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MAXIMAL HOLONOMY OF INFRA-NILMANIFOLDS WITH $\mathfrak{so}(3)\tilde{\times}\mathbb{R}^3$-GEOMETRY

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Abstract. Let $\mathfrak{so}(3)\tilde{\times}\mathbb{R}^3$ be the 6-dimensional nilpotent Lie group with group operation $(s,\mathbf{x})(t,\mathbf{y}) = (s+t+\mathbf{x}\mathbf{y}^t - \mathbf{y}\mathbf{x}^t,\ \mathbf{x}+\mathbf{y})$. We prove that the maximal order of the holonomy groups of all infra-nilmanifolds with $\mathfrak{so}(3)\tilde{\times}\mathbb{R}^3$-geometry is 16.

1. Introduction

Let $\mathfrak{so}(n)$ be the additive group of skew-symmetric matrices. This is the Lie algebra of the special orthogonal group $\mathrm{SO}(n)$. There is a bilinear map

$$\mathcal{I} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathfrak{so}(n)$$

defined as follows: For

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix},\ \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix},$$

$$\mathcal{I}(\mathbf{x},\mathbf{y}) = \mathbf{x}\mathbf{y}^t - \mathbf{y}\mathbf{x}^t,$$

where $()^t$ denotes transpose of the matrix $()$. With this $\mathcal{I}$, the set $\mathfrak{so}(n)\times \mathbb{R}^n$ gets a group structure given by

$$(s,\mathbf{x})(t,\mathbf{y}) = (s+t+\mathcal{I}(\mathbf{x},\mathbf{y}),\ \mathbf{x}+\mathbf{y}).$$

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This group is denoted by

$$\mathfrak{N}^n = \mathfrak{so}(n) \tilde{\times} \mathbb{R}^n.$$

Then $\mathfrak{N}^n$ is a simply connected nilpotent Lie group, of nilpotency class 2, with the center $\mathfrak{so}(n)$. Note that $\mathfrak{so}(n)$ is viewed as a commutative Lie group isomorphic to $\mathbb{R}^{\frac{n(n-1)}{2}}$ (not as a Lie algebra). It fits the short exact sequences of Lie groups

$$0 \to \mathfrak{so}(n) \to \mathfrak{N}^n \to \mathbb{R}^n \to 1.$$

Let M be an infra-nilmanifold with $\mathfrak{N}^n$-geometry; that is, $M = \Pi \backslash \mathfrak{N}^n$, where Π is a torsion free, discrete, cocompact subgroup of $\mathfrak{N}^n \rtimes C$ for some compact subgroup C of $\mathrm{Aut}(\mathfrak{N}^n)$. Such a subgroup Π is called an *almost Bieberbach group* (=AB group). It is well known that Π contains a cocompact lattice Γ of $\mathfrak{N}^n$ of finite index, and the quotient group Π/Γ is called the holonomy group of M.

Our aim is to understand those infra-nilmanifolds modelled on the 6-dimensional nilpotent Lie group

$$\mathfrak{N}^3 = \mathfrak{so}(3) \tilde{\times} \mathbb{R}^3.$$

In particular, we are interested in finding the possible maximal order for the holonomy groups. We shall prove that *the holonomy group of maximal order is $D_8 \times \mathbb{Z}_2$, of order 16.*

2. The automorphism group of $\mathfrak{N}^n$

Since the center, $\mathcal{Z}(\mathfrak{N}^n) = \mathfrak{so}(n)$, is a characteristic subgroup of $\mathfrak{N}^n$, every automorphism of $\mathfrak{N}^n$ restricts to an automorphism of $\mathfrak{so}(n)$. Consequently an automorphism of $\mathfrak{N}^n$ induces an automorphism on the quotient group $\mathbb{R}^n$. Thus there is a natural homomorphism $\mathrm{Aut}(\mathfrak{N}^n) \to \mathrm{Aut}(\mathfrak{so}(n)) \times \mathrm{Aut}(\mathbb{R}^n)$, $\theta \mapsto (\hat{\theta}, \bar{\theta})$. Here $\mathrm{Aut}(\mathfrak{so}(n))$ is the group of linear automorphisms of the $\mathbb{R}$-vector space $\mathfrak{so}(n)$, not the Lie algebra automorphisms.

LEMMA 2.1. $\mathrm{Aut}(\mathfrak{N}^n) \to \mathrm{Aut}(\mathbb{R}^n) = \mathrm{GL}(n, \mathbb{R})$ *is surjective. Moreover, the exact sequence* $\mathrm{Aut}(\mathfrak{N}^n) \to \mathrm{GL}(n, \mathbb{R}) \to 1$ *splits.*

Proof. First we define

$$J : \mathrm{GL}(n, \mathbb{R}) \longrightarrow \mathrm{Aut}(\mathfrak{so}(n))$$

by

$$J(C)(s) = CsC^t \tag{2.1}$$

for $C \in \mathrm{GL}(n,\mathbb{R})$ and $s \in \mathfrak{so}(n)$. Then clearly, J is a homomorphism. We denote $J(C)$ by $\widehat{C}$.

We claim that, for any $C \in \mathrm{GL}(n,\mathbb{R})$,

$$(s,\mathbf{x}) \mapsto (\widehat{C}s, C\mathbf{x}) \tag{2.2}$$

is an automorphism of $\mathfrak{N}^n$. From

$$\begin{aligned}(\widehat{C}s, C\mathbf{x})\cdot(\widehat{C}t, C\mathbf{y}) &= (\widehat{C}s + \widehat{C}t + \mathcal{I}(C\mathbf{x}, C\mathbf{y}), C\mathbf{x} + C\mathbf{y})\\ &= (\widehat{C}(s+t) + \mathcal{I}(C\mathbf{x}, C\mathbf{y}), C(\mathbf{x}+\mathbf{y}))\\ (\widehat{C}(s+t+\mathcal{I}(\mathbf{x}+\mathbf{y})), C(\mathbf{x}+\mathbf{y})) &= (\widehat{C}(s+t) + \widehat{C}\mathcal{I}(\mathbf{x}+\mathbf{y}), C(\mathbf{x}+\mathbf{y})),\end{aligned}$$

for the assignment (2.2) to be an automorphism, it is necessary and sufficient that

$$\widehat{C}(\mathcal{I}(\mathbf{x},\mathbf{y})) = \mathcal{I}(C\mathbf{x}, C\mathbf{y}) \tag{2.3}$$

holds for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$. Now this follows from

$$C(\mathbf{x}\mathbf{y}^t - \mathbf{y}\mathbf{x}^t)C^t = C\mathbf{x}(C\mathbf{y})^t - C\mathbf{y}(C\mathbf{x})^t$$

for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$. □

THEOREM 2.2 (Structure of $\mathrm{Aut}(\mathfrak{N}^n)$).

$$\mathrm{Aut}(\mathfrak{N}^n) = \mathrm{Hom}(\mathbb{R}^n, \mathfrak{so}(n)) \rtimes \mathrm{GL}(n,\mathbb{R}),$$

where $\mathrm{Hom}(\mathbb{R}^n, \mathfrak{so}(n))$ *is the additive group of linear transformations of vector spaces, and an element* $(\eta, C) \in \mathrm{Hom}(\mathbb{R}^n, \mathfrak{so}(n)) \rtimes \mathrm{GL}(n,\mathbb{R})$ *acts by*

$$(\eta, C)(s,\mathbf{x}) = (\widehat{C}s + \eta(\mathbf{x}), C\mathbf{x}).$$

Proof. Let $\theta \in \mathrm{Aut}(\mathfrak{N}^n)$. Then we have the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathfrak{so}(n) & \longrightarrow & \mathfrak{N}^n & \longrightarrow & \mathbb{R}^n & \longrightarrow & 1\\ & & \downarrow{\scriptstyle\hat\theta} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\bar\theta} & & \\ 1 & \longrightarrow & \mathfrak{so}(n) & \longrightarrow & \mathfrak{N}^n & \longrightarrow & \mathbb{R}^n & \longrightarrow & 1\end{array}$$

Thus $\theta(s,\mathbf{x}) = (\hat\theta(s) + \eta(s,\mathbf{x}), \bar\theta(\mathbf{x}))$ for $(s,\mathbf{x}) \in \mathfrak{N}^n$, where $\eta : \mathfrak{N}^n \to \mathfrak{so}(n)$. Since θ is a homomorphism, one can show that η is a homomorphism, i.e.,

$$\eta((s,\mathbf{x})(t,\mathbf{y})) = \eta(s,\mathbf{x}) + \eta(t,\mathbf{y}).$$

In particular, $(\hat\theta(s), \mathbf{0}) = \theta(s,\mathbf{0}) = (\hat\theta(s) + \eta(s,\mathbf{0}), \mathbf{0})$ implies that $\eta(s,\mathbf{0}) = 0$ for all $s \in \mathfrak{so}(n)$, and thus $\eta(s,\mathbf{x}) = \eta((s,\mathbf{0})(0,\mathbf{x})) = \eta(s,\mathbf{0}) + \eta(0,\mathbf{x}) = \eta(0,\mathbf{x})$. Hence $\eta \in \mathrm{Hom}(\mathbb{R}^n, \mathfrak{so}(n))$.

Let's find out the kernel of the surjective homomorphism of Lemma 2.1:

$$\mathrm{Aut}(\mathfrak{N}^n) \longrightarrow \mathrm{GL}(n,\mathbb{R}), \quad \theta \mapsto \bar{\theta}.$$

Suppose that $\theta \in \mathrm{Aut}(\mathfrak{N}^n)$ with $\bar{\theta} = \mathrm{id}_{\mathbb{R}^n}$. Then $\hat{\bar{\theta}} = \mathrm{id}_{\mathfrak{so}(n)}$ (see the equality (2.1)) and thus $\theta(s,\mathbf{x}) = (s+\eta(\mathbf{x}),\mathbf{x})$ for some $\eta \in \mathrm{Hom}(\mathbb{R}^n, \mathfrak{so}(n))$. Conversely given $\eta \in \mathrm{Hom}(\mathbb{R}^n,\mathfrak{so}(n))$, define $\theta \in \mathrm{Aut}(\mathfrak{N}^n)$ by $\theta(s,\mathbf{x}) = (s+\eta(\mathbf{x}),\mathbf{x})$. Clearly this θ lies in the kernel of the homomorphism. Hence we have a short exact sequence

$$1 \to \mathrm{Hom}(\mathbb{R}^n,\mathfrak{so}(n)) \to \mathrm{Aut}(\mathfrak{N}^n) \to \mathrm{GL}(n,\mathbb{R}) \to 1.$$

By Lemma 2.1, this sequence is split. □

Note that

$$\begin{gathered}\mathrm{Hom}(\mathbb{R}^n,\mathfrak{so}(n)) \rtimes \mathrm{GL}(n,\mathbb{R}) \subset \mathrm{Hom}(\mathbb{R}^n,\mathfrak{so}(n)) \rtimes (\mathrm{Aut}(\mathfrak{so}(n)) \times \mathrm{GL}(n,\mathbb{R})) \\ (\eta, A) \mapsto (\eta,(\widehat{A},A))\end{gathered}$$

and the action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathrm{Hom}(\mathbb{R}^n,\mathfrak{so}(n))$ is ${}^{A}\eta(\mathbf{x}) = \hat{A}\cdot\eta(A^{-1}\mathbf{x})$. The group operation on $\mathfrak{N}^n \rtimes \mathrm{GL}(n,\mathbb{R})$ is given by

$$\begin{aligned} & ((s,\mathbf{x}),A)((t,\mathbf{y}),B) \\ = \; & ((s,\mathbf{x})\cdot {}^{A}(t,\mathbf{y}), AB) \\ = \; & ((s,\mathbf{x})\cdot(\hat{A}t, A\mathbf{y}), AB) \\ = \; & ((s+\hat{A}t+\mathcal{I}(\mathbf{x},A\mathbf{y}), \mathbf{x}+A\mathbf{y}), AB). \end{aligned}$$

3. The structure of AB-groups for $\mathfrak{N}^n$

Let $\Pi \subset \mathfrak{N}^n \rtimes \mathrm{Aut}(\mathfrak{N}^n)$ be an AB-group. Then it is well known that $\Gamma = \Pi \cap \mathfrak{N}^n$, the pure translations in Π, is the maximal normal nilpotent subgroup, and $\Phi = \Pi/\Gamma$, the holonomy group of Π, is finite. Since Γ is a lattice of $\mathfrak{N}^n$, $Z = \Gamma \cap \mathcal{Z}(\mathfrak{N}^n)$ is a lattice of $\mathcal{Z}(\mathfrak{N}^n) = \mathfrak{so}(n)$, and Γ/Z is a lattice of $\mathfrak{N}^n/\mathcal{Z}(\mathfrak{N}^n) = \mathbb{R}^n$. Thus $Z = \Gamma \cap \mathcal{Z}(\mathfrak{N}^n) \cong \mathbb{Z}^{\frac{n(n-1)}{2}}$ and $\Gamma/\Gamma \cap \mathcal{Z}(\mathfrak{N}^n) \cong \mathbb{Z}^n$.

Consider the following natural commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathfrak{so}(n) & \xrightarrow[=]{} & \mathfrak{so}(n) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathfrak{N}^n & \longrightarrow & \mathfrak{N}^n \rtimes \mathrm{Aut}(\mathfrak{N}^n) & \longrightarrow & \mathrm{Aut}(\mathfrak{N}^n) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
1 & \longrightarrow & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \rtimes \mathrm{Aut}(\mathfrak{N}^n) & \longrightarrow & \mathrm{Aut}(\mathfrak{N}^n) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

Recall from Theorem 2.2 that an element

$$(\eta, A) \in \mathrm{Aut}(\mathfrak{N}^n) = \mathrm{Hom}(\mathbb{R}^n, \mathfrak{so}(n)) \rtimes \mathrm{GL}(n, \mathbb{R})$$

acts on $(s, \mathbf{x}) \in \mathfrak{N}^n$ by

$$(\eta, A)(s, \mathbf{x}) = (\hat{A}s + \eta(\mathbf{x}), A\mathbf{x}).$$

Thus $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathfrak{so}(n)$ via the homomorphism $\hat{\ } : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{Aut}(\mathfrak{so}(n))$, and $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication $\mathrm{GL}(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$.

Let $Q = \Pi / Z$. Then the above diagram induces the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Z & \xrightarrow[=]{} & Z & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
1 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & Q & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
\tag{3.1}
$$

Here an element $(\eta, A) \in \Phi \subset \mathrm{Aut}(\mathfrak{N}^n) = \mathrm{Hom}(\mathbb{R}^n, \mathfrak{so}(n)) \rtimes \mathrm{GL}(n,\mathbb{R})$ acts on $\mathbb{Z}^n$ by A, and on Z by $\hat{A}$.

Recall from [4, Proposition 2] that a virtually free abelian group $1 \to \mathbb{Z}^n \to Q \to \Phi \to 1$ is a crystallographic group if and only if the centralizer of $\mathbb{Z}^n$ in Q has no torsion elements. Suppose an element of Φ acts trivially on $\mathbb{Z}^n$. Then it is of the form $(\eta, A) \in \mathrm{Aut}(\mathfrak{N}^n)$, where $A = I \in \mathrm{GL}(n,\mathbb{R})$. Then $\hat{A} = I$ also, see equation (2.1). Since $\mathrm{Hom}(\mathbb{R}^n, \mathfrak{so}(n))$ is isomorphic to the additive group of $n \times \frac{n(n-1)}{2}$ matrices ($\approx \mathbb{R}^{\frac{n(n+1)}{2}}$), it has no elements of finite order. Consequently, $(\eta, A) \in \mathrm{Aut}(\mathfrak{N}^n)$ is trivial. This shows that Φ acts effectively on $\mathbb{Z}^n$. It follows that Q is naturally an n-dimensional crystallographic group.

The finite group Φ must be in a maximal compact subgroup $\mathrm{O}(n)$ of $\mathrm{Aut}(\mathfrak{N}^n) = \mathrm{Hom}(\mathbb{R}^n, \mathfrak{so}(n)) \rtimes \mathrm{GL}(n,\mathbb{R})$.

PROPOSITION 3.1. *Let $Q \subset \mathbb{R}^n \rtimes O(n)$ be a crystallographic group, and $Q' \subset \mathbb{R}^n \rtimes \mathrm{GL}(n,\mathbb{R})$ be an affine crystallographic group isomorphic to Q. Then there exists an* AB*-group Π obtained from Q if and only if there exists an* AB*-group Π' obtained from Q'.*

Proof. Let

$$\begin{aligned} Q &= \langle (\mathbf{v}_1, I), (\mathbf{v}_2, I), \ldots, (\mathbf{v}_n, I), (\mathbf{a}_1, A_1), (\mathbf{a}_2, A_2), \ldots, (\mathbf{a}_p, A_p) \rangle, \\ Q' &= \langle (\mathbf{v}'_1, I), (\mathbf{v}'_2, I), \ldots, (\mathbf{v}'_n, I), (\mathbf{a}'_1, A'_1), (\mathbf{a}'_2, A'_2), \ldots, (\mathbf{a}'_p, A'_p) \rangle \end{aligned}$$

and $(d, D) \in \mathbb{R}^n \rtimes \mathrm{GL}(n,\mathbb{R})$ with

$$(\mathbf{v}'_i, I) = (d, D)^{-1}(\mathbf{v}_i, I)(d, D), \quad i = 1, 2, \ldots, n$$

$$(\mathbf{a}'_j, A'_j) = (d, D)^{-1}(\mathbf{a}_j, A_j)(d, D), \quad j = 1, 2, \ldots, p.$$

Suppose there exist $t_1, t_2, \ldots, t_n,\ s_1, s_2, \ldots, s_p \in \mathfrak{so}(n)$ such that

$$\begin{aligned} \Pi = \langle &(t_1, \mathbf{v}_1, I), (t_2, \mathbf{v}_2, I), \ldots, (t_n, \mathbf{v}_n, I), \\ &(s_1, \mathbf{a}_1, A_1), (s_2, \mathbf{a}_2, A_2), \ldots, (s_p, \mathbf{a}_p, A_p) \rangle \end{aligned}$$

is an AB-group. Then clearly, conjugate of Π by $(\mathbf{0}, d, D) \in \mathfrak{N}^n \rtimes \mathrm{Aut}(\mathfrak{N}^n)$ is an AB-group Π' obtained from Q'. The converse is similar. □

From this Proposition, we need not put our abstract crystallographic group into $\mathbb{R}^n \rtimes \mathrm{O}(n)$, but it is enough to use any convenient affine embedding.

LEMMA 3.2. *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis of $\mathbb{R}^n$. Then the set $\{\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) \mid i < j\}$ forms a lattice of $\mathfrak{so}(n)$.*

Proof. First we set notation. Let E_{ij} be the skew-symmetric matrix whose (i,j) entry is 1, (j,i) entry is -1, and 0 elsewhere. Suppose $\mathbf{v}_i = \mathbf{e}_i$ for all $i = 1,2,\ldots,n$. Then $\mathcal{I}(\mathbf{e}_i,\mathbf{e}_j) = \mathbf{e}_i\mathbf{e}_j^t - \mathbf{e}_j\mathbf{e}_i^t = E_{ij}$. Therefore $\{\mathcal{I}(\mathbf{e}_i,\mathbf{e}_j) \mid i<j\} = \{E_{ij} \mid i<j\}$ is a basis for the vector space $\mathfrak{so}(n)$. For a general basis $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$, let C be the matrix whose jth column is $\mathbf{v}_j$. Then $\mathcal{I}(\mathbf{v}_i,\mathbf{v}_j) = J(C)(E_{ij})$, and hence the set $\{\mathcal{I}(\mathbf{v}_i,\mathbf{v}_j) \mid i<j\}$ forms a basis of $\mathfrak{so}(n)$. □

This lemma tells us that the lattice $\mathbb{Z}^n$ of $\mathbb{R}^n$ generated by $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ yields a lattice generated by $\{\mathcal{I}(\mathbf{v}_i,\mathbf{v}_j) \mid i<j\}$, which must be contained in Z in the above diagram. However, the Z in the diagram can be finer than the lattice generated by $\{\mathcal{I}(\mathbf{v}_i,\mathbf{v}_j) \mid i<j\}$.

PROPOSITION 3.3. *Let $Q \subset \mathbb{R}^n \rtimes \mathrm{SO}(n)$ be a crystallographic group generated by*

$$(\mathbf{v}_1,I),(\mathbf{v}_2,I),\ldots,(\mathbf{v}_n,I),(\mathbf{a}_1,A_1),(\mathbf{a}_2,A_2),\ldots,(\mathbf{a}_p,A_p),$$

where the subgroup $\langle(\mathbf{v}_1,I),(\mathbf{v}_2,I),\ldots,(\mathbf{v}_n,I)\rangle$ is the maximal normal free abelian subgroup of Q. If there exists an AB-group *obtained from Q, then there is an AB-group $\Pi_{(s_1,s_2,\ldots,s_p)}$ generated by*

$$(0,\mathbf{v}_1,I),(0,\mathbf{v}_2,I),\ldots,(0,\mathbf{v}_n,I),$$
$$(s_1,\mathbf{a}_1,A_1),(s_2,\mathbf{a}_2,A_2),\ldots,(s_p,\mathbf{a}_p,A_p),$$

for some $s_1,s_2,\ldots,s_p \in \mathfrak{so}(n)$.

Proof. Let $\Pi \subset \mathfrak{N}^n \rtimes \mathrm{Aut}(\mathfrak{N}^n)$ be an AB-group obtained from Q. Then

$$\Pi \cap \mathfrak{so}(n) = \langle(v_1,\mathbf{0},I),(v_2,\mathbf{0},I),\ldots,(v_{\frac{n(n-1)}{2}},\mathbf{0},I)\rangle$$

for some $v_1,v_2,\ldots,v_{\frac{n(n-1)}{2}} \in \mathfrak{so}(n)$. Let us take any pre-images in Π as follows:

$$(w_1,\mathbf{v}_1,I),(w_2,\mathbf{v}_2,I),\ldots,(w_n,\mathbf{v}_n,I),$$
$$(t_1,\mathbf{a}_1,A_1),(t_2,\mathbf{a}_2,A_2),\ldots,(t_p,\mathbf{a}_p,A_p)$$

for some $w_1,w_2,\ldots,w_n,\ t_1,t_2,\ldots,t_p \in \mathfrak{so}(n)$. Since $[(w_i,\mathbf{v}_i,I),(w_j,\mathbf{v}_j,I)] = (2\mathcal{I}(\mathbf{v}_i,\mathbf{v}_j),\mathbf{0},I)$, the group generated by

$$(w_1,\mathbf{v}_1,I),(w_2,\mathbf{v}_2,I),\ldots,(w_n,\mathbf{v}_n,I)$$

becomes a lattice of $\mathfrak{N}^n$ already (see Lemma 3.2), and hence the group generated by

$$(w_1,\mathbf{v}_1,I),(w_2,\mathbf{v}_2,I),\ldots,(w_n,\mathbf{v}_n,I),$$
$$(t_1,\mathbf{a}_1,A_1),(t_2,\mathbf{a}_2,A_2),\ldots,(t_p,\mathbf{a}_p,A_p)$$

(without $(v_1, \mathbf{0}, I), (v_2, \mathbf{0}, I), \ldots, (v_{\frac{n(n-1)}{2}}, \mathbf{0}, I)$) will be a subgroup of Π of finite index, and is an AB-group (since it is torsion free). Let us call this smaller group Π'.

Let $\eta \in \mathrm{Hom}(\mathbb{R}^n, \mathfrak{so}(n))$. Consider the product in $\mathfrak{N}^n \rtimes \mathrm{Aut}(\mathfrak{N}^n)$:

$$((0, \mathbf{0}), \eta)^{-1}((w_i, \mathbf{v}_i), I)((0, \mathbf{0}), \eta) = ((w_i - \eta(\mathbf{v}_i), \mathbf{v}_i), I).$$

Certainly, there exists $\eta \in \mathrm{Hom}(\mathbb{R}^n, \mathfrak{so}(n))$ such that

$$\eta(\mathbf{v}_i) = w_i$$

for all $i = 1, 2, \ldots, n$.

Consequently, the group Π'' obtained by conjugating Π' by $(0, \mathbf{0}, \eta)$ is another AB-group, satisfying the condition $w_1 = w_2 = \cdots = w_n = 0$ (without changing the second and third slots). Thus we have obtained an AB-group $\Pi'' = \Pi_{(s_1, s_2, \ldots, s_p)} \subset \mathfrak{N}^n \rtimes \mathrm{Aut}(\mathfrak{N}^n)$ generated by

$$(0, \mathbf{v}_1, I), (0, \mathbf{v}_2, I), \ldots, (0, \mathbf{v}_n, I),$$

$$(s_1, \mathbf{a}_1, A_1), (s_2, \mathbf{a}_2, A_2), \ldots, (s_p, \mathbf{a}_p, A_p)$$

for some $s_1, s_2, \ldots, s_p \in \mathfrak{so}(n)$. □

The Proposition says that: If there is an AB-group Π constructed from Q, then $\Pi_{(s_1, s_2, \ldots, s_p)}$ is conjugate to a subgroup of Π of finite index, which is another AB-group constructed from Q. Conversely, if $\Pi_{(s_1, s_2, \ldots, s_p)}$ is an AB-group (i.e., is torsion free), then we are done. Therefore, the existence/non-existence of construction is solely determined by the group of the form $\Pi_{(s_1, s_2, \ldots, s_p)}$; that is, whether there exist $s_1, s_2, \ldots, s_p$ for which $\Pi_{(s_1, s_2, \ldots, s_p)}$ is torsion free.

4. The group $\mathfrak{N}^3$

Our group $\mathfrak{N}^3 = \mathfrak{so}(3) \widetilde{\times} \mathbb{R}^3$ has group law

$$(s, \mathbf{x})(t, \mathbf{y}) = (s + t + \mathcal{I}(\mathbf{x}, \mathbf{y}), \mathbf{x} + \mathbf{y}),$$

where

$$\begin{aligned}\mathcal{I}(\mathbf{x}, \mathbf{y}) &= \mathbf{x}\mathbf{y}^t - \mathbf{y}\mathbf{x}^t \\ &= \begin{bmatrix} 0 & x_1y_2 - x_2y_1 & x_1y_3 - x_3y_1 \\ x_2y_1 - x_1y_2 & 0 & x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 & x_3y_2 - x_2y_3 & 0 \end{bmatrix}.\end{aligned}$$

If we identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ by

$$\begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

then clearly,

$$\mathcal{I}(\mathbf{x}, \mathbf{y}) = \mathbf{x} \times \mathbf{y} \text{ (cross product)}$$

so that $\mathfrak{N}^3 = \mathfrak{so}(3) \times \mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$ has the group operation

$$(s, \mathbf{x})(t, \mathbf{y}) = (s + t + \mathbf{x} \times \mathbf{y},\ \mathbf{x} + \mathbf{y}).$$

In our case, $\widehat{C} = J(C)$ has a very simple description. Let $\mathbf{e}_i \in \mathbb{R}^n$ whose ith component is 1 and 0 elsewhere. From the condition (2.3), we have

$$\begin{aligned} \widehat{C}(\mathcal{I}(\mathbf{e}_i, \mathbf{e}_j)) &= \mathcal{I}(C\mathbf{e}_i, C\mathbf{e}_j) \\ &= \mathcal{I}(\mathbf{c}_i, \mathbf{c}_j) \end{aligned}$$

for all i, j, where $\mathbf{c}_i$ is the ith column of the matrix C. Note that $\mathcal{I}(\mathbf{e}_i, \mathbf{e}_j)$ is the skew-symmetric matrix whose (i, j) entry is 1, (j, i) entry is -1, and 0 elsewhere.

By the identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$ as above, the above equality becomes

$$\begin{aligned} \widehat{C}(\mathbf{e}_1) &= \mathbf{c}_2 \times \mathbf{c}_3 \\ \widehat{C}(\mathbf{e}_2) &= \mathbf{c}_3 \times \mathbf{c}_1 \\ \widehat{C}(\mathbf{e}_3) &= \mathbf{c}_1 \times \mathbf{c}_2 \end{aligned}$$

so that

$$\widehat{C} = \det(C)(C^{-1})^t.$$

For each 3-dimensional crystallographic group Q, we shall check if there exists an AB-group Π constructed from Q; that is, a torsion free $\Pi \subset \mathfrak{N}^3 \rtimes \mathrm{O}(3) \subset \mathfrak{N}^3 \rtimes \mathrm{Aut}(\mathfrak{N}^3)$ fitting the short exact sequence

$$1 \longrightarrow Z \longrightarrow \Pi \longrightarrow Q \longrightarrow 1.$$

This is the key notion for our arguments and construction. We have a complete classification of 3-dimensional crystallographic groups (Q's in the above statement). We shall use the representations of the 3-dimensional crystallographic groups given in the book [1].

Every Q has an explicit representation $Q \longrightarrow \mathbb{R}^3 \rtimes \mathrm{GL}(3, \mathbb{Z})$ (not into $\mathbb{R}^3 \rtimes \mathrm{O}(3)$) in this book.

Our goal is to determine which Q will give rise to a *torsion free* Π that fits the diagram (3.1). When Q is torsion free, then Π will be automatically torsion free, but when Q contains a torsion subgroup Q_0, we need to check whether the lift Q_0 to Π will be torsion free.

Let Π_0 be the lift of Q_0 to Π. Thus

$$(4.1) \qquad 1 \longrightarrow Z \longrightarrow \Pi_0 \longrightarrow Q_0 \longrightarrow 1$$

is exact. If Π is torsion free, then so is Π_0, and is a 3-dimensional Bieberbach group. Therefore,

PROPOSITION 4.1. *Any finite subgroup of Q is one of the holonomy groups $1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ of 3-dimensional crystallographic groups.*

PROPOSITION 4.2. *There is no AB-group $\Pi \subset \mathfrak{N}^3 \rtimes \mathrm{Aut}(\mathfrak{N}^3)$ constructed from Q given in* **7/4/3/02**, **7/3/2/02**, **7/2/1/03**, **7/2/3/02**, (*all holonomy group of order 24*).

Proof. Let

$$Q = \langle (\mathbf{e}_1, I), (\mathbf{e}_2, I), (\mathbf{e}_3, I), (\mathbf{a}, A), (\mathbf{b}, B), (\mathbf{c}, C), (\mathbf{d}, D)\rangle,$$

and let, for some $s, t, u, v \in \mathfrak{so}(3)$,

$$t_1 = (0, \mathbf{e}_1, I), \quad t_2 = (0, \mathbf{e}_2, I), \quad t_3 = (0, \mathbf{e}_3, I),$$

$$\alpha = (s, \mathbf{a}, A), \quad \beta = (t, \mathbf{b}, B), \quad \gamma = (u, \mathbf{c}, C), \quad \delta = (v, \mathbf{d}, D).$$

By Proposition 3.3, there is no AB-group from Q, if and only if, for any $s, t, u, v \in \mathfrak{so}(3)$, the group Π generated by $t_1, t_2, t_3, \alpha, \beta, \gamma, \delta$

$$(4.2) \qquad \Pi_{(s,t,u,v)} = \langle t_1, t_2, t_3, \alpha, \beta, \gamma, \delta\rangle$$

has a nontrivial torsion element. Now let $Z = \Pi \cap \mathfrak{so}(3)$. Then there is no AB-group from Q if and only if the following holds: for any $s, t, u, v \in \mathfrak{so}(3)$, there exists a nontrivial torsion element of the form

$$z t_1^{n_1} t_2^{n_2} t_3^{n_3} \alpha^p \beta^q \gamma^r \delta^o$$

where $z \in Z$ and $n_i, p, q, r, o, k \in \mathbb{Z}$ with $0 \le p <$ order of A, etc.

(**7/4/3/02**). For any choice of $s, t, u, v \in \mathfrak{so}(3)$, $\alpha^{-1}\gamma^3\alpha\gamma$ is a torsion element of order 3.

(**7/3/2/02**). Let $z = (16(1,-1,-1), \mathbf{0}, I)$. Then $z \in Z$. For any choice of $s, t, u, v \in \mathfrak{so}(3)$, $z\alpha^{-1}\gamma^3\alpha\gamma$ is a torsion element of order 3.

(**7/2/1/03**). For any choice of $s, t, u, v \in \mathfrak{so}(3)$, $\alpha^{-1}\gamma^3\alpha\gamma$ is a torsion element of order 3.

(**7/2/3/02**). For any choice of $s, t, u, v \in \mathfrak{so}(3)$, $\alpha^{-1}\gamma^3\alpha\gamma$ is a torsion element of order 3. □

5. A construction of an AB-group with holonomy order 16

THEOREM 5.1 (Existence). *There exists an almost Bieberbach group* $\Pi \subset \mathfrak{N}^3 \rtimes \mathrm{Aut}(\mathfrak{N}^3)$ *whose holonomy group is* $D_8 \times \mathbb{Z}_2$*, of order 16.*

Proof. Consider the 3-dimensional crystallographic group Q (Case **4/7/1/10**) generated by the following elements:

$$(\mathbf{e}_1, I), \quad (\mathbf{e}_2, I), \quad (\mathbf{e}_3, I), \quad (\mathbf{a}, A), \quad (\mathbf{b}, B), \quad (\mathbf{c}, C), \quad (\mathbf{d}, D),$$

where I is the identity matrix, $\mathbf{e}_i$ is the unit vector with 1 in the ith coordinate, and

$$(\mathbf{a}, A) = \left(\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right),$$
$$(\mathbf{b}, B) = \left(\frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right),$$
$$(\mathbf{c}, C) = \left(\frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right),$$
$$(\mathbf{d}, D) = \left(\frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right).$$

Note that

$$A^2 = I,\ B^2 = A,\ C^2 = I,\ D^2 = I,\ [B, C] = I,\ [B, D] = I,\ [C, D] = I$$

and that Q has the holonomy group Φ of order 16.

We consider the subgroup Π of $(\mathfrak{so}(3)\tilde{\times}\mathbb{R}^3) \rtimes \mathrm{SO}(3) = (\mathbb{R}^3\tilde{\times}\mathbb{R}^3) \rtimes \mathrm{SO}(3)$ generated by

$$t_1 = (0, \mathbf{e}_1, I), \quad t_2 = (0, \mathbf{e}_2, I), \quad t_3 = (0, \mathbf{e}_3, I),$$

$$\alpha = \left(\begin{bmatrix} 0 \\ 0 \\ \frac{1}{4} \end{bmatrix}, \mathbf{a}, A \right), \quad \beta = \left(\begin{bmatrix} 0 \\ 0 \\ \frac{1}{8} \end{bmatrix}, \mathbf{b}, B \right),$$

$$\gamma = \left(\begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \end{bmatrix}, \mathbf{c}, C \right), \quad \delta = \left(\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}, \mathbf{d}, D \right).$$

Let $Z = \Pi \cap \mathfrak{so}(3)$. We know Z is a lattice of $\mathfrak{so}(3)$ from Lemma 3.2. By calculation, we see that Z is actually generated by the vectors

$$z_1 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{4} \\ 0 \end{bmatrix}, \quad z_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

In particular, Z is discrete and hence Π is discrete. [Notice that we are abusing the notation: $z_i = (z_i, \mathbf{0}, I) \in (\mathfrak{so}(3)\tilde{\times}\mathbb{R}^3) \rtimes \mathrm{SO}(3)$].

Next we claim that Π is torsion free. Every element of Π can be written as

$$z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} t_1^{n_1} t_2^{n_2} t_3^{n_3} \alpha^p \beta^q \gamma^r \delta^s,$$

where $\ell_1, \ell_2, \ell_3,\ n_1, n_2, n_3,\ p, q, r, s$ are integers such that $0 \le p < 2,\ 0 \le q < 4,\ 0 \le r < 2$ and $0 \le s < 2$. By calculation, it can be shown that the equation

$$(z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} t_1^{n_1} t_2^{n_2} t_3^{n_3} \alpha^p \beta^q \gamma^r \delta^s)^d = (0, \mathbf{0}, I)$$

has no integral solution for $\ell_1, \ell_2, \ell_3,\ n_1, n_2, n_3, p, q, r, s$ with $d > 0$. This proves that Π is indeed torsion free, and thus is an AB-group. □

6. Non-existence

Using notations from [1], we list all three-dimensional crystallographic groups with holonomy order greater than 16 below. Most of these groups are eliminated by the fact that they contain torsion subgroups which are not in the list $1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. See Proposition 4.1. The remaining 4 cases where this proposition does not apply are proved in Proposition 4.2. All the calculations were done using the program MATHEMATICA [7] and hand-checked.

• <u>48</u>:

7/5/1/02: $\langle (a, A), (b, B), (e, E) \rangle \cong (\mathbb{Z}_2)^3$.
7/5/1/03: $\langle (a, A), (c, C) \rangle$ non-commutative group of order 12.
7/5/1/04: $\langle (a, A), (c, C) \rangle$ non-commutative group of order 12.
7/5/2/02: $\langle (a, A)(d, D), (c, C) \rangle$ non-commutative group of order 6.
7/5/2/03: $\langle (a, A), (c, C) \rangle$ non-commutative group of order 12.
7/5/2/04: $\langle (a, A), (c, C) \rangle$ non-commutative group of order 12.
7/5/3/02: $\langle (a, A)(d, D), (c, C) \rangle$ non-commutative group of order 6.

- 24:

7/4/1/02: $\langle (a,A),(c,C)\rangle$ non-commutative group of order 12.
7/4/2/02: $\langle (a,A),(c,C)\rangle$ non-commutative group of order 12.
7/4/3/02: Proposition 4.2.

7/3/1/02: $\langle (a,A)(d,D),(b,B)(c,C)\rangle$ non-commutative group of order 6.
7/3/1/03: $\langle (a,A)(b,B),(a,A)(c,C)\rangle$ non-commutative group of order 12.
7/3/2/02: Proposition 4.2.
7/3/3/02: $\langle (a,A)(d,D),(c,C)\rangle$ non-commutative group of order 6.

7/2/1/02: $\langle (a,A),(c,C)\rangle$ non-commutative group of order 12.
7/2/1/03: Proposition 4.2.
7/2/2/02: $\langle (a,A),(c,C)^2\rangle$ non-commutative group of order 12.
7/2/3/02: Proposition 4.2.

6/7/1/02: $\langle (a,A),(c,C)\rangle$ non-commutative group of order 6.
6/7/1/03: $\langle (a,A),(c,C)\rangle$ non-commutative group of order 6.
6/7/1/04: $\langle (a,A),(c,C)\rangle$ non-commutative group of order 6.

- 16:

4/7/1/10: CONSTRUCTION! (Theorem 5.1)

References

[1] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek, and H. Zassenhaus, *Crystallographic Groups of Four-Dimensional Spaces*, John Wiley & Sons, Inc., New York, 1978.
[2] K. Y. Ha, J. B. Lee, and K. B. Lee, *Maximal holonomy of infra-nilmanifolds with 2-dimensional Quaternionic Heisenberg Geometry*, Trans. Amer. Math. Soc. **357** (2005), no. 1, 355–383.
[3] I. Kim and J. R. Parker, *Geometry of quaternionic hyperbolic manifolds*, Math. Proc. Cambridge Philos. Soc. **135** (2003), no. 2, 291–320.
[4] K. B. Lee and F. Raymond, *Topological, affine and isometric actions on flat Riemannian manifolds*, J. Differential Geom. **16** (1981), no. 2, 255–269.
[5] K. B. Lee, J. K. Shin, and S. Yokura, *Free actions of finite abelian groups on the 3-torus*, Topology Appl. **53** (1993), no. 2, 153–175.
[6] K. B. Lee and A. Szczepański, *Maximal holonomy of almost Bieberbach groups for* $\mathbf{Heis}_5$, Geom. Dedicata **87** (2001), no. 1-3, 167–180.
[7] S. Wolfram, *Mathematica*, Wolfram Research, 1993.

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