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ON PREHERMITIAN OPERATORS

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ABSTRACT. In this paper, we are concerned with the algebraic representation of the quasi-nilpotent part for prehermitian operators on Banach spaces. The quasi-nilpotent part of an operator plays a significant role in the spectral theory and Fredholm theory of operators on Banach spaces. Properties of the quasi-nilpotent part are investigated and an application is given to totally paranormal and prehermitian operators.

1. Introduction

Throughout this note, let X be a Banach space over the complex plane $\mathbb{C}$ and let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on X. For a given $T \in \mathcal{L}(X)$, let $\sigma(T)$ and $\rho(T)$ denote the spectrum and the resolvent set of T, respectively. An operator $T \in \mathcal{L}(X)$ is said to be *prehermitian* if

$$\sup_{t \in \mathbb{R}} \| \exp(itT) \| < \infty,$$

where $\mathbb{R}$ is the set of real numbers. An operator $T \in \mathcal{L}(X)$ is called *normal-equivalent* if there are two commuting prehermitian operators $A, B \in \mathcal{L}(X)$ such that $T = A + iB$. In [9], Lumer showed that an operator $T \in \mathcal{L}(X)$ is prehermitian if and only if there is an equivalent algebra norm $\| \cdot \|_1$ on $\mathcal{L}(X)$ such that T is a hermitian element of $\mathcal{L}(X)$ endowed with this norm. Also, it is known that $T \in \mathcal{L}(X)$ is normal-equivalent if and only if there is an equivalent algebra norm $\| \cdot \|_1$ on $\mathcal{L}(X)$ such that T is a normal element of $\mathcal{L}(X)$ endowed with this norm.

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We denote by $C^\infty(\mathbb{C})$ the Fréchet algebra of all infinitely differentiable complex valued functions defined on the complex plane $\mathbb{C}$ with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{C}$. An operator $T \in \mathcal{L}(X)$ is called a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty(\mathbb{C}) \to \mathcal{L}(X)$ satisfying $\Phi(1) = I$, the identity operator on X, and $\Phi(z) = T$, where z denotes the identity function on $\mathbb{C}$. Such a continuous homomorphism Φ is in fact an operator valued distribution and it is called a spectral distribution for T. An operator $T \in \mathcal{L}(X)$ is said to be of *class* $C^2(\mathbb{C})$ if there exists a continuous algebra homomorphism $\Phi : C^2(\mathbb{C}) \to \mathcal{L}(X)$ satisfying $\Phi(1) = I$ and $\Phi(z) = T$. Such Φ is then called a continuous functional calculus for T. It is clear from Lemma 3.5 of [2] that normal-equivalent operators are of class $C^2(\mathbb{C})$. Conversely, if $T \in \mathcal{L}(X)$ is of class $C^2(\mathbb{C})$ then T is normal-equivalent. For, if $\Phi : C^2(\mathbb{C}) \to \mathcal{L}(X)$ is a continuous functional calculus for T then $A := \Phi(\mathrm{Re}(\cdot))$ and $B := \Phi(\mathrm{Im}(\cdot))$ are commuting operators of class $C^2(\mathbb{C})$ and hence prehermitian.

An operator $T \in \mathcal{L}(X)$ is called *decomposable* if for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$ there exists a pair of T-invariant closed linear subspaces Y and Z of X such that

$$Y + Z = X, \qquad \sigma(T|Y) \subseteq U \quad \text{and} \quad \sigma(T|Z) \subseteq V,$$

where $T|Y$ denotes the restriction operator of T on Y. Although decomposable operators generally have no functional calculus of Riesz, these operators possess many of the spectral properties of normal operators. It is known that if T is a prehermitian operator then T is a generalized scalar operator and hence decomposable [8].

The *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ which satisfies $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$. The *local spectrum*

$$\sigma_T(x) := \mathbb{C} \setminus \rho_T(x).$$

Clearly, the local resolvent set $\rho_T(x)$ is open, and the local spectrum $\sigma_T(x)$ is closed. For each $x \in X$, the function $f(\lambda) : \rho(T) \to X$ defined by $f(\lambda) = (T - \lambda)^{-1}x$ is analytic on $\rho(T)$ and satisfies $(T - \lambda)f(\lambda) = x$ for all $\lambda \in \rho(T)$. Hence the resolvent set $\rho(T)$ is always subset of $\rho_T(x)$ and hence $\sigma_T(x)$ is always subset of $\sigma(T)$. The analytic solutions occurring in the definition of the local resolvent set may be thought of as local extensions of the function

$$(T-\lambda)^{-1}x : \rho(T) \to X.$$

There is no uniqueness implied. Thus we need the following definition.

An operator $T \in L(X)$ is said to have the *single-valued extension property*, abbreviated SVEP, if for every open set $U \subseteq \mathbb{C}$, the only analytic solution $f : U \to X$ of the equation $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U. Hence if T has the SVEP, then for each $x \in X$ there is the maximal analytic extension of $(T-\lambda)^{-1}x$ on $\rho_T(x)$.

Given an operator $T \in \mathcal{L}(X)$ and an element $x \in X$,

$$r_T(x) := \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}}$$

is called the *local spectral radius* of T at x. It is clear that

$$\max\{|\lambda| : \lambda \in \sigma_T(x)\} \le r_T(x)$$

for all $x \in X$ but for operators without the SVEP, this inequality may well be strict. It is well known that if T has the SVEP and $x \in X$ is a non-zero element, then the compact set $\sigma_T(x)$ is non empty and the local spectral radius formula

$$r_T(x) = \max\{\, |\lambda| \, : \, \lambda \in \sigma_T(x) \,\}$$

holds and the spectral radius

$$r(T) = \max\{r_T(x) \, : \, x \in X\}$$

by Proposition 1.2.16, Proposition 3.3.13 and Proposition 3.3.14 of [8].

Let $F \subseteq \mathbb{C}$, the *analytic spectral subspace* $X_T(F)$ of $T \in \mathcal{L}(X)$ is defined by

$$X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}.$$

It is easy to see that $X_T(F)$ is a hyperinvariant linear subspace of X but need not be closed. An operator $T \in \mathcal{L}(X)$ is said to have *Dunford's property (C)* if $X_T(F)$ is closed for every closed $F \subseteq \mathbb{C}$. This condition plays an important role in the theory of spectral operators. It is well known that Dunford's property (C) implies the SVEP. Let $\ker T$ denote the kernel of T. Then it is easy to see that

$$\ker(T-\lambda) \subseteq X_T(\{\lambda\})$$

for all $\lambda \in \mathbb{C}$.

For a $F \subseteq \mathbb{C}$, consider the class of all linear subspaces Z of X which satisfy $(T-\lambda)Z = Z$ for all $\lambda \notin F$ and let $E_T(F)$ denote the span of all such subspaces Z of X. $E_T(F)$ is called an *algebraic spectral subspace* of T. It is clear that

$$(T-\lambda)E_T(F) = E_T(F) \quad \text{for all} \quad \lambda \notin F$$

as well so that it is the largest linear subspace with this property.

Given an operator $T \in \mathcal{L}(X)$, the *quasi-nilpotent part* of T is the set

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

It is clear that $H_0(T)$ is a linear subspace of X and in fact hyperinvariant subspace of T. In general, $H_0(T)$ is not closed. It follows from Theorem 1.5 of [12] that T is quasi-nilpotent if and only if $H_0(T) = X$. Moreover, if T is invertible then $H_0(T) = \{0\}$. The systematic investigation of the space $H_0(T)$ was initiated by Mbekhta [10] after an earlier work of Vrbová [12]. As shown by Mbekhta, quasi-nilpotent part of an operator play a significant role in the local spectral and Fredholm theory of operators on Banach spaces.

2. Main results

PROPOSITION 1. *Let T be a prehermitian operator on a Banach space X and $x_0 \in X$. Then $\lim_{n\to\infty} \|T^n x_0\|^{\frac{1}{n}} = 0$ if and only if $Tx_0 = 0$. And $\ker T$ is the quasi-nilpotent part of T. Moreover,*

$$H_0(T) = X_T(\{0\}) = E_T(\{0\}) = \ker T = \{x \in X : r_T(x) = 0\}.$$

PROOF. It is easy to see that

$$\ker T \subseteq X_T(\{0\}).$$

Since prehermitian operators are of class $C^2(\mathbb{C})$, T is a generalized scalar operator. From Theorem 2.7 of [14], we have

$$X_T(\{0\}) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Since $X_T(\{0\})$ is a closed hyperinvariant subspace of T,

$$\sigma(T|X_T(\{0\})) = \sigma(T) \cap \{0\}.$$

It follows that $T|X_T(\{0\})$ is quasinilpotent and prehermitian. As there is an equivalent algebra norm on $\mathcal{L}(X_T(\{0\}))$ such that $T|X_T(\{0\})$ is a hermitian element in $\mathcal{L}(X_T(\{0\}))$ endowed with this norm. By Theorem 10.17 of [3] we have

$$T|X_T(\{0\}) = 0.$$

Hence

$$X_T(\{0\}) \subseteq \ker T.$$

Since T is a generalized scalar operator,

$$H_0(T) = \{x \in X : r_T(x) = 0\}$$

by Proposition 3.3.17 of [8]. This completes the proof. □

A bounded linear operator T on a Banach space X is said to be *totally paranormal* if

$$\|(T-\lambda)x\|^2 \leq \|(T-\lambda)^2 x\| \, \|x\|$$

for all $x \in X$. It is well known that if T is totally paranormal then

$$\ker (T-\lambda) = \ker (T-\lambda)^2$$

for all $\lambda \in \mathbb{C}$ and so T has the SVEP. Also it is clear that every hyponormal operator on a Hilbert space H is totally paranormal.

THEOREM 2. *Let T be a totally paranormal operator on a Banach space X and $x_0 \in X$. Then $\lim_{n\to\infty} \|T^n x_0\|^{\frac{1}{n}} = 0$ if and only if $Tx_0 = 0$. Moreover,*

$$X_T(\{\lambda\}) = \ker (T-\lambda) = H_0(T-\lambda)$$

for all $\lambda \in \mathbb{C}$. In particular, $\ker T$ is the quasi-nilpotent part of T.

PROOF. Let $x \in X$ be a unit vector. Since T is totally paranormal,

$$\|(T-\lambda)x\|^2 \leq \|(T-\lambda)^2 x\| \, \|x\|.$$

For any $n = 2, 3, \dots$, we obtain

$$\begin{aligned} \|(T-\lambda)^n x\|^2 &= \|(T-\lambda)(T-\lambda)^{n-1}x\|^2 \\ &\leq \|(T-\lambda)^2(T-\lambda)^{n-1}x\| \, \|(T-\lambda)^{n-1}x\| \\ &= \|(T-\lambda)^{n+1}x\| \, \|(T-\lambda)^{n-1}x\|. \end{aligned}$$

It follows from Lemma 1.2 of [4] that

$$\|(T-\lambda)x\|^n \leq \|(T-\lambda)^n x\|$$

for any unit vector $x \in X$ and $n \in \mathbb{N}$. Hence, if $\lim_{n\to\infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0$ then $(T-\lambda)x = 0$. Since T has the SVEP, we have

$$\begin{aligned} X_T(\{\lambda\}) &= X_{T-\lambda}(\{0\}) \\ &= \{x \in X : \lim_{n\to\infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\} \\ &= H_0(T-\lambda). \end{aligned}$$

Therefore we have

$$H_0(T-\lambda) = X_{T-\lambda}(\{0\}) \subseteq \ker(T-\lambda)$$

for all $\lambda \in \mathbb{C}$. This completes the proof. □

Let X and Y be complex Banach spaces over the complex field $\mathbb{C}$ and let $\mathcal{L}(X,Y)$ denote the space of all continuous linear operators from X to Y. For given operator $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, we consider the corresponding commutator $C(S,T) : \mathcal{L}(X,Y) \to \mathcal{L}(X,Y)$ defined by

$$C(S,T)(A) := SA - AT \quad \text{for all} \quad A \in \mathcal{L}(X,Y).$$

For a $n \in \mathbb{N}$, the set of all natural numbers, define $C(S,T)^n$ to be the n-th composition of the map $C(S,T)$. That is,

$$\begin{aligned} C(S,T)^n(A) &= C(S,T)^{n-1}(SA - AT) \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k S^{n-k} A T^k. \end{aligned}$$

Also we introduce the operators L_S, $R_T \in \mathcal{L}(\mathcal{L}(X,Y))$ by

$$L_S(A) := SA \quad \text{and} \quad R_T(A) := AT$$

for all $A \in \mathcal{L}(X,Y)$. An operator $A \in \mathcal{L}(X,Y)$ is said to *intertwine S and T asymptotically* if

$$\|C(S,T)^n(A)\|^{\frac{1}{n}} \to 0 \quad \text{as} \quad n \to \infty.$$

This condition has been investigated by Colojoară and C. Foiaş [5]. It is clear that an operator $A \in \mathcal{L}(X,Y)$ intertwines S and T asymptotically if and only if its adjoint $A^* \in \mathcal{L}(Y^*, X^*)$ intertwines T^* and S^* asymptotically. Two operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are said to be *asymptotically similar* if there exists a bijection $A \in \mathcal{L}(X,Y)$ such that A intertwines S and T asymptotically and its inverse A^{-1} intertwines T and S asymptotically. In fact, asymptotic similarity generalizes slightly the notion of quasinilpotent equivalence, denoted by $T \sim^q S$, where $X = Y$ and $A = I$ is the identity operator on X in the definition of asymptotically similarity [5].

COROLLARY 3. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be prehermitian operators. Assume that $A \in \mathcal{L}(X,Y)$ intertwines S and T asymptotically. Then $SA = AT$ and $AH_0(T) \subseteq H_0(S)$.*

PROOF. Since L_S and R_T are commuting prehermitian operators, the commutator $C(S,T) = L_S - R_T$ is also a prehermitian operator by Lemma 4.19 of [6]. By Proposition 1, we have

$$A \in H_0(C(S,T)) = \ker C(S,T).$$

Hence $SA = AT$. Let $y = Ax$ and $x \in H_0(T)$. Then, by Proposition 1,

$$\begin{aligned} Sy &= SAx \\ &= ATx \\ &= 0 \end{aligned}$$

and hence

$$y \in \ker S = H_0(S).$$

This completes the proof. □

An operator $A \in \mathcal{L}(X,Y)$ is said to be *quasi-affinity* if A is injective and has dense range. If the operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are intertwined asymptotically by quasi-affinities $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$, that is A and B are quasi-affinities for which

$$\|C(S,T)^n(A)\|^{\frac{1}{n}} \to 0 \quad \text{and} \quad \|C(T,S)^n(B)\|^{\frac{1}{n}} \to 0 \quad \text{as} \quad n \to \infty,$$

then we say that T and S are *asymptotically quasi-similar*. Two operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are called *quasi-similar* if there exist quasi-affinities $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$ so that $AT = SA$ and $TB = BS$.

An operator $T \in \mathcal{L}(X)$ is said to be a *Fredholm operator* if $\ker T$ and X/TX are both of finite dimension. The *essential spectrum* $\sigma_e(T)$ of $T \in \mathcal{L}(X)$ is defined by

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a Fredholm operator}\}.$$

It is clear that

$$\sigma_e(T) \subseteq \sigma(T)$$

and will be empty when X is finite dimensional.

COROLLARY 4. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be prehermitian operators and let T and S be asymptotically quasi-similar. Then T and S are quasi-similar. Moreover, $\sigma(T) = \sigma(S)$ and $\sigma_e(T) = \sigma_e(S)$.*

PROOF. Assume that $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are intertwined asymptotically by quasi-affinities $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$. Then, by Proposition 1, we have

$$A \in \ker C(S,T) \quad \text{and} \quad B \in \ker C(T,S)$$

and hence T and S are quasi-similar. The final assertion is a consequence of Theorem 3.5 of [7] and the main Theorem of [11]. □

COROLLARY 5. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be prehermitian operators and let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be asymptotically similar. Then T and S are similar. Moreover, there exists an invertible operator $A \in \mathcal{L}(X,Y)$ for which $AH_0(T) = H_0(S)$ and $\sigma_T(x) = \sigma_S(x)$ for all $x \in X$.*

PROOF. If T and S are asymptotically similar then there exists an invertible operator $A \in \mathcal{L}(X, Y)$ such that A intertwines S and T asymptotically and its inverse A^{-1} intertwines T and S asymptotically. Then, by Proposition 1, we have

$$A \in \ker C(S,T) \quad \text{and} \quad A^{-1} \in \ker C(T,S),$$

and hence T and S are similar. The final assertion is a consequence of Proposition 1.2.17 of [8]. □

COROLLARY 6. *Let T, $S \in \mathcal{L}(X)$ be prehermitian operators. If T is quasinilpotent equivalent to S then $S = T$.*

THEOREM 7. *Let $T \in \mathcal{L}(X)$ be prehermitian operator and let $S \in \mathcal{L}(Y)$ be a normal-equivalent such that $S = A + iB$ for some prehermitian operators A, $B \in \mathcal{L}(Y)$ with $AB = BA$. Assume that T and S are asymptotically similar. Then there exists an invertible operator $L \in \mathcal{L}(X, Y)$ such that*

$$LH_0(T) \subseteq \mathit{ker} A \cap \mathit{ker} B \subseteq H_0(A) \cap H_0(B).$$

PROOF. It is clear that

$$\ker A \cap \ker B \subseteq H_0(A) \cap H_0(B).$$

Assume that T and S are asymptotically similar and choose a corresponding bijection $L \in \mathcal{L}(X, Y)$ for the asymptotic intertwining of (S, T) and (T, S). Then there is a continuous algebra homomorphism $\Phi : \mathrm{C}^2(\mathbb{C}) \to \mathcal{L}(X)$ with $\Phi(1) = I$, $\Phi(\mathrm{Re}(\cdot)) = A$ and $\Phi(\mathrm{Im}(\cdot)) = B$. For a $x \in H_0(T)$, let $y = Lx$. Then by Corollary 3 we have

$$\begin{aligned} y \in LX_T(\{0\}) &= Y_A(\{0\}) \\ &\subseteq Y_A(Re^{-1}(\{0\})) \\ &= Y_A(\{0\}). \end{aligned}$$

Hence

$$\lim_{n\to\infty} \|A^n y\|^{\frac{1}{n}} = 0.$$

Thus by Proposition 1, $Ay = 0$ and hence $y \in \ker A$. In the same way we obtain $y \in \ker B$ and hence

$$LH_0(T) \subseteq \ker A \cap \ker B.$$

This completes the proof. □

COROLLARY 8. *Let* $S \in \mathcal{L}(Y)$ *be a normal-equivalent such that* $S = A + iB$ *for some prehermitian operators* A, $B \in \mathcal{L}(Y)$ *with* $AB = BA$. *Assume that* $T \in \mathcal{L}(X)$ *is quasinilpotent equivalent to* S. *Then* $H_0(T) = H_0(S)$ *and* $H_0(S) \subseteq \ker A \cap \ker B \subseteq H_0(A) \cap H_0(B)$.

PROOF. It is clear that if $\lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0$ then $Ax = Bx = 0$. □

THEOREM 9. *Let* $S_k = A_k + iB_k$ *be normal-equivalent operators for some prehermitian operators* A_k, $B_k \in \mathcal{L}(Y)$ *for* $k = 1, 2$. *And let* $T_1 \in \mathcal{L}(X)$, $T_2 \in \mathcal{L}(Y)$, $T_1 \sim^q S_1$ *and* $T_2 \sim^q S_2$. *Assume that* $A \in \mathcal{L}(X, Y)$ *intertwines* T_1 *and* T_2 *asymptotically. Then* $A_2 A = AA_1$, $B_2 A = AB_1$ *and* $S_2 A = AS_1$. *Moreover,* $AH_0(A_2) \subseteq H_0(A_1)$ *and* $AH_0(B_2) \subseteq H_0(B_1)$.

PROOF. Clearly for $k = 1, 2$, S_k is decomposable. Since $A \in \mathcal{L}(X, Y)$ intertwines T_1 and T_2 asymptotically, then by Theorem 2.4 in of [7] we have

$$AX_{T_1}(F) \subseteq Y_{T_2}(F)$$

for every closed $F \subseteq \mathbb{C}$. It follows from Proposition 3.4.12 of [8] and $T_k \sim^q S_k$ that

$$X_{T_1}(F) = X_{S_1}(F) \quad \text{and} \quad Y_{T_2}(F) = Y_{S_2}(F)$$

for all closed $F \subseteq \mathbb{C}$. Thus we conclude that

$$AX_{S_1}(F) = AX_{T_1}(F) \subseteq Y_{T_2}(F) = Y_{S_2}(F)$$

for all closed $F \subseteq \mathbb{C}$. As in the proof of Theorem 7, for $k = 1, 2$, let Φ_k be the continuous algebra homomorphism from $\mathrm{C}^2(\mathbb{C})$ to $\mathcal{L}(X)$ (respectively $\mathcal{L}(Y)$) with $\Phi_k(1) = I$ on X (respectively Y), $\Phi_k(\mathrm{Re}(\cdot)) = A_k$ and $\Phi_k(\mathrm{Im}(\cdot)) = B_k$. By Theorem 3.2.4 of [5], we have

$$\begin{aligned} AX_{A_1}(F) &= AX_{\Phi_1(Re(\cdot))}(F) \\ &= AX_{S_1}(Re^{-1}(F)) \\ &\subseteq Y_{S_2}(Re^{-1}(F)) \\ &= Y_{\Phi_2(Re(\cdot))}(F) \\ &= Y_{A_2}(F) \end{aligned}$$

for every closed $F \subseteq \mathbb{C}$. By Proposition 3.4.12 of [8], A intertwines A_1 and A_2 asymptotically. It follows from Proposition 1 that

$$C(A_2, A_1)A = 0 \quad \text{and} \quad AH_0(A_2) \subseteq H_0(A_1).$$

In the same way we have

$$C(B_2, B_1)A = 0 \quad \text{and} \quad AH_0(B_2) \subseteq H_0(B_1),$$

and hence

$$\begin{aligned} C(S_2, S_1)A &= C(A_2, A_1)A + iC(B_2, B_1)A \\ &= 0. \end{aligned}$$

This completes the proof. □

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