A Comparison Study of Real-Time Solution to All-Attitude Angles of an Aircraft

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In this paper, the quaternion, the dual Euler, and the direction cosine methods are numerically compared using a non-aerodynamic 6 degree-of-freedom rigid model at all-attitude angles of an aircraft. The dual Euler method turns out to be superior to the others in the applications because it shows better numerical accuracy, stability, and robustness in integration step sizes. The dual Euler method is affordably less efficient than the quaternion method in terms of computational cost. Numerical accuracy and stability, which allow larger integration step sizes, are more critical in modern real-time applications than computational efficiency because of today's increased computational power. If the quaternion method is required because of constraints in computation time, then a suppression mechanism should be provided for algebraic constraint errors which will eventually add computational burden.

Key Words : Quaternion, Dual Euler, Direction Cosine, Real-Time Solution, All Attitude

Nomenclature

A : Euler angle transformation matrix
a : Direction cosine element
e : Quaternion parameter
h : Integration step size
k : Constant for quaternion method
P,Q,R : Roll rate, pitch rate, yaw rate
t : Time
λ : Constraint equation to quaternion parameters
ϕ, θ, ψ : Roll angle, pitch angle, yaw angle

Subscripts

0,1,2,3 : Index of quaternion parameters
r : Reversed euler angle
x, y, z : Aircraft body axes

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1. Introduction

Euler angles (Rolfe and Staples, 1986) and quarternions (Cardullo, 1994; Robinson, 1958; Hyochoong Bang, Jung-shin Lee and Youn-Ju Eun, 2004 in Korea) have traditionally been used for typical flight simulations and control in the simulation community. Euler angles have their own merits in simple mathematical expression and physical interpretation of the parameters, but they suffer from singularities in vertical flights. Thus Euler angles are limited to simulation of commercial aircrafts. In order to overcome this singularity problem the quaternion method is applied to fighter simulations, since vertical or reversed flights are quite common in fighter aircrafts. The quaternion method has its own limits which include a more complex expression of algebraic differential equations and difficulties in physical interpretation of the parameters.

In the existing literature there are some other methods (Mebius, 1995; Huang, 1993), for solving all-attitude angles of an aircraft, claimed to be superior to traditional Euler angles and quaternions. The dual Euler method (Huang, 1993) and the direction cosine method are among those. The dual Euler method divides all-attitude angles into two zones, and applies two alternate sets of Euler angles to avoid singularities in relevant zones. The core concept of the dual Euler method is to take advantage of different sets of Euler angles and to switch to a non-singular set when rotational singularity points are crossed over when using a set of Euler angles. The direction cosine (Cardullo, 1994) method avoids the singularity problem by integrating the direction cosine rates and solving rotational kinematics equations resulting from orthogonality conditions of the direction cosine matrix. The direction cosine method is similar to the quaternion method in the sense that it is difficult to apply physical constraints to the parameters when necessary.

In this paper three methods the quaternions, the dual Euler, and the direction cosine methods — are compared experimentally to identify the most appropriate one for real-time applications. Real-time aspects including numerical accuracy, stability, and computational load are focused in this study. Test simulations say that the dual Euler method is superior to the others in accuracy and stability. The dual Euler method and the direction cosine methods are more accurate and stable than the quaternion method with small integration step sizes. The dual Euler method is more robust in step sizes than the quaternion and the direction cosine methods. The dual Euler methods affordably less efficient than the other methods in computational burden. However, its additional computational cost can be considered acceptable considering the overall amount of simulation codes in modern applications and the recent growth in computation capability. If the quaternion method is required because of constraints in computation time, then a suppression mechanism should be provided for algebraic constraint errors which will eventually add computational burden.

2. Numerical Solutions to All-Attitude Angles

In this comparison the traditional Euler method was excluded because it comprises intrinsic singularity and cannot be applicable to all-attitude angles. Three numerical solutions available in the literature are introduced: the quaternion, the dual Euler, and the direction cosine methods. Their natures are briefly explained here.

2.1 The quaternion method

Coordinate transformation from aircrafts' rotating body axes (x_0, y_0, z_0) to fixed inertial axes (x, y, z) is represented by Eq. (1), when quaternions (e_0, e_1, e_2, e_3) are adopted rather than Euler angles.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2+e_0e_3) & 2(e_1e_3-e_0e_2) \\ 2(e_1e_2-e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3+e_0e_1) \\ 2(e_0e_2+e_1e_3) & 2(e_2e_3-e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \quad (1)$$

Here quaternions are governed by the equations as follows:

$$\begin{aligned} \dot{e}_0 &= -\frac{1}{2}(e_1P+e_2Q+e_3R)+k\lambda e_0 \\ \dot{e}_1 &= \frac{1}{2}(e_0P+e_2R-e_3Q)+k\lambda e_1 \\ \dot{e}_2 &= \frac{1}{2}(e_0Q+e_3P-e_1R)+k\lambda e_2 \\ \dot{e}_3 &= \frac{1}{2}(e_0R+e_1Q-e_2P)+k\lambda e_3 \end{aligned} \quad (2)$$

where P, Q, and R are roll, pitch, and yaw rate respectively. Integration step size, h, must satisfy the condition of $kh \leq 1$, for the simulation not to diverge. k is a constant, and determined experimentally for an appropriate simulation result to be achieved.

A constraint equation applies to quaternion parameters:

$$\lambda = 1 - (e_0^2 + e_1^2 + e_2^2 + e_3^2) \tag{3}$$

Thus the quaternion approach comprises 4 1st-order differential kinematic equations and 1 algebraic constraint equation for the problem of 3 degrees of rotational freedom.

2.2 The direction cosine method

The direction cosine matrix defines the transformation between inertial and body axes. Each direction cosine represents the projection of a unit vector in one axis system onto a unit vector in the other axis system. The direction cosine method takes advantage of the relation :

$$[\dot{a}_{ij}] = \begin{bmatrix} 0 & R & -Q \\ -R & 0 & P \\ Q & -P & 0 \end{bmatrix} [a_{ij}] \tag{4}$$

The 9 Direction cosine rates $[\dot{a}_{ij}]$ in Eq. (4) are integrated, and obey the orthogonality conditions leading to the following constraint equations :

$$\begin{aligned} a_{11} &= a_{22}a_{33} - a_{23}a_{32} \\ a_{21} &= a_{13}a_{32} - a_{12}a_{33} \\ a_{31} &= a_{12}a_{23} - a_{13}a_{22} \\ a_{12} &= a_{23}a_{31} - a_{21}a_{33} \\ a_{22} &= a_{11}a_{33} - a_{13}a_{31} \\ a_{32} &= a_{13}a_{21} - a_{11}a_{23} \\ a_{13} &= a_{21}a_{32} - a_{22}a_{31} \\ a_{23} &= a_{12}a_{31} - a_{11}a_{32} \\ a_{33} &= a_{11}a_{22} - a_{12}a_{21} \end{aligned} \tag{5}$$

These equations result in the states of direction cosines at the present integration step, and the state values are used for integration of Eq. (4) at the next step. That is, the singularity problem is resolved by integrating 9 linear differential equations for rotational kinematics rather than 3.

2.3 The dual Euler method

The dual Euler method applies two different sets of Euler angles to avoid singularities by switching from one to the other. Two sets, ordinary and reversed, are related by Eq. (6) for the direction cosine.

$$\begin{aligned} A &= [\Phi]_x[\Theta]_y[\Psi]_z \\ &= [\Theta_r]_y[\Phi_r]_x[\Psi_r]_z \end{aligned} \tag{6}$$

An ordinary set of Euler equations shown in Eq. (7) are integrated until θ comes close to vertical points, $\theta = \pm\pi/2$.

$$\begin{aligned} \dot{\phi} &= P + \dot{\psi}\sin\theta \\ \dot{\theta} &= Q\cos\phi - R\sin\phi \\ \dot{\psi} &= (Q\sin\phi + R\cos\phi)/\cos\theta \end{aligned} \tag{7}$$

where ϕ, θ, and ψ are roll, pitch, and yaw angle respectively.

In the vicinity of the singular points the set of Euler equations switches to a different set of reversed Euler equations defined in Eq. (8).

$$\begin{aligned} \dot{\phi}_r &= P\cos\theta_r + R\sin\theta_r \\ \dot{\theta}_r &= Q - \dot{\psi}_r\sin\phi_r \\ \dot{\psi}_r &= (-P\sin\theta_r + R\cos\theta_r)/\cos\phi_r \end{aligned} \tag{8}$$

Here ϕ_r, ψ_r, and θ_r are reversed roll, yaw, and pitch angle respectively. As can be seen, this set also has singularities at $\phi_r = \pm\pi/2$. When the singular points of ϕ_r come closer during numerical simulation, the ordinary set of Euler equations is recovered. Here $\cos\phi_r = 0$ is equivalent to $\theta = 0$ by the relationship in Eq. (6). The switch-over criteria are typically recommended to be $\theta = \pm\pi/4$, $\theta = \pm 3\pi/4$.

3. Numerical Tests

The goal of numerical tests is to identify which algorithm among the quaternion, the dual Euler, and the direction cosine methods is the most appropriate in areal-time environment. The criteria are accuracy, stability, computational load, and robustness in integration step sizes. The three methods are applied to a non-aerodynamic 6 DOF rigid-body model (Huang, 1993), and compared under various conditions such as different integration algorithms and step sizes. Non-aerodynamic means that the 6 DOF rigid body is driven by Eq. (9), not by aerodynamics and other forces.

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \pi/4\sin\pi t \\ \pi \\ -P \end{bmatrix} \tag{9}$$

Since it is impossible to obtain an analytic solution to the problem, a Runge-Kutta 4th-order integration is used to yield a solution assumed to be exact. Errors in rotational state variables are monitored in the paper only when an Adams-Bashforth 2nd-order integration is applied with step sizes $h=0.05$ sec and 0.01 sec. However, it is observed that other integration methods result in similar trends.

It is not a trivial task to determine the appropriate value of k in the quaternion method. In the tests shown here $k=10.0$ is selected for best performance after scores of test runs. Fig. 1 through Fig. 3 show errors of three methods in the rotational variables when $h=0.01$ sec is adopted. As can be seen in the plots, both the dual Euler and the direction cosine method are acceptably accurate, and stable compared with the quaternion method. The quaternion method has a problem with algebraic-differential equations, and the divergence of numerical errors can be expected. Fig. 4 through Fig. 6 are regarding a larger step size, $h=0.05$ sec. The plots show that the dual Euler method is more robust in step sizes than the direction cosine method. The direction cosine method yields unacceptable errors in pitch angle around $t=6.0$ sec, which is equivalent to $\theta=-\pi/2$. The direction cosine method fails to compute the value of θ by the relation :

$$\theta=\sin^{-1}(-a_{13}) \tag{10}$$

Because of truncation errors, the absolute value of the argument in Eq. (10) at the time becomes larger than 1 when AB-2 and $t=1$ are combined. But RK-4 does not violate the limit even when $t=0.05$ is used. The integration tool, MATLAB,

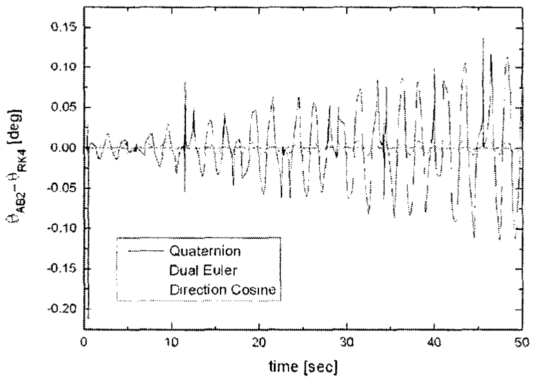

Fig. 1 Comparison of numerical errors in pitch angle θ ($h=0.01$, AB-2)

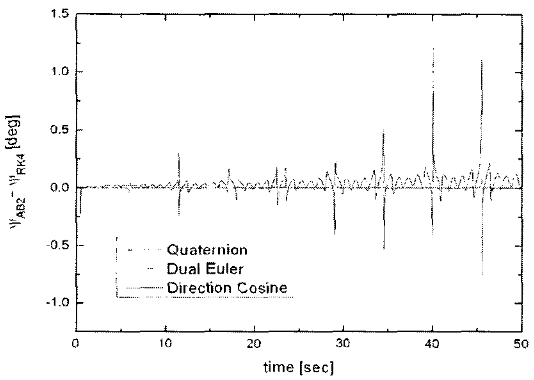

Fig. 2 Comparison of numerical errors in yaw angle ψ ($h=0.01$, AB-2)

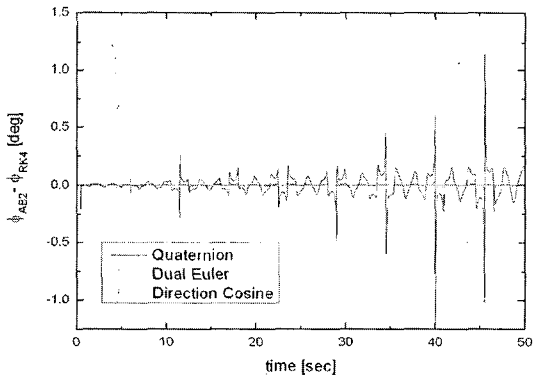

Fig. 3 Comparison of numerical errors in roll angle ϕ ($h=0.01$, AB-2)

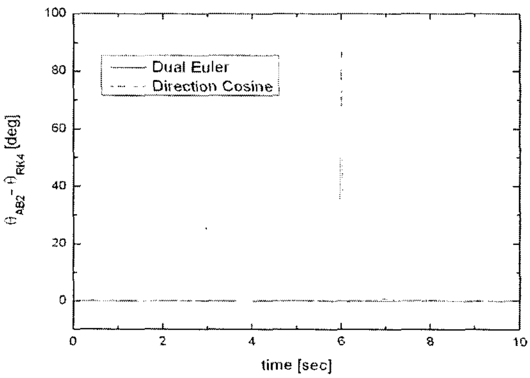

Fig. 4 Comparison of numerical errors in pitch angle θ ($h=0.05$, AB-2)

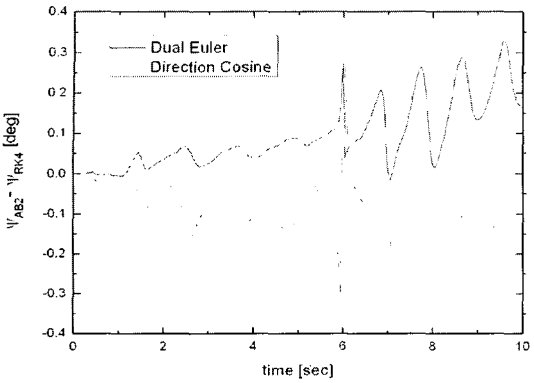

Fig. 5 Comparison of numerical errors in yaw angle ψ (h=0.05, AB-2)

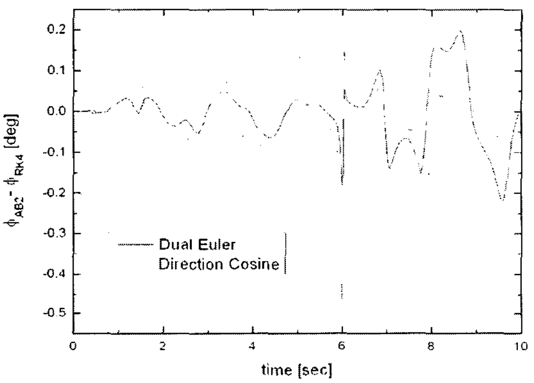

Fig. 6 Comparison of numerical errors in roll angle ϕ (h=0.05, AB-2)

automatically assigns 0 if it fails to compute $\sin^{-1}$. That is why we have unacceptable errors around t=6 sec. If the simualtion lasts for a longer time or if larger step sizes are used, which means larger truncation errors, then the direction cosine method inevitably reveals similar failures more often.

The dual Euler and the direction cosine methods are more accurate and stable than quaternions with small integration step sizes. The dual Euler method is more robust in step sizes than the quaternion and the direction cosine methods. The reason is that among the three methods only the dual Euler method is free from algebraic constraint equations. The constraints are constantly disturbed by truncation errors during integrations, and there is no mechanism applied in the

Table 1 Relative time consumptions of solutions to all-attitude angles (h=0.01 sec)

	Quaternion	Dual Euler	Direction Cosine
AB-2	1	1.22	1.11
RK-4	1.82	2.55	2.03

original methods to suppress the divergence of errors.

Computational burdens of the three methods are compared in Table 1. AB-2 and RK-4 are combined with a step size, h=0.01 sec in the comparison. Relative time consumptions of the others to the quaternion method's are presented in the table. The result shows that the dual Euler method is the least efficient among the three, but the differences are negligible considering today's rich computational power and additional computational load to integration of state variables in whole flight simulation.

Some additional test runs are performed to identify at what pitch angles switch-overs of Euler angles should be made for the best result in the dual Euler method. As mentioned earlier in this paper, recommended switch-over pitch angles are $\theta=\pm\pi/4$, $\theta=\pm 3\pi/4$. The test simulations show that the values of switch-over angles do not significantly effect the accuracy of simulations, if the angles are not close enough to the singular points.

4. Conclusions

The dual Euler method is affordably less efficient than the other methods in computational burden. However, its additional computational cost can be considered acceptable considering the overall amount of simulation codes in modern applications and the recent growth in computation capability.

In conclusion, the dual Euler method is superior to the other methods in accuracy, stability, and robustness in integration step sizes. If the quaternion method is required because of constraints in computation time, then a suppression mechanism should be provided for algebraic constraint errors which will eventually add computational burden. The same logic applies to the direction cosine method.

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