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ANALYSIS OF A MESHFREE METHOD FOR THE COMPRESSIBLE EULER EQUATIONS

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ABSTRACT. Mathematical analysis is made on a meshfree method for the compressible Euler equations. In particular, the Moving Least Square Reproducing Kernel (MLSRK) method is employed for space approximation. With the backward-Euler method used for time discretization, existence of discrete solution and it's L^2-error estimate are obtained under a regularity assumption of the continuous solution. The result of numerical experiment made on the biconvex airfoil is presented.

1. Introduction

The objective of this paper is to develop the mathematical background for the SUPG (stream-line upwind Petrov Galerkin) formulation of the MLSRK (moving least square reproducing kernel) method for the compressible Euler equations. The discretized MLSRK equations in the conservative variable form are solved using the backward-Euler time discretization. The resultant approximate discrete solution is shown to converge to the continuous solution under a smoothness assumption in this paper. Such smoothness is guaranteed for the local time interval by the works of Kato [7] and Lax [9]. The strong solvability shown for the compressible Euler equations can be generalized to the general hyperbolic equations. However, the global solvability of the compressible Euler equations in two or three dimension is not known until now.

Several meshfree approximations were proposed with a variety of applications. Examples are the Smoothed Particle Hydrodynamics (SPH) by Gingold and Monaghan [4], the Reproducing Kernel Particle Method (RKPM) by Liu et al. [13], [14], the Diffuse Element Method (DEM)

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by Nayrole et al. [20], the Element Free Galerkin Method (EFG) by Belytschko et al. [17], the Partition of Unity Finite Element Method (PUFEM) by Babuška and Melenk [19], the Moving Least Square Reproducing Kernel Galerkin Method (MLSRK) proposed by Liu et al. [15], and the Fast Moving Least Square Reproducing Kernel Method (FMLSRK) by Kim and Kim [8]. In this paper, the Moving Least Square Reproducing Kernel Galerkin Method (MLSRK) proposed by Liu et al. [15] is considered to solve the non-stationary compressible Euler equations. One distinct advantage of this method over the standard finite element method is that it requires simple distribution of nodes, not the complex mesh generation dependent on the geometry of the flow domain. Another advantage is that the desired regularity of the approximate solutions can be readily achieved by introducing suitable basis functions with sufficient regularity.

Though there have been much interest in developing meshfree methods for the Garlerkin formulation of the partial differential equations in engineering, mathematical analysis on the existence and convergence criterion of discrete solutions has been hardly made yet. In papers [1] and [2], the solvability and convergence of discrete solutions have been shown for the incompressible Stokes and Navier-Stokes equations. In this paper, the solvability and convergence property of the approximate solution is derived for the non-stationary compressible Euler equations in the short time period. It is noted that such a convergence analysis seems not possible in the framework of the continuous piecewise linear FEM.

As a benchmark test, a numerical example is made in two dimensions using the non-stationary compressible Euler equations. The model problem is the symmetric flow over a biconvex airfoil. To show the convergence, relative errors are depicted from the node distribution.

For the analysis to be followed, we define $B_r(x_0) = \{x : |x - x_0| < r\}$. We let k be an integer and $p \geq 1$ be a real number. We define the Sobolev norm $||u||_{W^{k,p}(\Omega)} = \left[\sum_{|\alpha|\leq k} \int_\Omega |\nabla^\alpha u|^p dx\right]^{1/p}$ and introduce the standard Sobolev space $W^{k,p}(\Omega)$ by the set of all L^p functions whose weak derivatives up to order k exist and $W^{k,p}$ norms are finite. Also $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ for the $W^{k,p}(\Omega)$ norm. When $p = 2$, we denote $H^k(\Omega) = W^{k,2}(\Omega)$ and $H_0^k(\Omega) = W_0^{k,2}(\Omega)$. Throughout this paper c stands for a generic constant which varies in each occurrence.

2. Compressible Euler equations

For a domain Ω in $\mathbb{R}^N$, $N = 2$ or 3, we consider the Euler equations

$$\mathbf{u}_t + \sum_{i=1}^{N} \left(\mathbf{F}^i(\mathbf{u})\right)_{x_i} = \mathbf{R}, \quad x \in \Omega,\ t > 0, \tag{2.1}$$

where $\mathbf{u}(x,0) = \mathbf{u}_0(x)$, $\mathbf{u}(x,t) = \rho(1, \mathbf{v}, e)^T$. Here ρ is the density, $\mathbf{v} = (v_1, \ldots, v_N)^T$ is the velocity, e is the total energy density, and $\mathbf{F}^i$ is the Euler flux in the form

$$\mathbf{F}^i = \rho v_i \begin{pmatrix} 1 \\ v_1 \\ \cdot \\ v_N \\ e \end{pmatrix} + p \begin{pmatrix} 0 \\ \delta_{1i} \\ \cdot \\ \delta_{Ni} \\ v_i \end{pmatrix},$$

where p is the pressure. The source vector $\mathbf{R}$ is defined by

$$\mathbf{R} = \rho \begin{pmatrix} 1 \\ b_1 \\ \cdot \\ b_N \\ b_i v_i + \gamma \end{pmatrix},$$

where $\mathbf{b} = (b_1, \ldots, b_N)^T$ is the body force vector and γ is the heat supply. The necessary constitutive relations are

$$e = \frac{1}{2}\left(\sum_{i=1}^{N} v_i^2\right) + c_\nu \theta,$$
$$p = (\gamma - 1)\rho c_\nu \theta,$$

for the ideal gas, where c_ν is the specific heat at constant volume, θ is the absolute temperature, and γ is the ratio of specific heat. The relation between the primitive variables and the state vector is, for $N = 3$,

$$\rho = \mathbf{u}_1, \quad v_1 = \frac{\mathbf{u}_2}{\mathbf{u}_1}, \quad v_2 = \frac{\mathbf{u}_3}{\mathbf{u}_1}, \quad v_3 = \frac{\mathbf{u}_4}{\mathbf{u}_1},$$
$$\theta = \frac{2\mathbf{u}_1\mathbf{u}_5 - (\mathbf{u}_2^2 + \mathbf{u}_3^2 + \mathbf{u}_4^2)}{2c_\nu \mathbf{u}_1^2},$$
$$p = (\gamma - 1)\left(\mathbf{u}_5 - \frac{\mathbf{u}_2^2 + \mathbf{u}_3^2 + \mathbf{u}_4^2}{2\mathbf{u}_1}\right).$$

We now state the local existence theorem based on the work of Kato [7] and Lax [9]. For this purpose, we introduce the integer Sobolev space $H^s(\mathbb{R}^N)$ with norm $\|g\|_s^2 = \sum_{|\alpha|\le s} \int_{\mathbb{R}^N} |D^\alpha g|^2 dx$. Also we introduce $L^\infty([0,T] : H^s)$ with norm $\|u\|_{s,T} = \max_{0\le t\le T} \|u(t)\|_s$, and $Lip(\mathbb{R}^N)$ with norm $\|u\|_{Lip} = \|u\|_{L^\infty} + \sup_{x,y} \frac{|u(x)-u(y)|}{|x-y|}$. Now we suppose that the flux vector $\mathbf{F}$ satisfies the assumption of symmetric hyperbolicity for all $\mathbf{u}$ such that there is a positive symmetric matrix $\mathbf{A}_0(\mathbf{u})$ smoothly varying with $\mathbf{u}$ so that

a) $c_0\mathbf{I} \le \mathbf{A}_0(\mathbf{u}) \le c_0^{-1}\mathbf{I}$, $\mathbf{A}_0 = \mathbf{A}_0^T$ with a constant c_0 uniformly for $\mathbf{u}$.

b) $\mathbf{A}_0(\mathbf{u})\mathbf{A}_j(\mathbf{u}) = \tilde{\mathbf{A}}_j(\mathbf{u})$, with $\tilde{\mathbf{A}}_j(\mathbf{u}) = \tilde{\mathbf{A}}_j(\mathbf{u})^T$ for $j = 1,\dots,N$, where $\mathbf{A}_j(\mathbf{u}) = \frac{D(\mathbf{F}_j)}{D\mathbf{u}}$.

THEOREM 1. (Kato [7] and Lax [9]) *Assume $\mathbf{u}_0 \in H^s$, $s > \frac{N}{2} + 1$. Then there is a time interval $[0,T]$ with $T > 0$, so that the Euler equations (2.1) has a unique classical solution $\mathbf{u}(x,t) \in C^1([0,T] : H^{s-1})$. Furthermore, $\mathbf{u} \in C([0,T] : H^s) \cap C^1([0,T] : H^{s-1})$ and T depends on $\|\mathbf{u}\|_s$.*

The compressible Euler equations satisfy the symmetric hyperbolicity conditions *a*) and *b*). Since we develop the convergence theory for the general hyperbolic equations, we do not specify the symmetrizer $\mathbf{A}_0$ for the compressible Euler equations. Since the general global existence is not known, some *a priori* regularity assumptions are imposed for our convergence theory.

3. Numerical scheme

We briefly introduce MLSRK approximation scheme. We refer [15] and [1] for the detail. Let Ω be a bounded domain with smooth boundary and $u(x)$ be a smooth function defined in Ω. For convenience, we let $u(x) = 0$ if $x \notin \Omega$. To achieve $m-$th order consistency, we introduce the set of all basis polynomials of order less than or equal to m

$$\mathcal{P}_m(x) = \{P_\alpha(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid |\alpha| = \alpha_1 + \cdots + \alpha_n \le m\}.$$

The number of all entries in $\mathcal{P}_m(x)$ is $\frac{(n+m)!}{n!m!}$. To obtain localized approximations of a function $u(x)$, we need a nonnegative window function Φ which has a compact support, say, $\operatorname{supp}\Phi \subset B_1(0)$. We set $\Omega_\varrho = \{x : \operatorname{dist}(x,\Omega) < \varrho\}$. We introduce a localized error residual functional

$$J(\mathbf{a}(\bar{x})) \equiv \int_\Omega \left| u(x) - \mathcal{P}_m\left(\frac{x-\bar{x}}{\varrho}\right) \cdot \mathbf{a}(\bar{x}) \right|^2 \Phi_\varrho(x-\bar{x})\, dx,$$

where $\Phi_\varrho(x-\bar{x}) = \dfrac{1}{\varrho^n}\Phi\left(\dfrac{x-\bar{x}}{\varrho}\right)$. Here $\bar{x}$ is a fixed point in Ω.

The coefficient vector $\mathbf{a}(\bar{x})$ at which the quadratic functional $J(\mathbf{a}(\bar{x}))$ is minimized, satisfies

$$\int_\Omega \mathcal{P}_m^T\left(\frac{x-\bar{x}}{\varrho}\right)\left(u(x) - \mathcal{P}_m\left(\frac{x-\bar{x}}{\varrho}\right)\cdot \mathbf{a}(\bar{x})\right)\Phi_\varrho(x-\bar{x})\, dx = \mathbf{0}.$$

For further development we define the moment matrix $M(\bar{x})$ such that

$$M(\bar{x}) \equiv \int_\Omega \mathcal{P}_m^T\left(\frac{x-\bar{x}}{\varrho}\right)\mathcal{P}_m\left(\frac{x-\bar{x}}{\varrho}\right)\Phi_\varrho(x-\bar{x})\, dx. \tag{3.2}$$

Since the polynomial basis $\mathcal{P}_\alpha(x)$'s are linearly independent, $M(\bar{x})$ is always invertible and $\det M(\bar{x}) > 0$. Consequently we find the minimizer of the moving least functional $J(\mathbf{a}(\bar{x}))$ such that

$$\mathbf{a}(\bar{x}) = M^{-1}(\bar{x}) \int_\Omega \mathcal{P}_m^T\left(\frac{x-\bar{x}}{\varrho}\right) u(x)\Phi_\varrho(x-\bar{x})\, dx,$$

and obtain the following approximation of $u(x)$,

$$\begin{aligned} U(x,\bar{x}) &\equiv \mathcal{P}_m\left(\frac{x-\bar{x}}{\varrho}\right)\cdot \mathbf{a}(\bar{x}) \\ &= \mathcal{P}_m\left(\frac{x-\bar{x}}{\varrho}\right) M^{-1}(\bar{x}) \int_\Omega \mathcal{P}_m^T\left(\frac{y-\bar{x}}{\varrho}\right) u(y)\Phi_\varrho(y-\bar{x})\, dy. \end{aligned}$$

For $\bar{x} \in \Omega$, an arbitrary point in Ω for the weighted least square procedure, we let $\bar{x}$ be equal to x and obtain a global approximation of $u(x)$. The novel point in MLSRK is the choice $\bar{x} = x$. More precisely, the global approximation operator is defined by

$$\begin{aligned} G\,u(x) &\equiv U(x,x) \\ &= \mathcal{P}_m(0) M^{-1}(x) \int_\Omega \mathcal{P}_m^T\left(\frac{y-x}{\varrho}\right) u(y)\Phi_\varrho(y-x)\, dy. \end{aligned} \tag{3.3}$$

Introducing the correction function

$$C(\varrho, y-x, x) = \mathcal{P}_m(0) M^{-1}(x) \mathcal{P}_m^T\left(\frac{y-x}{\varrho}\right),$$

we can define the shape function

$$K_\varrho(y-x,x) \equiv C(\varrho,y-x,x)\Phi_\varrho(y-x).$$

The global approximation (3.3) can be written in a convolution form

$$G\,u(x) = \int_\Omega K_\varrho(y-x,x)u(y)\,dy. \tag{3.4}$$

This formulation is the so-called reproducing kernel formulation by Liu et al. [13]. It satisfies $m-$th order consistency, namely,

$$G\,u(x) = u(x) \tag{3.5}$$

if $u(x)$ is a polynomial of order less than or equal to m.

REMARK. The $m-$th order consistency can be generalized by inserting any function ϕ into the set of basis polynomials $\mathcal{P}_m$. Then the reproducing kernel approximation (3.4) reproduces linear combination of $m-$th order polynomials and ϕ.

We define the discretized shape function to analyze the MLSRK method for the Euler equations. For a given set of nodes $\Lambda = \{x_i | i = 1,\dots,NP\}$, employing discretized moment matrix

$$M^h(x) = \sum_{i=1}^{NP} \mathcal{P}_m^T\left(\frac{x-x_i}{\varrho}\right)\mathcal{P}_m\left(\frac{x-x_i}{\varrho}\right)\Phi_\varrho(x-x_i),$$

we define the discretized shape function

$$\begin{aligned} K_\varrho^h(x-x_i,x) &= C^h(\varrho,x-x_i,x)\Phi_\varrho(x-x_i) \\ &= \mathcal{P}(0)\left(M^h\right)^{-1}(x)\mathcal{P}^T\left(\frac{x-x_i}{\varrho}\right)\Phi_\varrho(x-x_i). \end{aligned} \tag{3.6}$$

Now this set of discrete shape functions are used as MLSRK shape functions, which is called simply shape functions if there is no confusion. Also we will denote briefly $K_\varrho^h(x-x_i,x)$ as $\phi_i(x)$ for the window function Φ.

In the finite element method, the mesh generation follows *quasi-uniform* or *regular* condition. Similarly, we have the following condition. Since there is a close relation between node distribution and shape functions, we consider regular node set and admissible set of shape functions.

DEFINITION 1. Let $\Lambda = \{x_i | i = 1,\dots,NP\}$ be the set of nodes. We define Λ be a regular node set if the followings hold.

i) There exists $C_1 > 0$ independent of NP such that

$$\min_i h_{x_i} \geq C_1 \max_i h_{x_i},$$

where $h_{x_i} = \min_{j \neq i} |x_i - x_j|$.

ii) There exists $C_\varrho > 0$ depend only on $\varrho > 1$ such that

$$\min_i N(i, \varrho) \geq C_\varrho \max_i N(i, \varrho),$$

where $N(i, \varrho)$ be the number of nodes contained in $B_{\varrho h_{x_i}}(x_i)$.

Definition 2. Let $A = \{\phi_i | i = 1, \ldots, NP\}$ be the set of MLSRK shape functions generated by the window function Φ for the regular node set $\Lambda = \{x_i | i = 1, \ldots, NP\}$. Then A and Λ are admissible if there is a positive constant β_0 such that

$$\sum_{\alpha=1}^{n} \sum_{i,j=1}^{NP} \int_\Omega \phi_i \phi_j \, dx \, a_i^\alpha \, a_j^\alpha \geq \beta_0 \|a\|^2 \tag{3.7}$$

for all $a^\alpha \in \mathbb{R}^{NP}$, $\alpha = 1, \ldots, n$.

Note that the above *regular* condition for node set implies an overlapping condition of shape functions, that is, sufficient number of node points belong to the support of each shape function ϕ_i to ensure the stability of the moment matrix. In short, the *regular* condition implies certain uniform condition for the node distance and support radius of shape functions. In making shape function, dilation parameter ϱ and node distance h are depend on each x_i, but with assuming *regular* node distribution, we may consider ϱ and h as independent parameter with respect to each x_i.

For the convergence analysis, we need interpolation error estimate between the solution space and the projection generated by the set of shape functions. We introduce the discrete projection and projection error estimate, which are studied in [8, 15].

Definition 3. Let $A = \{\phi_i | i = 1, \ldots, NP\}$ be the set of MLSRK shape functions generated by the window function Φ for the regular node set $\Lambda = \{x_i | i = 1, \ldots, NP\}$. Let $u(x) \in C^0(\Omega)$ be a function and $\varrho > 0$ a real number. We define the discrete projection as

$$\mathcal{R}_{\varrho,h}^m u(x) \equiv \sum_{i=1}^{NP} u(x_i) \, \phi_i(x) = \sum_{x_i \in \Lambda(x)} u(x_i) \, \phi_i(x),$$

where $\phi_i(x) = K_\varrho^h(x - x_i, x)$ as in (3.6) and $\Lambda(x) = \{x_i \in \Lambda | x \in \operatorname{supp} \phi_i \cap \bar{\Omega}\}$. Here, m denotes the order of generating polynomial basis $\mathcal{P}_m$, ϱ is the dilation parameter which is the characteristic radius of the support of window function Φ_ϱ, and h stands for the distance between the nodes.

THEOREM 2. *Assume the window function $\Phi(x) \in C_0^m(\mathbb{R}^n)$ and $v(x) \in C^{m+1}(\overline{\Omega})$, where Ω is a bounded open set in $\mathbb{R}^n$. Let $\Lambda = \{x_i | i = 1, \dots, NP\}$ be a regular node set and $A = \{\phi_i | i = 1, \dots, NP\}$ be the set of admissible shape functions. Suppose the boundary of Ω is smooth and supp$\phi_i \cap \bar{\Omega}$ is convex for each i. If m and p satisfy*

$$m > \frac{n}{p} - 1,$$

then the following interpolation estimate holds

(3.8)

$$\|v - \mathcal{R}_{\varrho,h}^m v\|_{W^{k,p}(\Omega)} \le C_k \, \varrho^{m+1-k} \, \|v\|_{W^{m+1,p}(\Omega)}, \text{ for all } 0 \le k \le m.$$

4. Application of the MLSRK method to the Euler equations

We consider the non-stationary compressible Euler equations with zero body force. We will show the existence of MLSRK discrete solution for all time. Also we show the L^2-convergence theorem under the regularity assumption of the true solution. From Theorem 1, we can say that the true solution has sufficient regularity for a short time period. Hence our convergence theorem is valid for that time period, too. Throughout this section, we assume that Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N = 2, 3$) and T_0 represents the time for existence of regular solution from Theorem 1. Let $A = \{\phi_i \,|\, i = 1, \dots, NP\}$ be the set of MLSRK shape functions generated from the window function Φ with m-th order consistency for the *regular* node set $\Lambda = \{x_i \,|\, i = 1, \dots, NP\}$. Using MLSRK method in space and backward Euler method in time, we want to find a discrete solution $\mathbf{U}^n = \sum_{i=1}^{NP} \mathbf{C}_i^n \phi_i$ satisfying

$$\int_\Omega \mathbf{U}^{n+1} \phi_k dx - \Delta t \int_\Omega \mathbf{F}^i(\mathbf{U}^{n+1})(\phi_k)_{x_i} dx = \int_\Omega \mathbf{U}^n \phi_k dx, \tag{4.9}$$

$$\mathbf{U}^0(x) = \sum_{i=1}^{NP} \mathbf{u}_0(x_i)\phi_i(x).$$

Here we do not specify the boundary condition. Since a solution to hyperbolic system has a finite propagation speeds, we assume our solution has compact support, namely, supp$(\mathbf{U}(t)) \subset\subset \Omega$ and supp$(\mathbf{u}(t)) \subset\subset \Omega$.

For computational simplicity, we assume $\mathbf{F}^i(\mathbf{0}) = \mathbf{0}$, which can be achieved by adding a constant vector.

Now we prove the existence and the convergence theorem under the smoothness assumption of $\mathbf{u}$. By Theorem 1 again, such a smoothness assumption is guaranteed for a short time as long as the initial data is smooth. Global convergence theorem is not known for two or three dimensional Euler equations. We introduce the CFL (Courant-Friedrichs-Lewy) condition

$$\lambda = \frac{\Delta t}{\varrho} \ll 1, \tag{4.10}$$

where ϱ is the dilation parameter of the shape function. Firstly, we show the existence of discrete solution to equation (4.9), which follows from simple application of the implicit function theorem.

THEOREM 3. *Suppose $\mathbf{F}^i$'s, $(i = 1, \ldots, N)$ are Lipschitz functions with $\sum_{i=1}^{N} \|\mathbf{F}^i\|_{Lip} \leq M$ for some $M > 0$. Then if $\lambda < C$, where C is independent of ϱ and Δt, then there is a unique solution to* (4.9).

Proof. If $\mathbf{U}^n = \sum_{i=1}^{NP} \mathbf{C}_i^n \phi_i$ is a solution to (4.9), then we have

$$\begin{aligned} &\sum_{i=1}^{NP} \left(\int_\Omega \phi_k \phi_j \, dx \right) \mathbf{C}_j^{n+1} - \Delta t \int_\Omega \mathbf{F}^i \left(\sum_{j=1}^{NP} \mathbf{C}_j^{n+1} \phi_j \right) (\phi_k)_{x_i} \, dx \\ &= \sum_{i=1}^{NP} \left(\int_\Omega \phi_k \phi_j \, dx \right) \mathbf{C}_j^n, \end{aligned} \tag{4.11}$$

for $i = 1, \ldots, N$ and $k = 1, \ldots, NP$. From the *admissibility* assumption on $A = \{\phi_j \mid j = 1, \ldots, NP\}$, we get $\mathbf{M}_{kj} = \left(\int_\Omega \phi_k \phi_j dx \right)$ is symmetric positive definite so that

$$\mathbf{C}^{n+1^T} \mathbf{M}_{kj} \, \mathbf{C}^{n+1} \geq \alpha_0 |\mathbf{C}^{n+1}|^2, \tag{4.12}$$

for some $\alpha_0 > 0$. Moreover, since $|\nabla \phi_k| \leq \frac{c}{\varrho}$, we get

$$\begin{aligned} \left| \Delta t \int_\Omega \mathbf{F}^i \left(\sum_{j=1}^{NP} \mathbf{C}_j^{n+1} \phi_j \right) (\phi_k)_{x_i} \, dx \right| &\leq c \frac{\Delta t}{\varrho} \| \sum_{i=1}^{N} \mathbf{F}^i \|_{Lip} \int_\Omega \left| \sum_{j=1}^{NP} \mathbf{C}_j^{n+1} \phi_j \right| dx \\ &\leq \frac{\Delta t}{\varrho} M |\Omega|^{\frac{1}{2}} |\mathbf{C}^{n+1}|. \end{aligned}$$

Now, using this implicit function theorem, we prove that there is a solution to (4.9), if $c\frac{\Delta t}{\varrho}M|\Omega|^{\frac{1}{2}} \leq \frac{\alpha_0}{4}$ or equivalently $\lambda \leq \frac{C\alpha_0}{M|\Omega|^{\frac{1}{2}}}$. The uniqueness follows from the implicit function theorem. □

To show convergence of the discrete solution to the true solution, we assume that

$$(4.13) \qquad \sum_{i=1}^{N}\left(\|\mathbf{F}^i_{\mathbf{u}}\|_{L^\infty} + \|\mathbf{F}^i_{\mathbf{uu}}\|_{L^\infty}\right) \leq M,$$

where $\mathbf{F}_{\mathbf{u}} = \frac{\partial \mathbf{F}}{\partial \mathbf{u}}$. Physically, we may assume that density ρ of the fluid has lower bound in the Euler equations (2.1). Hence the assumption (4.13) is reasonable at least for the Euler equations (2.1). Since the existence of the true solution is not known in higher dimensional case, we consider only the case of short time interval from Theorem 1. Then we have,

$$(4.14) \qquad \sup_{0\leq t\leq T_0}\left(\|\mathbf{u}(\cdot,t)\|_{L^\infty}, \|\mathbf{u}_t(\cdot,t)\|_{L^\infty}, \|\nabla\mathbf{u}(\cdot,t)\|_{L^\infty}, \|\nabla\mathbf{u}_t(\cdot,t)\|_{L^\infty}\right) < C_0,$$

for sufficiently smooth $\mathbf{u}_0$, say $\mathbf{u}_0 \in C^\infty$. We denote $\mathbf{u}((n+1)\Delta t) = \mathbf{u}^{n+1}$ and assume

$$(4.15) \qquad \max_{0\leq n\leq N_0}\left(\|\mathbf{U}^n(\cdot,t)\|_{L^\infty}, \|\nabla\mathbf{U}^n(\cdot,t)\|_{L^\infty}\right) < C_0,$$

where $\Delta t N_0 = T_0$. Since our MLSRK discrete solution is smooth in space and we are considering fixed small interval of time, the above assumption (4.15) is justified. By defining

$$\begin{aligned}\mathbf{R}_n : &= \mathbf{u}((n+1)\Delta t) - \mathbf{u}(n\Delta t) + \Delta t\left[\mathbf{F}^i(\mathbf{u}((n+1)\Delta t))\right]_{x_i} \\ &= \int_{n\Delta t}^{(n+1)\Delta t}\left[\mathbf{F}^i(\mathbf{u}((n+1)\Delta t)) - \mathbf{F}^i(\mathbf{u}(\tau))\right]_{x_i} d\tau,\end{aligned}$$

we get our error equation

$$(4.16) \qquad \begin{aligned}&\left[\mathbf{U}^{n+1} - \mathbf{u}^{n+1}\right] + \Delta t\left[\mathbf{F}^i(\mathbf{U}^{n+1}) - \mathbf{F}^i(\mathbf{u}^{n+1})\right]_{x_i} \\ &= \mathbf{U}^n - \mathbf{u}^n - \mathbf{R}_n\,.\end{aligned}$$

From the mean value theorem, we have

$$
\begin{aligned}
&\left[\mathbf{F}^i(\mathbf{U}^{n+1}) - \mathbf{F}^i(\mathbf{u}^{n+1})\right]_{x_i} \\
&= \left[\mathbf{F}^i_{\mathbf{u}}(\mathbf{u}^{n+1}) \cdot (\mathbf{U}^{n+1} - \mathbf{u}^{n+1})\right]_{x_i} \\
&\quad + \left[\int_0^1 \int_0^1 \mathbf{F}^i_{\mathbf{uu}}(\alpha s \mathbf{U}^{n+1} + (1-\alpha s)\mathbf{u}^{n+1})\, ds\, d\alpha \cdot \right. \\
&\quad \left. \cdot (\mathbf{U}^{n+1} - \mathbf{u}^{n+1})(\mathbf{U}^{n+1} - \mathbf{u}^{n+1})\right]_{x_i}.
\end{aligned} \tag{4.17}
$$

Hence by introducing the symmetrizer $\mathbf{A}_0(\mathbf{u}^{n+1})$ for $\mathbf{F}^i_{\mathbf{u}}(\mathbf{u}^{n+1})$, and from (4.16) and (4.17) we obtain

(4.18)

$$
\begin{aligned}
&\int_\Omega (\mathbf{U}^{n+1} - \mathbf{u}^{n+1}) \cdot \mathbf{A}_0(\mathbf{u}^{n+1})(\mathbf{U}^{n+1} - \mathbf{u}^{n+1})\, dx \\
&= -\Delta t \int_\Omega (\mathbf{U}^{n+1} - \mathbf{u}^{n+1}) \cdot \mathbf{A}_0(\mathbf{u}^{n+1}) \left[\mathbf{F}^i_{\mathbf{u}}(\mathbf{u}^{n+1}) \cdot (\mathbf{U}^{n+1} - \mathbf{u}^{n+1})\right]_{x_i} \\
&\quad - \Delta t \int_\Omega (\mathbf{U}^{n+1} - \mathbf{u}^{n+1}) \cdot \mathbf{A}_0(\mathbf{u}^{n+1}) \left[\int_0^1 \int_0^1 \mathbf{F}^i_{\mathbf{uu}}(\alpha s \mathbf{U}^{n+1}\right. \\
&\quad \left. + (1-\alpha s)\mathbf{u}^{n+1})\, ds\, d\alpha \cdot (\mathbf{U}^{n+1} - \mathbf{u}^{n+1})(\mathbf{U}^{n+1} - \mathbf{u}^{n+1})\right]_{x_i} dx \\
&\quad + \int_\Omega (\mathbf{U}^{n+1} - \mathbf{u}^{n+1}) \cdot \mathbf{A}_0(\mathbf{u}^{n+1})(\mathbf{U}^n - \mathbf{u}^n - \mathbf{R}_n)\, dx \\
&= I + II + III.
\end{aligned}
$$

We introduce the weighted L^2-norm

$$
\|\mathbf{u}\|_{L^2, \mathbf{A}_0^{n+1}} = \left[\int_\Omega \mathbf{u}\mathbf{A}_0(\mathbf{u}^{n+1})\mathbf{u}\, dx\right]^{\frac{1}{2}}. \tag{4.19}
$$

Estimation of $\|\mathbf{U}^{n+1} - \mathbf{u}^{n+1}\|_{L^2, \mathbf{A}_0^{n+1}}$ shall be made using the equation (4.18). Note that the weighted L^2-norm $\|\cdot\|_{L^2, \mathbf{A}_0^{n+1}}$ is equivalent to the standard L^2-norm $\|\cdot\|_{L^2}$, since the matrix norm of $\mathbf{A}_0$ is bounded below and above.

Now, we are ready to state and prove our main theorem.

THEOREM 4. *Suppose* $\mathbf{F}^i$*'s,* $(i = 1, \ldots, N)$ *satisfy* (4.13) *for some* $M > 0$. *Let* T_0 *be the time for the existence of regular solution from*

Theorem 1, for $\mathbf{u}_0 \in C^\infty$. *We assume* (4.15) *for the MLSRK discrete solution. If* $\lambda = \frac{\Delta t}{\varrho} < \frac{c\alpha_0}{M|\Omega|^{\frac{1}{2}}}$, *then there is* C_1 *depending only on* λ *and* C_0 *such that*

$$\|(\mathbf{U}^n - \mathbf{u}^n)(\cdot, T)\|_{L^2} \le C_1 \varrho e^{cT} \left[\|\mathbf{u}^0\|_{H^1} + c\right],$$

where C_0 *is in* (4.15) *and* α_0 *is in* (4.12).

Proof. From the equation (4.18), since $\mathbf{A}_0(\mathbf{u}^{n+1})\mathbf{F}^i_{\mathbf{u}}(\mathbf{u}^{n+1})$ is a symmetric matrix, we get

(4.20)
$$\begin{aligned}
I =& -\Delta t \int_\Omega (\mathbf{U}^{n+1} - \mathbf{u}^{n+1}) \cdot \mathbf{A}_0(\mathbf{u}^{n+1}) \left[\mathbf{F}^i_{\mathbf{u}}(\mathbf{u}^{n+1}) \cdot (\mathbf{U}^{n+1} - \mathbf{u}^{n+1})\right]_{x_i} dx \\
=& \frac{\Delta t}{2} \int_\Omega (\mathbf{U}^{n+1} - \mathbf{u}^{n+1}) \frac{\partial}{\partial x_i} \left[\mathbf{A}_0(\mathbf{u}^{n+1})\mathbf{F}^i_{\mathbf{u}}(\mathbf{u}^{n+1})\right] \cdot (\mathbf{U}^{n+1} - \mathbf{u}^{n+1})\, dx \\
& - \Delta t \int_\Omega (\mathbf{U}^{n+1} - \mathbf{u}^{n+1})\, \mathbf{A}_0(\mathbf{u}^{n+1})\, \mathbf{F}^i_{\mathbf{uu}}(\mathbf{u}^{n+1})\, (\mathbf{u}^{n+1})_{x_i}\, (\mathbf{U}^{n+1} - \mathbf{u}^{n+1})\, dx \\
\le & \; cM \|\nabla \mathbf{u}\|_{L^\infty} \Delta t \|\mathbf{U}^{n+1} - \mathbf{u}^{n+1}\|^2_{L^2}.
\end{aligned}$$

Here the constant c only depends on $\|\mathbf{A}_0\|_{C^1}$. In a similar way, the term II in (4.18) becomes

(4.21)
$$\begin{aligned}
II = -\Delta t \int_\Omega & (\mathbf{U}^{n+1} - \mathbf{u}^{n+1}) \cdot \mathbf{A}_0(\mathbf{u}^{n+1}) \Big[\int_0^1 \int_0^1 \mathbf{F}^i_{\mathbf{uu}}(\alpha s \mathbf{U}^{n+1} \\
& + (1 - \alpha s)\mathbf{u}^{n+1})\, ds\, d\alpha \cdot (\mathbf{U}^{n+1} - \mathbf{u}^{n+1})(\mathbf{U}^{n+1} - \mathbf{u}^{n+1})\Big]_{x_i} dx \\
& \le c\Delta t M \left(\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla \mathbf{U}^{n+1}\|_{L^\infty}\right) \|\mathbf{U}^{n+1} - \mathbf{u}^{n+1}\|^2_{L^2}.
\end{aligned}$$

Here, the constant c depends only on $\|\mathbf{A}_0\|_{C^1}$. Since $\mathbf{A}_0(\mathbf{u}^{n+1})$ is symmetric positive definite, the term III in (4.18) becomes

$$\int_\Omega (\mathbf{U}^{n+1} - \mathbf{u}^{n+1}) \cdot \mathbf{A}_0(\mathbf{u}^{n+1})\, (\mathbf{U}^n - \mathbf{u}^n)\, dx$$

$$\leq \left[\int_\Omega (\mathbf{U}^{n+1}-\mathbf{u}^{n+1})\cdot \mathbf{A}_0(\mathbf{u}^{n+1})\,(\mathbf{U}^{n+1}-\mathbf{u}^{n+1})\,dx\right]^{\frac{1}{2}}$$

$$\left[\int_\Omega (\mathbf{U}^{n}-\mathbf{u}^{n})\cdot \mathbf{A}_0(\mathbf{u}^{n+1})\,(\mathbf{U}^{n}-\mathbf{u}^{n})\,dx\right]^{\frac{1}{2}}.$$

From the Hölder inequality, we have

$$\int_\Omega (\mathbf{U}^{n+1}-\mathbf{u}^{n+1})\cdot \mathbf{A}_0(\mathbf{u}^{n+1})\mathbf{R}_n dx \leq \|\mathbf{R}_n\|_{L^2}\|\mathbf{U}^{n+1}-\mathbf{u}^{n+1}\|_{L^2},$$

and from the mean value theorem, we have

$$\begin{aligned}|\mathbf{R}_n| &= \left|\int_{n\Delta t}^{(n+1)\Delta t}\left[\mathbf{F}^i(\mathbf{u}((n+1)\Delta t))-\mathbf{F}^i(\mathbf{u}(\tau))\right]_{x_i}d\tau\right|\\ &\leq cM\left(\|\mathbf{u}_t\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^\infty}+\|\nabla\mathbf{u}_t\|_{L^\infty}\right)\Delta t^2,\end{aligned}$$

where c is the absolute constant. Since $\|\cdot\|_{L^2,\mathbf{A}_0^{n+1}}$ is equivalent to $\|\cdot\|_{L^2}$, we have

$$III \leq \left(\|\mathbf{U}^n-\mathbf{u}^n\|_{L^2,\mathbf{A}_0^{n+1}}+cM\Delta t^2\right)\|\mathbf{U}^{n+1}-\mathbf{u}^{n+1}\|_{L^2,\mathbf{A}_0^{n+1}}, \tag{4.22}$$

where $c=c(\|\mathbf{A}_0\|,\|\mathbf{u}_t\|_{L^\infty},\|\nabla\mathbf{u}\|_{L^\infty},\|\nabla\mathbf{u}_t\|_{L^\infty},|\Omega|)$.

Combining (4.18), (4.20), (4.21), and (4.22), we have

$$\|\mathbf{U}^{n+1}-\mathbf{u}^{n+1}\|_{L^2,\mathbf{A}_0^{n+1}}(1-c_1M\Delta t)\leq \|\mathbf{U}^n-\mathbf{u}^n\|_{L^2,\mathbf{A}_0^{n+1}}+c_2M\Delta t^2,$$

where $c_1=c_1(\mathbf{u},\mathbf{A}_0)$ and $c_2=c_2(\mathbf{u},\mathbf{A}_0,|\Omega|)$. Using the bound

$$\begin{aligned}&\|\mathbf{U}^n-\mathbf{u}^n\|^2_{L^2,\mathbf{A}_0^{n+1}}\\ &=\int_\Omega(\mathbf{U}^n-\mathbf{u}^n)\mathbf{A}_0(\mathbf{u}^{n+1})(\mathbf{U}^n-\mathbf{u}^n)\,dx\\ &\leq \|\mathbf{U}^n-\mathbf{u}^n\|^2_{L^2,\mathbf{A}_0^n}+cM\Delta t\|\mathbf{u}_t\|_{L^\infty}\|\mathbf{U}^n-\mathbf{u}^n\|^2_{L^2,\mathbf{A}_0^n},\end{aligned}$$

where $c=c(\|\mathbf{A}_0\|)$, we have

$$\|\mathbf{U}^{n+1}-\mathbf{u}^{n+1}\|_{L^2,\mathbf{A}_0^{n+1}}\leq(1+c\Delta t)^{\frac{3}{2}}\|\mathbf{U}^n-\mathbf{u}^n\|_{L^2,\mathbf{A}_0^n}+c\lambda^2\varrho^2.$$

Iterating this, we get

$$\|\mathbf{U}^n-\mathbf{u}^n\|_{L^2,\mathbf{A}_0^n}\leq(1+c\Delta t)^{\frac{3n}{2}}\left[\|\mathbf{U}^0-\mathbf{u}^0\|_{L^2,\mathbf{A}_0^0}+c\lambda^2\varrho^2\right].$$

If we set $T_0 = N_0 \Delta t$, then for $n \leq N_0$

$$(1 + c\Delta t)^{\frac{3n}{2}} \simeq e^{cT_0},$$

and

$$\|\mathbf{U}^n - \mathbf{u}^n\|_{L^2, \mathbf{A}_0^n} \leq e^{cT_0} \left[\|\mathbf{U}^0 - \mathbf{u}^0\|_{L^2, \mathbf{A}_0^0} + c\lambda\varrho \right].$$

Hence by Theorem 2, the present theorem follows, in case $\mathbf{U}^0$ is H^1-projection,

$$\|\mathbf{U}^0 - \mathbf{u}^0\|_{L^2, \mathbf{A}_0^0} \leq c\varrho \|\mathbf{u}^0\|_{H^1}. \qquad \square$$

In Theorem 4, we have shown that the approximation errors of MLSRK discrete solution decrease as the number of nodes is increased. Though the present convergence analysis was made for a short time interval, the following numerical example shows that convergence of successive errors persists for a sufficiently long time. For one dimensional scalar conservation law, the existence of global solution in BV-space (functions of bounded variations) is known. In this case, the shock region is a measure zero set; our analysis thus appears to explain the convergence even for a long time.

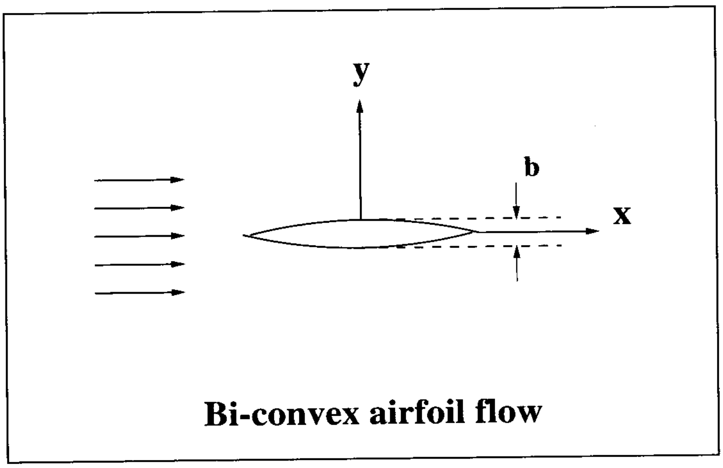

FIGURE 1. Problem statements

5. Numerical example

The numerical experiment consist of a supersonic flow over a symmetric biconvex airfoil. The MLSRK for space discretization and the backward Euler method for time discretization are employed. To solve the nonlinear vector equation, the Bi-CGM (Bi-Conjugate Gradient Method) is used.

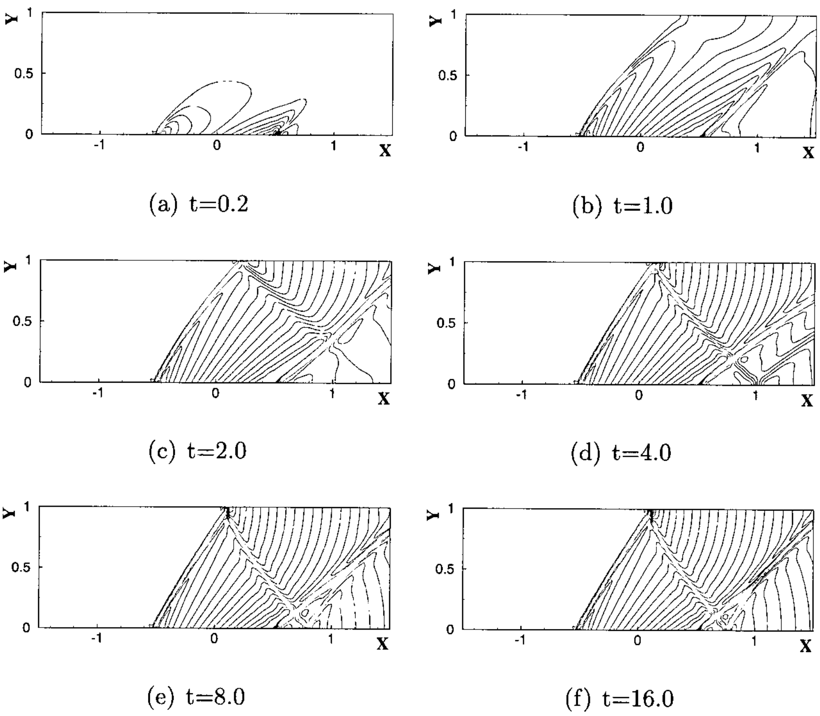

FIGURE 2. Pressure of biconvex airfoil flow

We consider a thin symmetric biconvex airfoil in an initial uniform flow field. The airfoil is given by a parabolic arc, $x_2 = \frac{1}{2}b(1-4x_1^2)$ as in Figure 1. The maximum airfoil thickness scaled by the chord length is $b = 0.08$.

Following free stream values are taken with $M_\infty = 1.4$:

$$\varrho_\infty = 1.0\,, \quad \mathbf{u}_{1,\infty} = 1.4, \quad \mathbf{u}_{2,\infty} = 0.0\,, \quad p_\infty = \frac{1.0}{\gamma},$$

where the specific heat ratio $\gamma = 1.4$. On the symmetric free boundary ($x_2 = 0$), we impose $\mathbf{u}_2 = 0$. On the surface of airfoil, the velocity vector is tangential to the surface and hence $\frac{\mathbf{u}_2}{\mathbf{u}_1} = \frac{dx_2}{dx_1}$. Assuming a thin airfoil, we replace $\mathbf{u}_1$ by its free stream value $\mathbf{u}_{1,\infty}$, and $\mathbf{u}_2 = \mathbf{u}_{1,\infty}\frac{dx_2}{dx_1} = -4bx_1\mathbf{u}_{1,\infty}$ on the surface of airfoil. We impose $\varrho = \varrho_\infty$, $\mathbf{u}_1 = \mathbf{u}_{1,\infty}$,

$\mathbf{u}_2 = \mathbf{u}_{2,\infty}$, $e = e_\infty$ on the inflow boundary, and $\mathbf{u}_2 = \mathbf{u}_{2,\infty}$ on the upper boundary. No boundary condition is imposed on the outflow boundary.

Figure 2 depicts the time development of stationary shock in the flow field. We observe that the leading edge shock is reflected from the upper boundary to interfere in its course with the trailing edge shock. This leading edge shock is reflected once again on the central free boundary to merge with the trailing edge shock. For this computation, a uniform 121×41 node set is used in the computational domain. To show convergence of the present scheme graphically, the relative errors are plotted in Figure 3. Four different uniform node sets are used: case $I(31 \times 11$ nodes), case $II(61 \times 21$ nodes), case $III(91 \times 31$ nodes), and case $IV(121 \times 41$ nodes).

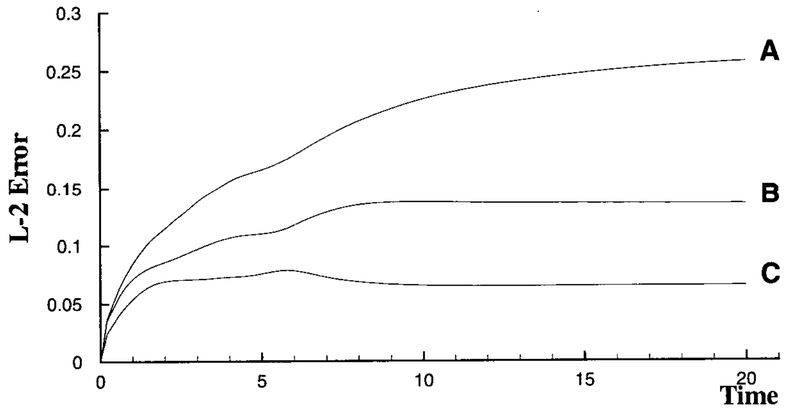

FIGURE 3. Relative errors depending on the node numbers

The three curves A, B and C in Figure 3 represent the L^2-errors $\|\mathbf{U}^I - \mathbf{U}^{II}\|_{L^2}$, $\|\mathbf{U}^{II} - \mathbf{U}^{III}\|_{L^2}$, and $\|\mathbf{U}^{III} - \mathbf{U}^{IV}\|_{L^2}$, respectively. Here $\mathbf{U}^I$, $\mathbf{U}^{II}$, $\mathbf{U}^{III}$ and $\mathbf{U}^{IV}$ stand for the solutions obtained using the indicated node set. Since the shock region has a small measure in the computational domain, the present results are consistent with our convergence theory given by Theorem 4.

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