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TUBES IN SINGULAR SPACES OF NONPOSITIVE CURVATURE

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Abstract. In this paper, we estimate area of tube in a CBA(0)-space with extendible geodesics. As its application, we obtain an upper bound of systole in a nonsimply connected space of nonpositive curvature. Also, we determine a relative growth of a ball in a CBA(0)-space to the corresponding ball in Euclidean plane.

1. Introduction

Bishop-Günther inequality ([8]) states that volume of n-ball in Riemannian manifolds of non-positive sectional curvature is not less than that of the ball with same radius in Euclidean n-space. This enables us to characterize Riemannian manifolds in terms of local behavior of volume functional.

The aim of this paper is to study tubes in CBA(k)-space in terms of definition of area of surface by Nikolaev ([12]). A CBA(k)-space is a singular metric space of curvature bounded above by k in the sense of Alexandrov ([1]). Riemannian manifolds of sectional curvature bounded above by k are CBA(k)-spaces and trees with intrinsic metric, locally convex polyhedrons and many other topological spaces are any other CBA(k)-spaces. CBA(k)-spaces share many valuable geometric properties with Riemannian manifolds. Recently Nagano ([11]) proved a sphere theorem for 2-dimensional CBA(1)-space and Mese ([10]) obtained a universal constant C so that $L^2 - CA \geq 0$, where L is the length of the boundary and A is the area of minimal surface in 2-dimensional CBA(k)-space.

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On the other hand, Nikolaev ([12]) introduced *area of surface* in 2-dimensional CBA(k)-space and gave the solution of Plateau's problem using his definition.

In this paper, we take Nikolaev's definition for the area of surface and generalize Bishop-Günther inequality in 2-dimensional singular Hadamard space by comparing the area of surface of a parallel set of a geodesic segment to the Lebesgue measure of the corresponding set in $\mathbb{R}^2$.

In what follows, we denote by $\mathcal{A}$ area of surface in CBA(k)-space and by S Lebesgue measure in its model space. Our main result is the following theorem:

THEOREM 1.1. *Let X be a 2-dimensional singular Hadamard space with extendible geodesics, and let $\gamma : [0, \ell] \to X$ be a geodesic segment of given length parametrized by arclength. Let $\bar{\gamma}$ be a straight line segment in $\mathbb{R}^2$ with the same length as γ. Then*

$$\mathcal{A}(T(\gamma, r)) \geq S(T_o(\bar{\gamma}, r)),$$

where $T(\gamma, r)$ is an r-neighborhood of γ in X and $T_o(\bar{\gamma}, r)$ is an r-neighborhood of $\bar{\gamma}$ in $\mathbb{R}^2$.

As an application, we find an upper bound of systole of X(cf. Theorem 2.3). Also, we obtain a lower bound of area of surface of a tube about a convex closed curve in 2-dimensional singular Hadamard spaces and prove that relative growth of area of disc in X with respect to a fixed disc D is faster than that of area of disc in Euclidean plane $\mathbb{R}^2$.

2. Area comparison theorems in 2-dimensional singular Hadamard spaces

Let (X, d) be an arbitrary metric space. The length $L(\gamma)$ of a continuous curve $\gamma : [a, b] \to X$ is defined by

$$L(\gamma) = \sup \sum_{i=1}^{n-1} d(P_i, P_{i+1}),$$

where $P_1, P_2, \ldots, P_n$ is an arbitrary sequence of points of γ numbered in the order of their position on the curve, and the supremum is taken over all such sequence of points. The metric d on X is called intrinsic if for any $p, q \in X$

$$d(p, q) = \inf_{\gamma} L(\gamma),$$

where the infimum is taken over all continuous curves γ joining p and q. A continuous curve $\gamma : [a, b] \to X$ is called a unit speed geodesic in X if each $t \in [a,b]$ has an open neighborhood $V \subset [a,b]$ such that $d(\gamma(t_1),\gamma(t_2)) = |t_1 t_2|$ for all $t_1, t_2 \in V$. If $d(\gamma(t_1),\gamma(t_2)) = |t_1 t_2|$ for all $t_1, t_2 \in [a,b]$, then a geodesic $\gamma : [a,b] \to X$ is called a minimizer joining $\gamma(a)$ and $\gamma(b)$. If every geodesic in X can be extended infinitely in both direction, then X is said to be geodesically complete. A subset $U \subset X$ is said to be convex if any two points in U are joined by a minimizer of X lying inside U. A triangle $\triangle = (\sigma_1, \sigma_2, \sigma_3)$ in a metric space X is a set consisting of three minimizers $\sigma_1, \sigma_2, \sigma_3$ called the sides, which are pairwisely joining three points called the vertices. The perimeter of the triangle is the sum of the lengths of the sides $\sigma_1, \sigma_2, \sigma_3$.

For $k \in \mathbb{R}$, M_k denotes a Lobachevskii plane of curvature k when $k < 0$, a Euclidean plane when $k = 0$ and a sphere of radius $\frac{1}{\sqrt{k}}$ when $k > 0$. Let $r(k) = \infty$ if $k \le 0$, and $r(k) = \frac{1}{\sqrt{k}}$ if $k > 0$. We denote the intrinsic metric on M_k by $|\cdot|_k$. If $\triangle = (\sigma_1, \sigma_2, \sigma_3)$ is a triangle in X, a triangle $\bar{\triangle} = (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3)$ in M_k is called a comparison triangle for $\triangle$ if $L(\bar{\sigma}_i) = L(\sigma_i)$, for $1 \le i \le 3$. A triangle $\triangle$ in a metric space X is said to be k-thin if its perimeter $P(\triangle) < 2\pi\, r(k)$ and it satisfies

$$d(x,y) \le |\bar{x}\bar{y}|_k ,$$

for all points x, y on sides of $\triangle$ and the corresponding points $\bar{x}, \bar{y}$ on the sides of the comparison triangle $\overline{\triangle} \subset M_k$.

DEFINITION 2.1. A locally compact intrinsic metric space X is called a CBA(k)-space if each point $p \in X$ has a convex k-domain R_k, that is, every triangle $\triangle$ of the perimeter $P(\triangle) < 2\pi\, r(k)$ in R_k is k-thin.

DEFINITION 2.2. For $\delta > 0$, let $\gamma_1, \gamma_2 : [0,\delta] \to X$ be a pair of unit speed geodesics emanating from a point p in a complete CBA(k)-space X. For $s, t \in (0, \delta]$ let $\overline{\triangle}_{st} \subset M_k$ be the comparison triangle for the triangle $\triangle_{st} = (p, \gamma_1(s), \gamma_2(t))$. Then the angle between γ_1 and γ_2 is defined by

$$\angle(\gamma_1, \gamma_2) = \lim_{s,t \to 0} \alpha(s,t) ,$$

where $\alpha(s,t)$ is the angle of $\overline{\triangle}_{st}$ at $\overline{p}$ in M_k. Two geodesics are said to be equivalent if the angle between them is zero. The direction space D_pX at p in a complete CBA(k)-space is defined as the set of the equivalence classes of geodesics in X emenating from p.

The following proposition states that a direction space of a topological manifold is complete under angle metric.

PROPOSITION 2.1. *For a point p in a topological manifold X, a geodesic starts out in every direction at p, and the direction space D_pX with the angle metric at p is compact for each $p \in X$.*

Proof. See [2]. □

We are especially interested in complete simply connected spaces of nonpositive curvature, since they have the property that the distance function between geodesics is convex.

DEFINITION 2.3. (X, d) is called a singular Hadamard space if X is a complete simply connected metric space with an intrinsic metric d of nonpositive curvature in the sense of Alexandrov.

Hadamard manifolds, trees with complete intrinsic metrics and Euclidean buildings are the important examples of singular Hadamard space.

We take the definition of area of surface by Nikolaev ([12]). Let us be more concrete. Denoting by D^2 the closed unit disc on a Euclidean plane $\mathbb{R}^2$, we call a continuous mapping of D^2 into a convex k-domain R_k of a CBA(k)-space (X, d) a parametrized surface. A triangulation of an arbitrary polygon inscribed in D^2 is called a triangulation of D^2, and the vertices of the triangles of the triangulation are called the triangulation vertices. Consider a triangulated disc D^2 with a triangulation vertices v_i, $i = 1, 2, \ldots, n$, and a parametrized surface $f : D^2 \to R_k$. To each v_i we correspond a point $\tilde{v}_i \in R_k$ such that points $\tilde{v}_i$ and $\tilde{v}_j$ coincide if and only if $f(v_i)$ and $f(v_j)$ coincide. We join the points $\tilde{v}_i$ with each other by geodesics in the same order as the disc's triangulation vertices are joined to each other. If points $\tilde{v}_i$ and $\tilde{v}_j$ coincide, then we take this point itself as the geodesics. Thus we obtain a collection of triangles in a convex k-domain R_k, and we call it a generalized polyhedron of the parametrized surface f. Also, for a sequence Φ_n of generalized polyhedra, $(\Phi_n) \in \Phi(f)$ means that the maximum of $d(\tilde{v}_i, f(v_i))$ and the largest side of the triangles of the trangulation tend to zero as n increases. The area $\mathcal{A}(\Phi_n)$ of a generalized polyhedron Φ_n is defined as the sum of Lebesgue measures of the comparison triangles on a model surface M_k for the triangles of Φ_n.

DEFINITION 2.4. The area of a surface f is defined by

$$\mathcal{A}(f) = \inf\{\lim \mathcal{A}(\Phi_n)\},$$

where $(\Phi_n) \in \Phi(f)$.

Definition of area of surface in CBA(k)-space is completely analogous to Lebesgue's definition of surface area. Therefore, the properties of area of surface in CBA(k)-space can be proved completely analogously. The proofs of the analogous propositions for surfaces in $\mathbb{R}^3$ can be found in ([6]). Therefore, we present the following proposition without proof.

PROPOSITION 2.2. (*Kolmogorov's Principle*) *If h is a nonexpanding mapping from R_k into R_k, that is, $d(h(x_1), h(x_2)) \leq d(x_1, x_2)$ and f is a surface in R_k, then*

$$\mathcal{A}(h \circ f) \leq \mathcal{A}(f).$$

From now on, a 2-dimensional CBA(k)-space means a 2-dimensional topological manifold endowed with an intrinsic metric of curvature bounded above in the sense of Alexandrov. We define a geodesic hinge $(\gamma_1, \gamma_2, \alpha)$ at a point p in a 2-dimensional CBA(k)-space X as a configuration of two unit speed minimizers $\gamma_i : [0, \ell_i] \to M$, $i = 1, 2$, emanating from a common point p which meet at an angle α. Let r_p be the injectivity radius of $p \in X$, and suppose that a geodesic disc $D(p, r_p)$ is contained in a convex k-domain in X.

THEOREM 2.1. *Let M^2 be a 2-dimensional CBA(k)-space with an intrinsic metric d. Then*

$$\mathcal{A}(D(p, r_p)) \geq S(D_k(\overline{p}, r_p)),$$

where $D(p, r_p)$ is a geodesic disc centered at p of the injectivity radius r_p in M^2 and $S(D_k(\overline{p}, r_p))$ is the surface area of a closed disc $D_k(\overline{p}, r_p)$ centered at $\overline{p} \in M_k$ of radius r_p in the model surface M_k.

Proof. We will define a nonexpanding mapping h from $D(p, r_p)$ onto a ball $D_k(\overline{p}, r_p) \subset M_k$ with $h(p) = \overline{p}$. As a consequence, we obtain the inequality from Kolmogorov's principle, since M_k itself is also a convex k-domain.

Now we choose a point $\overline{q} \in D_k(\overline{p}, r_p)$. Then there exists a unique unit speed minimizer $\overline{\sigma} : [0, r_p] \to D_k(\overline{p}, r_p)$ joining $\overline{p} = \overline{\sigma}(0)$ and $\overline{q} = \overline{\sigma}(|\overline{p}\,\overline{q}|_k)$. We choose a point $q \in D(p, r_p)$ such that $d(p, q) = |\overline{p}\,\overline{q}|_k$. Then there exists a unit speed minimizer $\sigma : [0, r_p] \to D(p, r_p)$ joining $p = \sigma(0)$ and $q = \sigma(d(p, q))$. Since the length of the minimizer σ from $\sigma(0)$ to $\sigma(r_p)$ is equal to r_p, which is the same as the length of the minimizer $\overline{\sigma}$ from $\overline{\sigma}(0)$ to $\overline{\sigma}(r_p)$, we can define a mapping h from $\mathrm{Im}(\sigma)$ onto $\mathrm{Im}(\overline{\sigma})$ equilongally along $\mathrm{Im}(\sigma)$ such that $h(p) = \overline{p}$ and $h(q) = \overline{q}$. Since the total angle H of a point p is at least 2π, if we put $C = \frac{2\pi}{H}$, then C is less than or equal to 1. Let σ^e be the extension of σ passing through the point p. Then the angle $\angle(\sigma, \sigma^e)$ at p is π, since σ is a geodesic.

From now on, we denote $x = (\ell, \alpha)_p$ if $d(p,x) = \ell$ and $\alpha = \angle(\sigma, \gamma)$, where γ is the minimizer passing through p and x. Since the geodesic linking p and x is unique in $D(p, r_p)$, the expression is unique. Similarly, $\bar{x} = (\ell, \beta)_{\bar{p}}$ means that $|\bar{p}\,\bar{x}|_k = \ell$ and $\beta = \angle(\overline{\sigma}, \overline{\gamma})$, where $\overline{\gamma}$ is the minimizer passing through $\overline{p}$ and $\overline{x}$. To each $x = (\ell, \alpha)_p \in D(p, r_p)$, we correspond $\bar{x} = (\ell, C\alpha)_{\bar{p}} \in D_k(\bar{p}, r_p)$, where $|\bar{p}\bar{x}|_k = \ell = d(p,x)$ and $C\alpha$ is the angle between $\overline{\sigma}$ and the minimizer connecting $\overline{p}$ and $\overline{x}$. This define a mapping h from $D(p, r_p)$ into a ball $D_k(\overline{p}, r_p)$ by $h(x) = \overline{x}$. The mapping $h : D(p, r_p) \to D_k(\overline{p}, r_p)$ is clearly onto, since for given $\overline{y} = (\ell, \beta)_{\bar{p}} \in D_k(\overline{p}, r_p)$, there is $y = (\ell, \frac{1}{C}\beta)_p \in D(p, r_p)$ such that $h(y) = \overline{y}$.

Let $x = (\ell_1, \alpha_1)_p, y = (\ell_2, \alpha_2)_p \in D(p, r_p)$. Then there are geodesic hinges $(\sigma, \gamma_1, \alpha_1)$ and $(\sigma, \gamma_2, \alpha_2)$ at $p \in M$ such that $\gamma_1(\ell_1) = x, \gamma_2(\ell_2) = y$, where $\alpha_i = \angle(\sigma, \gamma_i), i = 1, 2$. Let $(\overline{\sigma}, \overline{\gamma}_1, C\alpha_1)$ and $(\overline{\sigma}, \overline{\gamma}_2, C\alpha_2)$ be the corresponding geodesic hinges in M_k such that $\overline{\gamma}_1(\ell_1) = \bar{x}, \overline{\gamma}_2(\ell_2) = \bar{y}$. Note that $\angle(\gamma_1, \gamma_2) \geq |\alpha_2 - \alpha_1|$, and so we have

$$\angle(\overline{\gamma}_1, \overline{\gamma}_2) = C|\alpha_2 - \alpha_1| \leq |\alpha_2 - \alpha_1| \leq \angle(\gamma_1, \gamma_2)\,.$$

Hence we obtain the desired result, i.e.,

$$d(x,y) \geq |h(x)\,h(y)|_k.$$

This implies that the mapping h from $D(p, r_p)$ onto $D_k(\overline{p}, r_p)$ is nonexpanding. This completes the proof. □

We will generalize Theorem 2.1 to the tube area in a 2-dimensional singular Hadamard space. From now on, we assume that X is a 2-dimensional singular Hadamard space with extendible geodesics, that is, a complete simply connected 2-dimensional topological manifold with extendible geodesics with an intrinsic metric d of nonpositive curvature in the sense of Alexandrov. By Hadamard-Cartan theorem, singular Hadamard spaces are strongly convex. We note that for any given geodesics $\gamma_1, \gamma_2 : I \to X$ in a singular Hadamard space X, the function $f : I \to \mathbb{R}$ defined by $f(t) = d(\gamma_1(t)\,, \gamma_2(t))$ is convex in t. Also, it is well-known that for a convex subset C in X, the function $d_C : X \to \mathbb{R}$ defined by $d_C(z) = d(z, C)$ is convex.

The following lemma generalizes the case of manifolds of nonpositive curvature to a singular Hadamard space.

LEMMA 2.1. *If $f : C \to \mathbb{R}$ is a convex function on a (strongly) convex subset C in a singular Hadamard space (X, d), then for any $s \in \mathbb{R}$, the subset $f_{\leq s}$ defined by*

$$f_{\leq s} \equiv \{p \in C : f(p) \leq s\}$$

is (strongly) convex in C. Moreover, any geodesic disc and any r-neighborhood of a geodesic segment in X are always (strongly) convex in X, respectively.

Proof. Let $p, q \in f_{\leq s}$ and let $\gamma : [a, b] \to C$ be the geodesic segment from p to q. By convexity of f on C,

$$f(\gamma(t)) = (f \circ \gamma)(t) \leq s\,,$$

since $f \circ \gamma(a) = f(p) \leq s$ and $f \circ \gamma(b) = f(q) \leq s$. So, γ is in $f_{\leq s}$. If we choose $C = \{p\}$ for $p \in X$, then $d_p : X \to \mathbb{R}$ defined by $d_p(z) = d(p, z)$ is convex. So, since $d_p : X \to \mathbb{R}$ is a convex function, a geodesic disc $D(p, s) = \{x \in X : d_p(x) \leq s\}$ is convex. By similar arguments, an r-neighborhood $T(\gamma, r)$ for a geodesic segment γ in X is convex in X, since γ itself is a convex set in X. □

Let C be a closed convex subset of a 2-dimensional singular Hadamard space X. Then for each $x \in X$ there exists a unique footpoint $p_C(x) \in C$ of x on C such that $d(x, p_C(x)) = d(x, C)$.

PROPOSITION 2.3. *Let C be a closed convex subset in a 2-dimensional singular Hadamard space X and $p_C(x) \in C$, the footpoint of $x \in X$. Then the metric projection $p_C : X \to C$ is a 1-Lipschitz retraction, and for the projective geodesic segment $x\,p_C(x)$ and a geodesic segment $p_C(x)\,y$ contained in C, we have*

$$\angle(p_C(x)x, p_C(x)y) \geq \frac{\pi}{2}\,,$$

for any $y \in C$.

Proof. See [3]. □

The following theorem compares the area of tube about a nontrivial geodesic segment of given length in singular Hadamard space with the area of tube about line segment in $\mathbb{R}^2$ with the same length.

THEOREM 2.2. *Let X be a 2-dimensional singular Hadamard space with extendible geodesics, and let $\gamma : [0, \ell] \to X$ be a nontrivial geodesic segment of given length ℓ parametrized by arclength. Let $\bar{\gamma}$ be a straight line segment in $\mathbb{R}^2$ with the length. Then*

$$\mathcal{A}(T(\gamma, r)) \geq S(T_o(\bar{\gamma}, r)),$$

where $T(\gamma, r)$ is an r-neighborhood of γ in X and $T_o(\bar{\gamma}, r)$ is an r-neighborhood of $\bar{\gamma}$ in $\mathbb{R}^2$.

Proof. We shall identify the map γ with its image $\gamma([0,\ell]) \subset X$. Note that r-neighborhood $T(\gamma, r)$ of γ defined as the set $\{x \in X : d(x,\gamma) \leq r\}$ is convex in X from Lemma 2.1. Without loss of generality, we can set $\bar{\gamma} = \{(x,0) \in \mathbb{R}^2 : x \in [0,\ell]\}$, and hence the r-neighborhood $T_o(\bar{\gamma}, r)$ of $\bar{\gamma}$ in $\mathbb{R}^2$ is the union of two half discs $D_L((0,0),r), D_R((\ell,0),r)$ and a rectangle $[0,\ell] \times [-r,r]$ in $\mathbb{R}^2$, where $D_L((0,0),r)$ means the intersection of the ball $D((0,0),r)$ and the left half plane of $\mathbb{R}^2$. Similarly $D_R((\ell,0),r)$ means the intersection of the ball $D((\ell,0),r)$ and the right half plane $\{(x,y) \in \mathbb{R}^2 : x \geq \ell\}$ of $\mathbb{R}^2$.

Now we divide $T(\gamma,r)$ by four parts T_U, T_D, T_L, T_R by using the uniqueness of the footpoint on γ of a point $x \in T(\gamma, r)$ as follows : Let $\gamma(0) = p$ and $\gamma(\ell) = q$ be the endpoints of the geodesic γ. For each $x \in T(\gamma,r)$, denote the footpoint of x on $\gamma([0,\ell])$ by $p_\gamma(x)$. Then at first we denote the set of points having $\gamma(0)$ as its footpoint by T_L, this is,

$$T_L \equiv \{x \in T(\gamma,r) : p_\gamma(x) = \gamma(0)\}.$$

Similarly we put

$$T_R \equiv \{x \in T(\gamma,r) : p_\gamma(x) = \gamma(\ell)\}.$$

They are disjoint, and hence $T(\gamma,r) - (T_L \cup T_R)$ is the tube $\{x \in T(\gamma,r) : p_\gamma(x) \in \gamma((0,\ell))\}$ of γ such that its interior is homeomorphic to an open disc in $\mathbb{R}^2$. Since this set is the union of two subsets which is divided by the geodesic segment $\gamma([0,\ell])$, we can denote, without loss of generality, the one by T_U and the other by T_D. Since the geodesic itself is convex, they overlap only on the geodesic segment $\gamma([0,\ell])$, and the intersection $T_U \cap T_D = \gamma([0,\ell])$ has measure 0.

At first, we define a nonexpanding mapping h_U from T_U to the rectangle $R_r^\ell \equiv [0,\ell] \times [0,r]$ in $\mathbb{R}^2$ as follows : for $x \in T_U$, there exists a unique $p_\gamma(x) \in \gamma((0,\ell))$. We correspond $x \in T_U$ to $h_U(x) \in R_r^\ell \subset \mathbb{R}^2$ by

$$h_U(x) = (\, d(\gamma(0), p_\gamma(x))\,,\, d(x, p_\gamma(x))\,).$$

We will show that $h_U : T_U \to R_r^\ell$ is a nonexpanding mapping: If x, y are two points in T_U such that $x\,p_\gamma(x) = y\,p_\gamma(y)$, then since the segment $x\,p_\gamma(x)$ joining x and $p_\gamma(x)$ is a geodesic, there exist two geodesics γ_1 and γ_2 in T_U such that $x\,p_\gamma(x)$ and $y\,p_\gamma(y)$ are contained in geodesics γ_1 and γ_2 respectively. Since the metric projection $p_\gamma : T_U \to \gamma$ is contractive, we have

$$d(x,y) \geq d(p_\gamma(x)\,,\, p_\gamma(y)\,) = |h_U(p_\gamma(x)).h_U(p_\gamma(y))| = |p_\gamma(x)\, p_\gamma(y)|\,,$$

where $|\cdot|$ means the Euclidean standard metric in $\mathbb{R}^2$. For $x, y \in T_U$ in general position, the inequality holds, since the function f defined by $f(t) = d(\gamma_1(t), \gamma_2(t))$ is convex in t.

Hence, by the Kolmogorov's principle(Proposition 2.2), we obtain that

$$\mathcal{A}(T_U) \geq S(R_r^\ell).$$

By the same argument as above, $h_V : T_V \to R_r^\ell$ defined by

$$h_V(x) = (d(p, p_\gamma(x)), d(x, p_\gamma(x)))$$

for $x \in T_V$, is nonexpanding, and so the area of surface of T_V is more than or equal to the Lebesgue measure of R_r^ℓ. Now we claim that

> the total angle of the direction space D_pT_L of $p \in T_L$ is not less than π.

Since the geodesic segment $\gamma([0,\ell])$ is a closed convex subset in X, the angle between the geodesic segments $x\,p_C(x)$ and $\gamma([0,\ell])$ is not less than $\frac{\pi}{2}$ for any $x \in T_U \cup T_D$ by Proposition 2.3. For sufficiently small $\varepsilon > 0$, consider the extended geodesic $\tilde{\gamma} : [-\varepsilon, \ell] \to X$ passing through $\gamma(0)$ of the geodesic $\gamma : [0,\ell] \to X$. Then there exist two geodesic segments z_1p and z_2p in T_L with length r such that $\gamma(0) = p_{\tilde{\gamma}}(z_1) = p_{\tilde{\gamma}}(z_2)$, for some $z_1, z_2 \in \partial T(\gamma, r) \cap T_L$. Since the geodesic segments z_1p and z_2p make the angle $\alpha \geq \frac{\pi}{2}$ with the extended geodesic $\tilde{\gamma}$, the angle $\angle(z_1p, z_2p)$ between z_1p and z_2p in T_L is greater than or equal to π. Hence we prove the claim.

Theorem 2.1 with the claim implies $\mathcal{A}(T_L) \geq \frac{1}{2}S(D((0,0),r))$. By the same argument, we obtain that $\mathcal{A}(T_R) \geq \frac{1}{2}S(D((0,0),r))$. Therefore, area of surface of the r-neighborhood $T(\gamma, r)$ of γ is greater than or equal to Lebesgue measure of the racetrack $T_o(\bar{\gamma}, r)$ in $\mathbb{R}^2$. This completes the proof of Theorem 2.2. □

The systole of a CBA(k)-space X, denoted by $sys(X)$ is, by definition, the infimum of lengths of closed curves in X which are not homotopic to zero. Related with $sys(X)$ to the area of a space X, there are several well-known inequalities ([4], [5], [7], [9]). Now, as an application of Theorem 2.2, we obtain the following inequality.

THEOREM 2.3. *Let X be a 2-dimensional nonsimply connected geodesically complete metric space of nonpositive curvature. Suppose that γ is a shortest closed geodesic in X with the property that the $\frac{1}{2}\ell(\gamma)$-neighborhood $N_\gamma \subset X$ of γ is homeomorphic to a cylinder, where $\ell(\gamma)$*

denotes the length of γ*. Then we have*

$$\mathcal{A}(X) \geq sys^2(X).$$

Proof. It is obvious that, for any point $p \in \gamma$, the open metric ball $B(p, \ell(\gamma)/2)$ does not contain γ. We claim that

any point $q \in B(p, \ell(\gamma)/2)$ is joined to p by the unique minimizer.

Suppose that two points p and q are joined by two minimizers σ_1 and σ_2 in $B(p, \ell(\gamma)/2)$. Since a minimizer is a convex subset of X, any point a in σ_1 has the unique footpoint $p_{\sigma_2}(a)$ in σ_2. If $p_{\sigma_2}(a)$ is either p or q for all $a \in \sigma_1$, then the concatenation $\sigma_1 \star \sigma_2^{-1}$ is a closed geodesic with length less than $\ell(\gamma)$. This is contrary to the assumption. On the other hand, if $p_{\sigma_2}(a)$ is neither p nor q for some $a \in \sigma_1$, then the angle $\angle(\overline{a\,p_{\sigma_2}(a)}, \overline{p\,p_{\sigma_2}(a)})$ between two geodesics $\overline{a\,p_{\sigma_2}(a)}$ and $\overline{p\,p_{\sigma_2}(a)}$ is not less than $\pi/2$ by Proposition 2.3. Also, we have $\angle(\overline{a\,p_{\sigma_2}(a)}, \overline{q\,p_{\sigma_2}(a)}) \geq \pi/2$. This implies that either the triangle $\triangle\, p\, a\, p_{\sigma_2}(a)$ or the triangle $\triangle\, q\, a\, p_{\sigma_2}(a)$ has an interior angle more than π, which is a contradiction. Hence, we prove the claim.

The claim implies that $B(p, \ell(\gamma)/2)$ is contractible. Hence, by Hadamard-Cartan theorem, the ball $B(p, \ell(\gamma)/2)$ does not contain any closed geodesic, and any two points in $B(p, \ell(\gamma)/2)$ is joined by the unique minimizer. This implies that the injectivity radius of X is half of $\ell(\gamma)$. Consider the half segments $\gamma^+ \equiv \{q \in \gamma : \gamma(0) < q < \gamma(\frac{\ell(\gamma)}{2})\}$ and $\gamma^- \equiv \{q \in \gamma : \gamma(\frac{\ell(\gamma)}{2}) < q < \gamma(\ell(\gamma)) = \gamma(0)\}$ of γ. Then $\frac{1}{2}\ell(\gamma)$-neighborhood $N_\gamma \subset X$ of γ is the union of the following four subsets

$$T_+ \equiv \{x \in N_\gamma : p_\gamma(x) = \gamma^+\}\ ,\ T_- \equiv \{x \in N_\gamma : p_\gamma(x) = \gamma^-\},$$

$$T_0 \equiv \{x \in N_\gamma : p_\gamma(x) = \gamma(0)\}\ ,\ T_{1/2} \equiv \{x \in N_\gamma : p_\gamma(x) = \gamma(\frac{\ell(\gamma)}{2})\}.$$

Note that they are pairwisely disjoint, from the assumption that N_γ is homeomorphic to a cylinder. Using the same argument as in the Theorem 2.2, the areas of T_+ and T_- are not less than $\frac{1}{2}\ell^2(\gamma)$, respectively. Therefore,

$$\mathcal{A}(N_\gamma) \geq \mathcal{A}(N_{\gamma^-}) + \mathcal{A}(N_{\gamma^+}) \geq \ell^2(\gamma) = sys^2(X),$$

and we obtain the desired result. □

3. A lower bound of a tube in 2-dimensional singular Hadamard spaces

As an application of Theorem 2.2, we obtain a generalized Steiner's formula in 2-dimensional singular Hadamard space with extendible geodesics. Consider a triangle $T = \triangle ABC$ in X with geodesic sides AB, BC, CA, and denote the exterior r-neighborhood of T by $N_r^e(T)$. And let ξ be the geodesic emanating from the vertex A such that the footpoint $p_T(x)$ of each point $x \in \xi$ is A, and for any point z in the region in $N_r^e(T)$ bounded by the geodesic segments AB and ξ, $p_T(z) \neq A$. Similarly, let ζ be the geodesic emanating from A such that the footpoint $p_T(x)$ of each point $x \in \zeta$ is A, and for any point y in the region in $N_r^e(T)$ bounded by the geodesic segments ζ and AC, $p_T(y) \neq A$. We call the angle $\angle(\xi, \zeta)$ the outer angle at the vertex A, which is the exactly same as the exterior angle in Euclidean plane.

LEMMA 3.1. *With ξ, ζ, AB and AC as above, we have*

$$\angle(\xi, AB) = \frac{\pi}{2} \text{ and } \angle(\zeta, AC) = \frac{\pi}{2}.$$

Proof. Since ξ is a projective geodesic, $\angle(\xi, AB) \geq \frac{\pi}{2}$ by Proposition 2.3. Suppose $\angle(\xi, AB) > \frac{\pi}{2}$. Then by Proposition 2.1 there exists a geodesic ρ between AB and ξ such that $\angle(\rho, AB) = \frac{\pi}{2}$, and $p_\gamma(w) \neq A$ for any $w \in \rho$ from assumption. Let $q = p_\gamma(w) \in AB$. Then the interior angle of the triangle $\triangle = (w, q, A)$ is more than π. By comparing triangle $\triangle = (w, q, A)$ to its comparison triangle in $\mathbb{R}^2$, we get a contradiction. Hence we have $\angle(\xi, AB) = \frac{\pi}{2}$. By the same argument, $\angle(\zeta, AC) = \frac{\pi}{2}$. □

THEOREM 3.1. *Consider a triangle $T = \triangle ABC$ with geodesic sides AB, BC and CA in a 2-dimensional singular Hadamard space X. Then, area of surface of the exterior r-neighborhood of T is more than or equal to Lebesgue measure of the corresponding exterior r-neighborhood of a comparison triangle T^0 in $\mathbb{R}^2$ with the same side lengths.*

Proof. We devide the exterior r-neighborhood $N_r^e(T)$ of the triangle $T = \triangle ABC$ in X by using the foot point of each point in $N_r^e(T)$ by pairwise disjoint six regions $D_{AB}, D_{BC}, D_{CA}, D_A, D_B, D_C$, where D_I means the set $\{x : p_T(x) \in I\}$. Denote their corresponding set in $\mathbb{R}^2$ by D_I^0. Using the same argument as in the Theorem 2.2, we have

$$\mathcal{A}(D_I) \geq S(D_I^0),$$

for geodesic segments $I = AB, BC, CA$ of the triangle T.

On the other hand, to compare another parts we denote the interior angles of the vertices A, B and C by a, b and c, respectively. Also we denote the outer angles of the vertices A, B and C by f, g and h, respectively. Then $a+b+c \leq \pi$, and $a+f+b+g+c+h \geq 3\pi$ by Lemma 3.1. Hence we have $f+g+h \geq 2\pi$. Therefore, we have

$$\mathcal{A}(D_A \cup D_B \cup D_C) \geq S(D_A^0 \cup D_B^0 \cup D_C^0),$$

since the total angle of a triangle in $\mathbb{R}^2$ is 2π. □

From the same argument as given in the proof of the previous theorem, we obtain the following general result for convex polygons in a 2-dimensional singular Hadamard space.

COROLLARY 3.1. *For any convex n-gon G_n in a 2-dimensional singular Hadamard space X, area of surface of the exterior r-neighborhood of G_n is not less than Lebesgue measure of the corresponding exterior r-neighborhood of a comparison n-gon G_n^0 in $\mathbb{R}^2$ with the same side lengths as the convex n-gon G_n.*

Proof. Since the sum of all interior angles at vertices of an n-gon G_n in X is less than $(n-2)\pi$, the sum of all outer angles at vertices is more than 2π by Lemma 3.1. Hence, we obtain the consequence by the same argument in the proof of Theorem 3.1. □

THEOREM 3.2. *Let Γ be a convex rectifiable closed curve in a 2-dimensional singular Hadamard space X. Then area of surface of the exterior r-neighborhood of Γ is greater than or equal to Lebesgue measure of the exterior r-neighborhood of a circle in $\mathbb{R}^2$ with the same perimeter as the length of Γ.*

Proof. Choose three points A, B, C on Γ such that the lengths of arcs $AB_\Gamma, BC_\Gamma, AC_\Gamma$ along the curve Γ is equal to $\frac{\ell(\Gamma)}{3}$. Then we make a triangle $T = \triangle ABC$ by connecting the points A, B, C by geodesic segments contained in Γ. Now we construct a sequence of polygons which are contained in Γ as follows: For arcs $AB_\Gamma, BC_\Gamma, AC_\Gamma$, we pick points D, E, F, as like $D \in AB_\Gamma, E \in BC_\Gamma, F \in AC_\Gamma$, such that

$$\ell_\Gamma(AD_\Gamma) = \ell_\Gamma(DB_\Gamma) = \frac{1}{2}\ell_\Gamma(AB_\Gamma) = \frac{1}{6}\ell(\Gamma).$$

Then we connect A and B with D, respectively, by shortest geodesics. By doing this procedure for arcs BC_Γ, AC_Γ, we obtain a hexagon $P_1 = ADBECF$ containing the triangle $T = \triangle ABC$ which is contained in Γ. Now we repeat the same process as above for six subarcs $AD_\Gamma, DB_\Gamma, BE_\Gamma, EC_\Gamma, CF_\Gamma, FA_\Gamma$. Then we obtain a 12-gon P_2 which is contained in Γ, and containing the hexagon $ADBECF$. Also, each subarc has the same arclength $\frac{1}{12}\ell(\Gamma)$.

By continuing this procedure, we obtain a sequence of polygons $\{P_n\}_{n=1}^{\infty}$ inscribed in Γ that gradually approach to Γ, and we denote $P_n \to \Gamma$. Note that P_n is a $3 \cdot 2^n$-gon, and if $P_n \to \Gamma$, then the sequential limit of area of surfaces of the convex bodies bounded by the convex polygons P_n is equal to area of surface of the region bounded by Γ, and $\lim_{n\to\infty} Length(P_n) = Length(\Gamma)$. For each convex polygon P inscribed in Γ, we correspond the circle C_P with perimeter $\ell(P)$ in $\mathbb{R}^2$.

By the construction given above, we have a sequence of convex polygons $\{P_n\}$ with $3 \cdot 2^n$ edges that are inscribed in Γ such that $\ell(P_n) \leq \ell(P_{n+1})$ and $P_n \to \Gamma$. Since $P_n \to \Gamma$, we have

$$\lim \mathcal{A}(N_r^e(P_n)) = \mathcal{A}(N_r^e(\Gamma)).$$

Since $\mathcal{A}(N_r^e(P_n)) \geq S(N_r^e(C_{P_n}))$ by Corollary 3.1, we obtain

$$\mathcal{A}(N_r^e(\Gamma)) \geq S(N_r^e(C_\Gamma)),$$

where C_Γ is the circle of perimeter $\ell(\Gamma)$. □

COROLLARY 3.2. *The relative area growth of balls in a 2-dimensional singular Hadamard space X with respect to a fixed disc D is faster than that in Euclidean plane $\mathbb{R}^2$.*

Proof. Consider a geodesic disc D which is bounded by a boundary curve ∂D of circumference L, and a disc D^0 with radius R in a Euclidean plane of length $\partial D^0 = L$. Then by the isoperimetric inequality in CBA(k)-space([2]), $\mathcal{A}(D) \leq S(D^0)$. Also by the generalized Steiner's tube formula above, $\mathcal{A}(N_r^e(\partial D)) \geq S(N_r^e(\partial D^0))$. This implies that

$$\frac{\mathcal{A}(N_r^e(\partial D))}{\mathcal{A}(D)} \geq \frac{S(N_r^e(\partial D^0))}{S(D^0)}.$$

Let B be the geodesic ball $B = D \cup N_r^e(\partial D)$, and let B^0 be the ball $B^0 = D^0 \cup N_r^e(\partial D^0)$ with radius $R + r$. Then $\mathcal{A}(B) = \mathcal{A}(D) + \mathcal{A}(N_r^e(\partial D))$. Hence

$$\begin{aligned} \frac{\mathcal{A}(B)}{\mathcal{A}(D)} &= \frac{\mathcal{A}(D) + \mathcal{A}(N_r^e(\partial D))}{\mathcal{A}(D)} \\ &\geq 1 + \frac{S(N_r^e(\partial D^0))}{S(D^0)} = \frac{S(B^0)}{S(D^0)}. \end{aligned}$$

□

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