Commun. Korean Math. Soc. **21** (2006), No. 4, pp. 787–794

SUBORDINATION, SELF-DECOMPOSABILITY AND SEMI-STABILITY

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ABSTRACT. Two main results are presented in relation to subordination, self-decomposability and semi-stability. One of the result is that strict semi-stability of subordinand process by self-decomposable subordinator gives semi-selfdecomposability of the subordinated process. The second result is a sufficient condition for any subordinated process arising from a semi-stable subordinand and a semi-stable subordinator to be semi- selfdecomposable.

1. Introduction and results

Denote the class of distributions (probability measures) on R^d and the class of all infinitely divisible distributions on R^d by $\mathcal{P}(R^d)$ and $I(R^d)$, respectively. $\mathcal{L}(X)$ is the distribution of a random variable or vector X. The characteristic function of $\mu\in \mathcal{P}(R^d)$ is denoted by $\hat{\mu}(z)$, $z \in R^d$.

A distribution $\mu\in \mathcal{P}(R^d)$ is called *self-decomposable* if, for every $b > 1$, there exists a $\mu_b \in \mathcal{P}(R^d)$ satisfying

$$\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\mu_b(z), \qquad z \in R^d(z). \tag{1.1}$$

The class of all self-decomposable distributions is denoted by $L(R^d)$. It is called *semi-selfdecomposable,* if there are some $b > 1$ and some μ_b $\in I(R^d)$ satisfying (1.1). This notion is introduced and characterized in Maejima and Naito [5] as an extension of self-decomposability on one hand and semi-stability on the other. Many distributions are known to be semi-selfdecomposable and their importance has been increasing in

Received January 3, 2006.

2000 Mathematics Subject Classification: Primary 60G51; Secondary 60E07.

Key words and phrases: Lévy processes, subordination, stability, self-decomposability, semi-stability, semi-selfdecomposability.

This work was supported by the Korea Research Foundation Grant (KRF-2002-070-C00013).

mathematical physics. See Maejima and Naito [5] and Sato [7]. The class of all semi-selfdecomposable distributions satisfying (1.1) is denoted by $L(b, R^d)$.

A distribution $\mu \in \mathcal{P}(R^d)$ is called *semi-stable* if, for some $a > 1$, there are $b > 0$ and $c \in R^d$ satisfying

$$\hat{\mu}(z)^a = \hat{\mu}(bz)e^{i<c,z>}, \quad z \in R^d(z). \tag{1.2}$$

It is called *strictly semi-stable* if c can be taken to be 0. If μ is nontrivial semi-stable, then b and c in the above definition are uniquely determined by a, and there is α such that $b^\alpha = a$. The class of all semi-stable distributions satisfying (1.2) and the class of all strictly semi-stable distributions satisfying (1.2) with $c = 0$ are denoted by $S(b, \alpha)$ and $S_0(b, \alpha)$, respectively. A distribution μ is called (b, α)*-semi-stable* (or α*-semi-stable* having a *span* $b > 0$) or *strictly* (b, α)*-semi-stable* (or *strictly* α*-semi-stable* having a *span* $b > 0$) if $\mu \in S(b, \alpha)$ or $\mu \in S_0(b, \alpha)$, respectively.

A distribution μ is called α*-stable* or *strictly* α*-stable* if, for any $b > 0$, $\mu \in S(b, \alpha)$ or $\mu \in S_0(b, \alpha)$, respectively. A Lévy process $\{Z(t)\}$ is called *self-decomposable, semi-selfdecomposable, strictly* α*-stable,* α*-stable, strictly* (b, α)*-semi-stable* or (b, α)*-semi-stable* if $\mathcal{L}(Z(1))$ is self-decomposable, semi-selfdecomposable, strictly α -stable, α-stable, strictly (b, α)-semi-stable or (b, α)-semi-stable, respectively.

A subordinator is an increasing Lévy process on R. Let $\{T(t)\}$ be a subordinator on R and $\{X(t)\}$ be a Lévy process on R^d. Subordination is a transformation of $\{X(t)\}$ to a new process $\{Y(t)\}$ defined by composition as $Y(t) = X(T(t))$through random time change by $\{T(t)\}$, where $\{X(t)\}$ and $\{T(t)\}$ are assumed to be independent. It is extensively studied in Sato [7, Chapter 6]. Its importance has been increasing in mathematical finance. See Barndorff-Nielsen, et.al [1] and Barndorff-Nielsen and Shephard [2]. An interesting problem is what properties of $\{X(t)\}$ and $\{T(t)\}$ give certain desirable properties of $\{Y(t)\}$. In this paper, we treat this problem in association with self-decomposability, semi-stability and stability.

Our basic results are the following two theorems.

THEOREM 1.1. *Let* $\{X(t)\}$ *be strictly semi-stable on* R^d. *If* $\{T(t)\}$ *is a self- decomposable subordinator on* R, *then the subordinated Lévy process* $\{Y(t)\}$ *is semi-selfdecomposable on* R^d.

THEOREM 1.2. *Let* $\{X(t)\}$ *be strictly* (b_1, α_1)*-semi-stable on* R^d *and let* $\{T(t)\}$ *be a* (b_2, α_2)*-semi-stable subordinator on* R *with* $\alpha_1 \log b_1/$

$\log b_2 \in Q$, the set of rational numbers. Then the subordinated Lévy process $\{Y(t)\}$ is semi- selfdecomposable on R^d.

It is known that if subordinand $\{X(t)\}$ is strictly stable and subordinator $\{T(t)\}$ is self-decomposable, then the subordinated process $\{Y(t)\}$ is self-decomposable. This is studied by Sato [8] and Bondesson [3]. A multivariate generalization of this fact is given in Theorem 6.1 of Barndorff-Nielsen et al. [1]. In Theorem 1.1, we consider this fact to the case where the subordinand $\{X(t)\}$ is a strictly semi-stable process. Proofs of Theorems 1.1 and 1.2 are given in Sections 2 and 3.

If $\{X(t)\}$ is strictly α- stable and subordinator $\{T(t)\}$ is strictly β-stable, then, as is well-know, the subordinated process $\{Y(t)\}$ is strictly $\alpha\beta$-stable. It is an interesting problem to see whether this can be generalized to semi-stable case. Proposition 3.1 is the answer to this problem under the additional assumption in Theorem 1.2, which is treated in Section 3. Theorem 1.2 is an extension of Proposition 3.1 to the case where the subordinand $\{X(t)\}$ is strictly semi-stable and subordinator $\{T(t)\}$ is semi-stable. In the case when $\alpha_1 \log b_1 / \log b_2$ is an irrational number, we do not know whether Theorem 1.2 holds.

2. Proof of Theorem 1.1

Any $\mu \in I(R^d)$ has the Lévy representation (A, ν, γ), which means that A is a symmetric nonnegative-definite operator on R^d, ν is a measure on R^d satisfying $\nu(\{0\}) = 0$ and $\int_{R^d - \{0\}} (|x|^2 \wedge 1)\nu(dx) < \infty$, and $\gamma \in R^d$ such that

$$\hat{\mu}(z) = \exp[i < \gamma, z > -\frac{1}{2} < Az, z > + \int_{R^d} G(z, x)\nu(dx)]$$

for $z \in R^d$, where $G(z, x) = \mathrm{e}^{i<z,x>} - 1 - i < z, x > I_{\{|x| \leq 1\}}$. These A, ν, and γ are uniquely determined by μ, A is called the Gaussian variance and ν is called the Lévy measure of μ. Here $I_D(x)$ is the indicator function of D. A Lévy process $\{Z(t)\}$ is said to have the Lévy representation (A, ν, γ) if $\mathcal{L}(Z(1))$ has the Lévy representation (A, ν, γ).

A distribution $\mu \in \mathcal{P}(R^d)$ with (A, ν, γ) belong to $L(R^d)$ if and only if the Lévy measure ν has the expression

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty \frac{k_\xi(u)}{u} I_B(u\xi)du, \quad B \in \mathcal{B}(R^d - \{0\}), \tag{2.1}$$

where $\mathcal{B}(R^d - \{0\})$ is the class of Borel subsets of $R^d - \{0\}$, $S = \{\xi \in R^d : |\xi| = 1\}$, λ is a finite measure on S, and $k_\xi(u)$ is measurable in ξ, decreasing in u, and nonnegative. We are using the words increase and decrease in the weak sense. The condition (2.1) is equivalent to saying that, for every $B \in \mathcal{B}(R^d)$ and $b > 1$,

$$\nu(B) - \nu(bB) \geq 0.$$

See Sato [7] for review on self-decomposable distributions.

It is a necessary and sufficient condition for $\mu \in \mathcal{P}(R^d)$ with (A, ν, γ) to be in $L(b, R^d)$ for some $b > 1$ that ν is expressed as

$$\nu(B) - \nu(bB) \geq 0 \qquad \text{for some } b > 1, \tag{2.2}$$

which is given in Sato [7] and Maejima and Naito [5].

It is well-known (see Theorem 30.1 of Sato [7]) that $\{T(t)\}$ is a subordinator with the Lévy representation (A, ρ, β) and $\kappa = \mathcal{L}(T(1))$ if and only if

$$\hat{\kappa}(z) = \exp\left[\int_{(0,\infty)} (e^{izs} - 1)\rho(ds) + i\beta_0 z\right] \tag{2.3}$$

with $\beta_0 = \beta - \int_{(0,1]} s\rho(ds) \geq 0$. Let $\kappa^t = \mathcal{L}(T(t))$. We note that (2.3) is equivalent to

$$\mathrm{E}[\mathrm{e}^{-uT(t)}] = \int_{[0,\infty)} \mathrm{e}^{-us}\kappa^t(ds) = e^{t\Psi(-u)}, \qquad u \geq 0,$$

where, for any complex w with $\mathrm{Re}w \leq 0$,

$$\Psi(w) = \beta_0 w + \int_{(0,\infty)} (\mathrm{e}^{ws} - 1)\,\rho(ds). \tag{2.4}$$

Let $\{X(t)\}$ be a Lévy process with the Lévy presentation (A, ν, γ) on R^d and $\{T(t)\}$ be a subordinator with the Lévy representation (A, ρ, β) on R. Then, from Theorem 30.1 of Sato [7], the subordinated process $\{Y(t)\} = \{X(T(t))\}$ is a Lévy process with the Lévy representation $(A^\sharp, \nu^\sharp, \gamma^\sharp)$ on R^d such that

$$A^\sharp = \beta_0 A,$$

$$\nu^\sharp(B) = \beta_0 \nu(B) + \int_{(0,\infty)} \mu^s(B)\rho(ds), \quad B \in \mathcal{B}(R^d - \{0\}), \tag{2.5}$$

$$\gamma^\sharp = \beta_0 + \int_{(0,\infty)} \rho(ds) \int_{|x|\le 1} x\mu^s(dx),$$

where $\mu^s = \mathcal{L}(X(s))$.

PROOF OF THEOREM 1.1. Let $\{X(t)\}$, $\{T(t)\}$, and $\{Y(t)\}$ be as above. By (2.5), we see that, for some $b > 1$,

$$\nu^\sharp(bB) = \beta_0 \nu(bB) + \int_{(0,\infty)} \mu^s(bB)\rho(ds), \quad B \in \mathcal{B}(R^d - \{0\}).$$

Let $\{X(t)\}$ be strictly (b,α)-semi-stable with $b > 1$. Then, we have that, for $B \in \mathcal{B}(R^d - \{0\})$,

$$\mu^s(bB) = \mu^{sb^{-\alpha}}(B),$$

which is equivalent to (1.2) with $\gamma = 0$. Suppose that $\{T(t)\}$ is self-decomposable. Then (2.1) shows that

$$\rho(B) = C \int_0^\infty \frac{k_1(u)}{u} I_B(u)du, \quad B \in \mathcal{B}(R(0,\infty)),$$

where $C = \lambda(1)$, $k_1(u)$ is decreasing in u and nonnegative. Thus, we have that

$$\begin{aligned} \int_{(0,\infty)} \mu^s(bB)\rho(ds) &= C \int_{(0,\infty)} \mu^{sb^{-\alpha}}(B) k_1(s) s^{-1} ds \\ &= C \int_{(0,\infty)} \mu^s(B) k_1(sb^\alpha) s^{-1} ds, \end{aligned}$$

which leads to saying that, for some $b > 1$, $\nu^\sharp(B) \ge \nu^\sharp(bB)$ from the facts that $\nu(bB) = b^{-\alpha}\nu(B)$ and $k_1(sb^\alpha) \le k_1(s)$. By (2.2), this means $\{Y(t)\}$ is semi-selfdecomposable. □

3. Proof of Theorem 1.2

Let $\{X(t)\}$, $\{T(t)\}$, and $\{Y(t)\}$ be as in section 1. Let γ, ρ, β_0, (A,ν,γ), $(A^\sharp,\nu^\sharp,\gamma^\sharp)$ and μ^s be as in section 2. We use C for constant.

PROOF OF THEOREM 1.2. Let $\{X(t)\}$, $\{T(t)\}$, and $\{Y(t)\}$ be processes in Theorem 1.2. Then there exist some positive integers M and N such that $b_1^{M\alpha_1} = b_2^N$. By equation (2.3) and Proposition 2.3 of Choi [4] (see also Proposition 14.5.3 of Sato [7] and Theorem 3.2.2 of Ramachandran and Lau [6]), the Lévy measure ρ of $\{T(t)\}$ is written as

$$\rho(ds) = d\{\frac{-h(s)}{s^{\alpha_2}}\}, \qquad 0 < \alpha_2 < 1,$$

where $h(s)$ is nonnegative, right-continuous in s and $h(s)s^{-\alpha_2}$ is decreasing in s, $h(b_2 s) = h(s)$ and $h(1) = C$. Using this and the fact that $\mu^s(b_1^M B) = \mu^{sb_1^{-M\alpha_1}}(B)$, we see that

$$\begin{aligned}\int_{(0,\infty)} \mu^s(b_1^M B)\rho(ds) &= \int_{(0,\infty)} \mu^{sb_1^{-M\alpha_1}}(B)d\{\frac{-h(s)}{s^{\alpha_2}}\}\\ &= b_2^{-N\alpha_2}\int_{(0,\infty)} \mu^s(B)d\{\frac{-h(s)}{s^{\alpha_2}}\}\end{aligned}$$

for $B \in \mathcal{B}(R^d - \{0\})$. This leads to

$$\begin{aligned}\nu^\sharp(b_1^M B) &= \beta_0 b_1^{-M\alpha_1}\nu(B)\\ &\quad + b_2^{-N\alpha_2}\int_{(0,\infty)} \mu^s(B)\rho(ds), \quad B \in \mathcal{B}(R^d - \{0\}),\end{aligned}$$

where we recall that ν is the Lévy measure of $\mu = \mathcal{L}(X(1))$. Noticing that $\nu(b_1^M B) = b_1^{-M\alpha_1}\nu(B)$. Thus we have that $\nu^\sharp(B) \geq \nu^\sharp(b_1^M B)$ for $B \in \mathcal{B}(R^d - \{0\})$. This says that $\{Y(t)\}$ is semi-selfdecomposable by (2.2). □

PROPOSITION 3.1. *Let $\{X(t)\}$ be strictly (b_1, α_1)-semi-stable and let $\{T(t)\}$ be a strictly (b_2, α_2)-semi-stable subordinator. If $\alpha_1 \log b_1/\log b_2$ is a rational number, then the subordinated Lévy process $\{Y(t)\}$ is strictly $\alpha_1\alpha_2$-semi-stable.*

PROOF. Let $\{X(t)\}$, $\{T(t)\}$, and $\{Y(t)\}$ be the processes in Proposition 3.1. Then we have that

$$E[e^{i<z,Y(t)>}] = \exp(t\Psi(\log\hat{\mu}(z))), \quad z \in R^d,$$

where $\mu = \mathcal{L}(X(1))$ and Ψ defined by (2.4). This is shown in Theorem 30.1 of Sato [7]. Let M and N be as in the proof of Theorem 1.2. Then, we have that

$$b_2^{\alpha_2}\Psi(w) = \Psi(b_2 w)$$

for $w \in C$ with $\mathrm{Re}\, w \le 0$. Using the fact that, for some b_1, $\hat{\mu}(z)^{b_1{}^{\alpha_1}} = \hat{\mu}(b_1 z)$, we get that

$$\begin{aligned} b_2^{N\alpha_2}\Psi(\log\hat{\mu}) &= \Psi(b_2^N \log\hat{\mu}(z)) \\ &= \Psi(b_1^{M\alpha_1}\log\hat{\mu}(z)) \\ &= \Psi(\log\hat{\mu}(b_1^M z)). \end{aligned}$$

This shows that $\{Y(t)\}$ is strictly $\alpha_1\alpha_2$- semi-stable having a span b_1^M.□

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