xi,Y^T)\xi,W^T) \\
&\quad + \eta(X)\eta(W)g(C(\xi,Y^T)Z^T,\xi) + \eta(Y)\eta(Z)g(C(X^T,\xi)\xi,W^T) \\
&\quad + \eta(Y)\eta(W)g(C(X^T,\xi)Z^T,\xi).
\end{aligned}$$

Furthermore, from (5.22), by using (5.2), (5.3), (5.21) and straightforward calculations, we obtain (5.7). □

From (5.7), by using (2.4) and (2.5), we find for the Ricci tensors:

(5.23)

$$\begin{aligned}
&\rho(X,Y)(= \sum_i g(R(e_i,X)Y,e_i)) \\
&= \frac{1}{2}\Big((n+1)H(p) - 4\Big)\Big(g(X,Y) - \eta(X)\eta(Y)\Big) \\
&\quad + (2n-1)g(hX,Y) + g(hX,hY) - \eta(X)\sum_i g(\varphi P(e_i,Y),e_i) \\
&\quad + \eta(Y)\sum_i g(\varphi P(X,e_i),e_i) + g(\varphi P(\xi,X),Y) \\
&\quad + \eta(X)\eta(Y)(2n + \text{tr } h^2)
\end{aligned}$$

for all vectors X and Y in T_pM, where $\{e_i\}$ $(i = 1, 2, \ldots, 2n+1)$ is an arbitrary local orthonormal basis for T_pM. Since the trace of h vanishes, from (5.23), we have for the scalar curvature:

$$\tau(= \sum_i \rho(e_i,e_i)) = n\Big((n+1)H - 4\Big) + 2n - \text{tr } h^2,$$

where we have used $\sum_i g(\varphi P(e_i,\xi),e_i) = \text{tr } h^2$.

6. Schur-type theorem for a contact strongly pseudo-convex CR-space form

Let M be a pseudo-parallel contact strongly pseudo-convex CR-manifold. Then, since we already know that the pseudo-parallel h is equivalent to the η-parallel h, it follows that

$$\begin{aligned}0 &= g((\nabla_{X^T}h)Y^T, Z^T)\\ &= g((\nabla_{X-\eta(X)\xi}h)(Y-\eta(Y)\xi, Z-\eta(Z)\xi)\\ &= g((\nabla_X h)Y, Z) - \eta(X)g((\nabla_\xi h)Y, Z) - \eta(Y)g((\nabla_X h)\xi, Z)\\ &\quad - \eta(Z)g((\nabla_X h)Y,\xi) + \eta(X)\eta(Y)g((\nabla_\xi h)\xi, Z)\\ &\quad + \eta(Y)\eta(Z)g((\nabla_X h)\xi,\xi) + \eta(Z)\eta(X)g((\nabla_\xi h)Y,\xi)\\ &\quad - \eta(X)\eta(Y)\eta(Z)g((\nabla_\xi h)\xi,\xi).\end{aligned}$$

From the above equation, by using (2.3), (2.4) and (2.5), we have

$$\begin{aligned}(\nabla_X h)Y = g((h-h^2)\varphi X, Y)\xi + \eta(Y)(h-h^2)\varphi X\\ + \eta(X)(\varphi Y - \varphi l Y - \varphi h^2 Y)\end{aligned} \tag{6.1}$$

for all vector fields X and Y. Before we prove the Schur-type theorem we prepare [6].

LEMMA 6.2. *Let M be a pseudo-parallel contact strongly pseudo-convex CR-manifold. Then the eigenvalues of h are constant.*

Moreover, from (6.1), we have

$$\begin{aligned}P(X,Y) = &-g((\varphi h^2 + h^2\varphi)X, Y)\xi\\ &+ \eta(Y)(h\varphi - \varphi + \varphi l)X - \eta(X)(h\varphi - \varphi + \varphi l)Y,\end{aligned} \tag{6.2}$$

$$\varphi P(X,Y) = \eta(Y)(h - \varphi^2 - l)X - \eta(X)(h - \varphi^2 - l)Y. \tag{6.3}$$

We prove a Schur-type theorem for this class. Namely,

THEOREM 6.3. *Let $(M^{2n+1};\eta, L)$ $(n>1)$ be a pseudo-parallel contact strongly pseudo-convex CR-manifold. If the pseudo-holomorphic sectional curvature (with respect to the Tanaka-Webster connection) at* *any point of M is independent of the choice of pseudo-holomorphic section, then it is constant c on M and the curvature tensor is given by*

(6.4)

$$\begin{aligned}
&g(R(X,Y)Z,W)\\
&=\frac{1}{4}\Big\{c\Big[\big(g(Y,Z)-\eta(Y)\eta(Z)\big)\big(g(X,W)-\eta(X)\eta(W)\big)\\
&-\big(g(X,Z)-\eta(X)\eta(Z)\big)\big(g(Y,W)-\eta(Y)\eta(W)\big)\Big]\\
&+(c-4)\Big[g(\varphi Y,Z)g(\varphi X,W)-g(\varphi X,Z)g(\varphi Y,W)\\
&-2g(\varphi X,Y)g(\varphi Z,W)\Big]\Big\}\\
&+g(hY,Z)\big(g(X,W)-\eta(X)\eta(W)\big)\\
&-g(hX,Z)\big(g(Y,W)-\eta(Y)\eta(W)\big)\\
&+g(hX,W)\big(g(Y,Z)-\eta(Y)\eta(Z)\big)-g(hY,W)\big(g(X,Z)-\eta(X)\eta(Z)\big)\\
&-g(\varphi hY,Z)g(\varphi hX,W)+g(\varphi hX,Z)g(\varphi hY,W)\\
&-\eta(X)\eta(Z)g(lY,W)+\eta(X)\eta(W)g(lY,Z)\\
&+\eta(Y)\eta(Z)g(lX,W)-\eta(Y)\eta(W)g(lX,Z)
\end{aligned}$$

for all vector fields X, Y, Z, W in M.

Proof. Suppose that M has pointwise constant pseudo-holomorphic sectional curvature H. Then, taking account of (6.1), (6.2) and (6.3), from (5.23) we obtain

(6.5)

$$\begin{aligned}
\rho(X,Y)=&\frac{1}{2}\Big((n+1)H-4\Big)\Big(g(X,Y)-\eta(X)\eta(Y)\Big)\\
&+2(n-1)g(hX,Y)+g(h^2X,Y)+g((\varphi^2+l)X,Y)\\
&+\eta(X)\eta(Y)(2n-\text{tr } h^2),
\end{aligned}$$

$$\tau=n\Big((n+1)H-4\Big)+2n-\text{tr } h^2. \tag{6.6}$$

From (6.1) and by using (2.4) and Lemma 6.2, we have

$$\begin{aligned}
&(\nabla_X\rho)(Y,Z)\\
&=\frac{1}{2}\Big((n+1)(XH)\Big)\Big(g(Y,Z)-\eta(Y)\eta(Z)\Big)\\
&-\frac{1}{2}\Big((n+1)H-4\Big)\Big((\nabla_X\eta)(Y)\eta(Z)+\eta(Y)(\nabla_X\eta)(Z)\Big)\\
&+2(n-1)(g(\nabla_Xh)Y,Z)+g((\nabla_Xh^2)Y,Z)
\end{aligned}$$

$$+ (\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z) + g((\nabla_X l)Y, Z)$$
$$+ (2n - \text{tr } h^2)\Big((\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z)\Big),$$

which yields

$$\begin{aligned}
&\sum_i (\nabla_{e_i}\rho)(X, e_i) \\
&= \frac{1}{2}(n+1)\{(XH) - (\xi H)\eta(X)\} + \sum_i g((\nabla_{e_i} l)X, e_i) \\
&= \frac{1}{2}(n+1)\{(XH) - (\xi H)\eta(X)\} - \sum_i g((\nabla_X R)(\xi, e_i)\xi, e_i) \\
&\quad - \sum_i g((\nabla_\xi R)(e_i, X)\xi, e_i) \\
&= \frac{1}{2}(n+1)\{(XH) - (\xi H)\eta(X)\} + (\nabla_X \rho)(\xi, \xi) - (\nabla_\xi \rho)(X, \xi) \\
&= \frac{1}{2}(n+1)\{(XH) - (\xi H)\eta(X)\},
\end{aligned} \tag{6.7}$$

where we have used the 2nd Bianchi identity. By the well-known formula

$$\nabla_X \tau = 2\sum_i (\nabla_{e_i}\rho)(X, e_i)$$

for any local orthonormal frame field $\{e_i\}$ $(i = 1, 2, \dots, 2n+1)$ and by using (6.6), (6.7) and Lemma 6.2, we have

$$(n+1)\{XH - (\xi H)\eta(X)\} = n(n+1)XH.$$

This says that $\xi H = 0$ and $(n-1)XH = 0$. Since $n > 1$, we see that H is constant, say c. By applying (6.1), (6.2) and (6.3) in Proposition 5.2, we obtain (6.4). □

So, from the proofs of Proposition 5.2 and Theorem 6.3, we have

THEOREM 6.4. *Let M be a complete and simply connected pseudo-parallel contact CR-space. Then M is a contact strongly pseudo-convex CR-space form if and only if the curvature tensor R is given by* (6.4).

We note that a contact strongly pseudo-convex CR-space form is a proper extension of a Sasakian space form ($h = 0$). Since we already know that a pseudo-parallel contact CR-space is a (k,μ)-space, from the results in [4], we see that a pseudo-parallel contact pseudo-convex CR-space form has a locally homogeneous contact Riemannian structure and is a locally φ-symmetric space in the strong sense. (We refer to [4] or [7] for the definition of a locally φ-symmetric space in the strong sense.)

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