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An Interacting Wave Profile of Three Trains of Gravity Waves on Finite Depth by Contraction Method

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ABSTRACT: *Superposition of three wave trains on finite depth is investigated. This paper is focused on how to improve the linear superposition of three waves. This was realized by introducing the scheme. The idea of the scheme is based on a fixed point approach. Application of the scheme to the superposition makes it possible to obtain a wave profile of wave-wave interaction. With the help of FFT, it was possible to analyze high-order nonlinear frequencies for three interacting Stokes' waves on finite depth.*

1. Introduction

The wave interaction theory has been studied by other researchers. Longuett-Higgins (1963) began research on deepwater problems. Dalzell (1999) employed the symbolic computation to neaten the complicated looking second-order wave-wave interaction coefficients. Oscillatory third-order perturbation solutions of sums of interacting Stokes wave on deep water was shown by Pierson (1993). The treatment of this nonlinear wave-wave interaction problem is based on the perturbation theory. Because the perturbation theory has a solid mathematical foundation, it can be a powerful tool for solving problems. However, the derivation of solutions becomes very lengthy.

A many of the applications of nonlinear mechanics in the field of engineering and science may be based directly on fixed-point methods (Zeidler, 1986; Deimling, 1985). One example of the application of the fixed point theorem to the wave problem is given in Bona and Bose (Bona et al., 1974). They examined the question of the existence of solitary wave solutions to simple one-dimensional models for long waves in nonlinear dispersive systems. Recently, Jang and Kwon (2005) have proposed about a scheme, which utilizes fixed-point theory to calculate the nonlinear wave profiles (Jang and Kwon, 2005; Jang, 2005b; Jang, 2005c). In this paper, based on the scheme, wave profiles for more than wave-wave progressive waves on finite depth is investigated. Only a few studies have been done on nonlinear interacting two Stokes' waves. Pierson (1993) has studied about perturbation solutions for sums of interacting Stokes' waves in deep water. However, the derivation of high-order solutions gets too lengthy to proceed, even when three Stokes' waves are considered to be summed.

In this paper, a numerical study on the interacting wave profiles is illustrated. In addition, using FFT, it was possible to analyze high-order nonlinear frequencies for the interaction. A fixed point approach was introduced in section 2. The Bernoulli's operator for wave interaction as well as the contraction coefficient was investigated in section 3. Numerical calculations for a various cases of wave slopes were carried out in Section 4. FFT analysis was also given.

2. Fixed Point Approach to Wave Profile

The fluid is assumed to be homogeneous, incompressible and inviscid. In addition, the fluid motion is irrotational, such that a velocity potential function exists. Suppose that we consider a free surface flow. A Cartesian coordinate system (x, y, z) is adopted, with $z=0$ the plane of the undisturbed free surface and the z-axis positive upwards. The vertical elevation of any point on the free surface may be defined by a function $z=\eta(x, y, t)$. The surface tension being negligible, then, Bernoulli's equation applied on the free surface is

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$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}\nabla\Phi\cdot\nabla\Phi+\frac{P_a}{\rho}+g\eta=f(t) \quad (1)$$

where Φ, P_a and ρ stand for the velocity potential, the pressure of the atmosphere, and the constant fluid density, respectively. Taking Bernoulli's constant $f(t)=P_a/\rho$, we have the expression for the free surface:

$$\eta=-\frac{1}{g}\left[\frac{\partial\Phi}{\partial t}+\frac{1}{2}\nabla\Phi\cdot\nabla\Phi\right]\bigg|_{z=\eta} \quad (2)$$

The right-hand side of (2) may be viewed as an operator for the free surface η in such a way that we can define an operator B, which is the Bernoulli's operator (Jang, T.S., Kwon, S.H. and Kinoshita, T., 2005):

$$B(\eta)=-\frac{1}{g}\left[\frac{\partial\Phi}{\partial t}+\frac{1}{2}\nabla\Phi\cdot\nabla\Phi\right]\bigg|_{z=\eta} \quad (3)$$

Then Bernoulli's operator B can be easily conformed to be nonlinear, and (2) can be simply written as

$$\eta=B(\eta) \quad (4)$$

If the operator B satisfies the following inequality

$$|B(\eta_1)-B(\eta_2)|_\infty\le\beta|\eta_1-\eta_2|_\infty,\ \beta<1 \quad (5)$$

then B is considereda contraction and the fixed point is realized as the limit of the following sequence

$$\eta_{k+1}=B(\eta_k) \quad (6)$$

with the initial condition for zero function $\eta_0=0$ (Roman, P., 1975; Jang and Kwon, 2005):

3. Superposition

We begin with three linear progressive wave potentials on finite depth with different wave numbers $k_i>0$ for $i=1,2,3$ and consider their linear sum Φ_{sum}:

$$\Phi_{sum}=\frac{a_1 g}{\omega_1}\frac{\cosh k_1(z+h)}{\cosh k_1 h}\sin\Theta_1+\frac{a_2 g}{\omega_2}\frac{\cosh k_2(z+h)}{\cosh k_2 h}\sin\Theta_2+\frac{a_3 g}{\omega_3}\frac{\cosh k_3(z+h)}{\cosh k_3 h}\sin\Theta_3 \quad (7)$$

where g, a_i and $\omega_i (i=1,2,3)$ represent the gravity acceleration, the wave amplitude and frequency, respectively. The product $k_i a_i$ the wave slope, is assumed small for $i=1,2,3$. The symbol Θ_i $(i=1,2,3)$ denotes the phase functions of progressive waves, that is, $k_i x-\omega_i t$ for $i=1,2,3$. The linear dispersion relation on finite depth is assumed:

$$k_i \tanh k_i h=\frac{\omega_i^2}{g} \text{ for } i=1,2,3 \quad (8)$$

If the linear sum (7) is substituted into (6), we obtain the following iteration for free surface η

$$\eta_{k+1}=-\frac{1}{g}\left[\frac{\partial\Phi_{sum}}{\partial t}+\frac{1}{2}\nabla\Phi_{sum}\cdot\nabla\Phi_{sum}\right]\bigg|_{z=\eta_k} \quad (9)$$

If wave condition is satisfied with the norm inequality (5), then the iteration scheme (9) works and yields wave profiles of an interacting Stokes' waves on finite depth (Jang, T.S., Kwon, S.H. and Kinoshita, T., 2005).

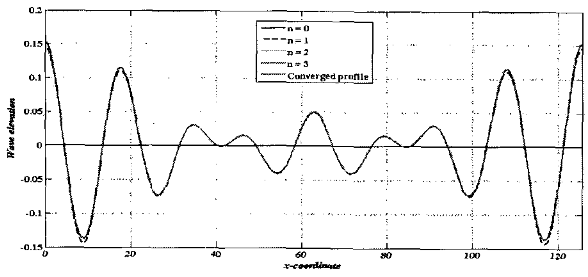
4. Numerical Results

In this section, we will present the numerical results of wave profiles of three interacting Stokes' waves. For solutions of the wave profile, (9) is iterated with an initial condition for zero function of mean water level (for *n = 0), that is,* $\eta_0=0$. For numerical study, we examine four different wave profiles. Their wave information is tabulated in Table 1.

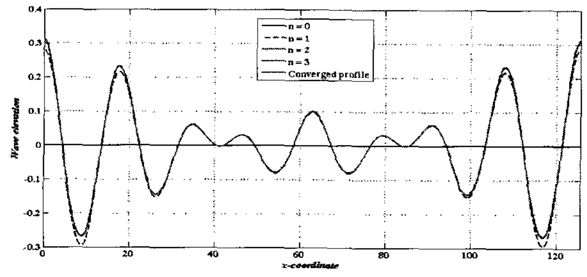
Table 1 Various Wave-Parameters investigated

Case No	k_1	a_1	k_2	a_2	k_3	a_3	h
Case 1	0.3	0.05	0.35	0.05	0.4	0.05	5
Case 2	0.3	0.1	0.35	0.1	0.4	0.1	5
Case 3	0.3	0.2	0.35	0.2	0.4	0.2	5
Case 4	0.3	0.3	0.35	0.3	0.4	0.3	5

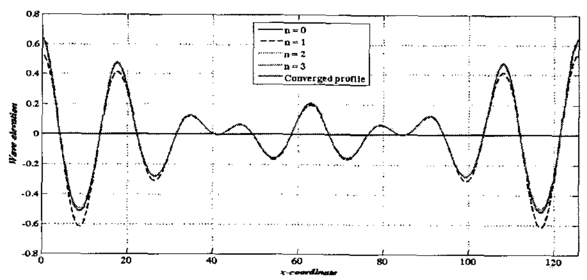
Case 1 represents a mild slope and case 4 the other extreme.



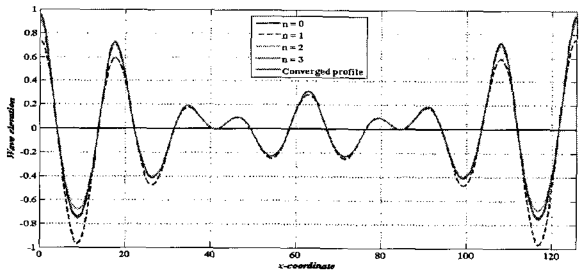
(a) Case 1



(b) Case 2

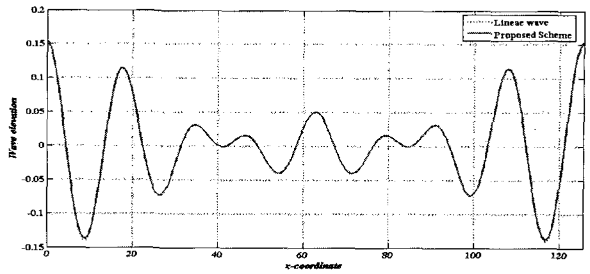


(c) Case 3

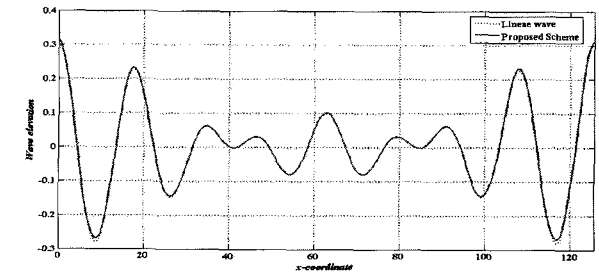


(d) Case 4

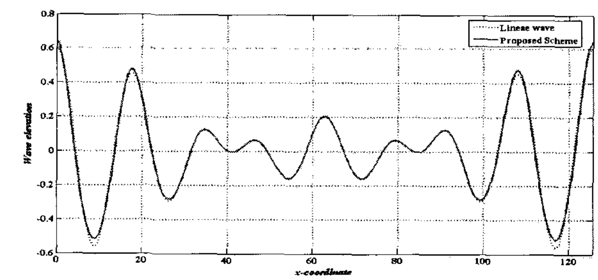
Fig. 1 Convergence behavior of η_n



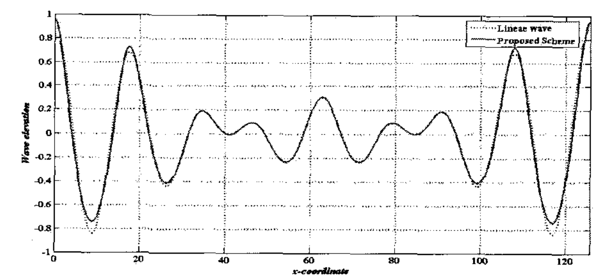
(a) Case 1



(b) Case 2



(c) Case 3



(d) Case 4

Fig. 2 Comparison of wave profiles

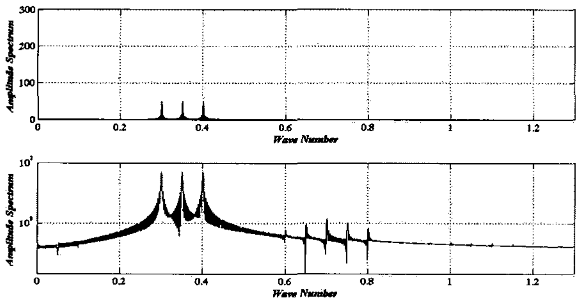
In Fig. 1 the process by which the wave profiles change is shown. The change is shown in terms of the number of iterations.

In Fig. 2, the obtained converged solutions are compared with the corresponding linear wave profiles. The figure shows an excellent agreement between the results obtained by the proposed scheme and the linear ones for small wave slope. However, some nonlinear features of wave profile or wave interaction can be observed for a relatively large wave slope.

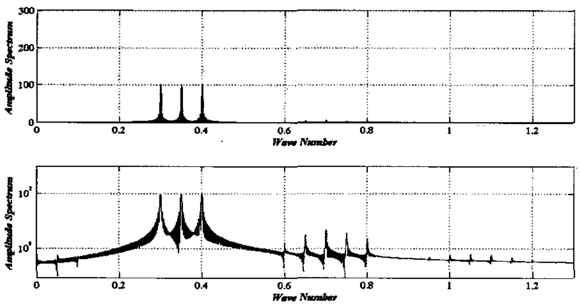
To examine the nonlinear behavior of the solution, the Fourier transform was introduced. The amplitude spectra of the solutions are presented in Fig. 3. To highlight the peaks in the amplitude spectrum, linear-log coordinates are also illustrated. For the four cases, three dominant peaks at k_1, k_2 and k_3 appear clear as is expected. They are the fundamental wave number components in this study.

In the figure, we can see the peaks of the proposed scheme due to the double wave number components at $2k_1$

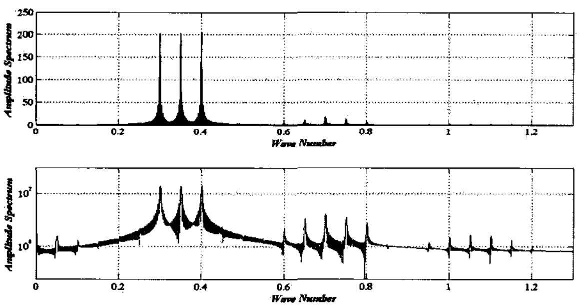
$(k=0.6)$, $2k_2(k=0.7)$ and $2k_3(k=0.8)$, the sum wave numbers at $k_1+k_2(k=0.65)$, $k_1+k_3(k=0.7)$ and k_2+k_3 $(k=0.8)$ even though their magnitudes are small compared to those of the fundamental wave number components.



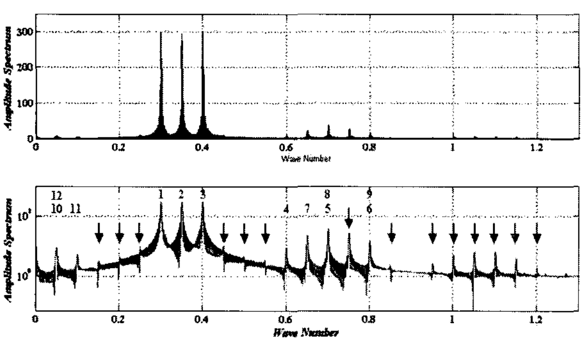
(a) Case 1



(b) Case 2



(c) Case 3



(d) Case 4

Fig. 3 Comparison of wave amplitude spectra

However, in Fig. 3(a), it may be hard to observe the peaks due to the difference wave numbers at k_2-k_1 $(k=0.05)$, $k_3-k_1(k=0.1)$ and $k_3-k_2(k=0.5)$ in the amplitude spectrum: the hidden peaks are recovered when the wave slopes are relatively larger cases as shown in Fig. 3 (b), (c) and (d). The interacting components are illustrated in Table 2. The numbers corresponding to the frequency components as shown in Table 2 are written in Fig. 4(d).

An interesting phenomenon was observed: the proposed scheme yields peaks at higher frequencies, as shown in red-colored arrows in Fig. 3(d). Their existence cannot be explained by the Pierson's (1999) perturbation solution of second order. They may be corresponding to higher order nonlinear frequencies for three interacting Stokes' waves, which should be investigated further.

Table 2 Peak frequencies in amplitude spectra

Indicated No	Frequency Components	Numerical Value of Frequency
1	k_1	0.3
2	k_2	0.35
3	k_3	0.4
4	$2k_1$	0.6
5	$2k_2$	0.7
6	$2k_3$	0.8
7	k_1+k_2	0.65
8	k_1+k_3	0.7
9	k_2+k_3	0.8
10	k_2-k_1	0.05
11	k_3-k_1	0.1
12	k_3-k_2	0.05

5. Conclusions

By applying Banach fixed-point theorem to Bernoulli's Equation, we have proposed a nonlinear iterative scheme to realize an interacting wave profile for three Stokes' waves on finite depth. The formulation and process of the computation involved are very handy even though three Stokes' wave interactions are taken into account. This is a completely different point of view when compared to the perturbation approach of Pierson (1993). It is interesting that the iteration, based on linear progressive potential solutions, enabled us to observe the higher-order nonlinear frequencies for three interacting Stokes' waves that Pierson's second order solution could not predict.

Acknowlegement

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