ON A GENERALIZED TRIF'S MAPPING IN BANACH MODULES OVER A $C^*$-ALGEBRA

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ABSTRACT. Let $X$ and $Y$ be vector spaces. It is shown that a mapping $f : X \to Y$ satisfies the functional equation

$$
\begin{align*}
&mn \ mn - 2C_{k-2}f\left(\frac{x_1 + \cdots + x_{mn}}{mn}\right) \\
&\quad + m \ mn - 2C_{k-1} \sum_{i=1}^{n} f\left(\frac{x_{mi-1} + \cdots + x_{mi}}{m}\right) \\
&= k \sum_{1 \leq i_1 < \cdots < i_k \leq mn} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right)
\end{align*}
$$

if and only if the mapping $f : X \to Y$ is additive, and we prove the Cauchy-Rassias stability of the functional equation (†) in Banach modules over a unital $C^*$-algebra. Let $A$ and $B$ be unital $C^*$-algebras or Lie $JC^*$-algebras. As an application, we show that every almost homomorphism $h : A \to B$ of $A$ into $B$ is a homomorphism when $h(2^d u y) = h(2^d u) h(y)$ or $h(2^d u \circ y) = h(2^d u) \circ h(y)$ for all unitaries $u \in A$, all $y \in A$, and $d = 0, 1, 2, \ldots$, and that every almost linear almost multiplicative mapping $h : A \to B$ is a homomorphism when $h(2x) = 2h(x)$ for all $x \in A$.

Moreover, we prove the Cauchy-Rassias stability of homomorphisms in $C^*$-algebras or in Lie $JC^*$-algebras, and of Lie $JC^*$-algebra derivations in Lie $JC^*$-algebras.

1. Introduction

In 1940, S. M. Ulam [20] raised the following question: Under what conditions does there exist an additive mapping near an approximately
additive mapping?

Let $X$ and $Y$ be Banach spaces with norms $|| \cdot ||$ and $\cdot$, respectively. Hyers [3] showed that if $\epsilon > 0$ and $f : X \rightarrow Y$ such that

$$||f(x + y) - f(x) - f(y)|| \leq \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$||f(x) - T(x)|| \leq \epsilon$$

for all $x \in X$.

Consider $f : X \rightarrow Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$(*) \quad ||f(x + y) - f(x) - f(y)|| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th. M. Rassias [12] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \rightarrow Y$ such that

$$||f(x) - T(x)|| \leq \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in X$. The inequality $(*)$ that was introduced for the first time by Th. M. Rassias [12] we call Cauchy-Rassias inequality and the stability of the functional equation Cauchy-Rassias stability. This inequality has provided a lot of influence in the development of what is now known as Hyers-Ulam-Rassias stability of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was taken up by a number of mathematicians (cf. [4], [10], [14]–[18]). Th. M. Rassias [13] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Z. Gajda [1] following the same approach as in Th. M. Rassias [12], gave an affirmative solution to this question for $p > 1$.

Găvruta [2] generalized the Rassias’ result: Let $G$ be an abelian group and $Y$ a Banach space. Denote by $\varphi : G \times G \rightarrow [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty$$
for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying
\[ \| f(x + y) - f(x) - f(y) \| \leq \varphi(x, y) \]
for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that
\[ \| f(x) - T(x) \| \leq \frac{1}{2} \tilde{\varphi}(x, x) \]
for all $x \in G$. C. Park [9] applied the Gǎvruta’s result to linear functional equations in Banach modules over a $C^*$-algebra.

Jun and Lee [5] proved the following: Denote by $\varphi : X \setminus \{0\} \times X \setminus \{0\} \to [0, \infty)$ a function such that
\[ \tilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 3^{-j} \varphi(3^j x, 3^j y) < \infty \]
for all $x, y \in X \setminus \{0\}$. Suppose that $f : X \to Y$ is a mapping satisfying
\[ \| 2f(\frac{x+y}{2}) - f(x) - f(y) \| \leq \varphi(x, y) \]
for all $x, y \in X \setminus \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that
\[ \| f(x) - f(0) - T(x) \| \leq \frac{1}{3} (\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)) \]
for all $x \in X \setminus \{0\}$. C. Park and W. Park [11] applied the Jun and Lee’s result to the Jensen’s equation in Banach modules over a $C^*$-algebra.

Recently, T. Trif [19, Theorem 2.1] proved that, for vector spaces $V$ and $W$, a mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation
\[ n_{n-2} C_{k-2} f \left( \frac{x_1 + \cdots + x_n}{n} \right) + n_{n-2} C_{k-1} \sum_{i=1}^{n} f(x_i) = k \sum_{1 \leq i_1 < \cdots < i_k \leq n} f \left( \frac{x_{i_1} + \cdots + x_{i_k}}{k} \right) \]
for all $x_1, \ldots, x_n \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x + y) = f(x) + f(y)$ for all $x, y \in V$. He
proved the stability of the functional equation (T) (see [19, Theorems 3.1 and 3.2]).

Throughout this paper, assume that $m, n, k$ are integers with $1 < m < k < mn$, and that $s, q$ are integers with $1 \leq s \leq \left[ \frac{n}{2} \right]$ and $1 < 2q \leq m$, where $[ \cdot ]$ denotes the Gauss symbol.

In this paper, we solve the following functional equation

\begin{equation}
\begin{aligned}
mn \sum_{i=1}^{n} 
\frac{f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right)}{mn} 
+ m \sum_{i=1}^{n} 
\frac{f\left(\frac{x_{i_{m+1}} + \cdots + x_{mi}}{m}\right)}{mn} 
\end{aligned}
\end{equation}

We moreover prove the Cauchy-Rassias stability of the functional equation (1.i) in Banach modules over a unital $C^*$-algebra. The main purpose of this paper is to investigate homomorphisms between $C^*$-algebras and between Lie $JC^*$-algebras, and to prove their Cauchy-Rassias stability.

2. A generalized Trif's mapping

Throughout this section, assume that $X$ and $Y$ are linear spaces.

**Lemma 2.1.** A mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation (1.i) for all $x_1, \ldots, x_{mn} \in X$ if and only if the mapping $f : X \to Y$ satisfies the additive Cauchy equation $f(x + y) = f(x) + f(y)$ for all $x, y \in X$.

**Proof.** Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation (1.i).

Replacing $x_m$ and $x_{m+1}$ by $x_{m+1}$ and $x_m$ in (1.i), respectively, we get

\begin{equation}
\begin{aligned}
mn \sum_{i=1}^{n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) 
+ m \sum_{i=1}^{n} f\left(\frac{x_{i_{m+1}} + \cdots + x_{mi}}{m}\right) 
+ m \sum_{i=1}^{n} f\left(\frac{x_{m} + x_{m+2} + \cdots + x_{2m}}{m}\right)
\end{aligned}
\end{equation}
for all $x_1, \ldots, x_{mn} \in X$. It follows from (1.i) and (2.1) that

$$f\left(\frac{x_1 + \cdots + x_{m-1} + x_m}{m}\right) + f\left(\frac{x_{m+1} + x_{m+2} + \cdots + x_{2m}}{m}\right)$$

$$= f\left(\frac{x_1 + \cdots + x_{m-1} + x_{m+1}}{m}\right) + f\left(\frac{x_m + x_{m+2} + \cdots + x_{2m}}{m}\right)$$

for all $x_1, \ldots, x_{2m} \in X$. Letting $x_{m-1} = x_m = x$, $x_{m+1} = x_{m+2} = y$ and $x_1 = \cdots = x_{m-2} = x_{m+3} = \cdots = x_{2m} = 0$ in (2.2), we get

$$f\left(\frac{2x}{m}\right) + f\left(\frac{2y}{m}\right) = 2f\left(\frac{x + y}{m}\right)$$

for all $x, y \in X$. Putting $y = 0$ in (2.3), we obtain

$$f\left(\frac{2x}{m}\right) = 2f\left(\frac{x}{m}\right)$$

for all $x \in X$. So

$$2f\left(\frac{x}{m}\right) + 2f\left(\frac{y}{m}\right) = 2f\left(\frac{x + y}{m}\right)$$

for all $x, y \in X$. Replacing $x$ and $y$ by $mx$ and $my$ in (2.4), respectively, we get

$$2f(x) + 2f(y) = 2f(x + y)$$

for all $x, y \in X$. Thus the mapping $f : X \rightarrow Y$ satisfies the additive Cauchy equation $f(x + y) = f(x) + f(y)$ for all $x, y \in X$.

The converse is obvious. □

3. Cauchy-Rassias stability of the generalized Trif’s mapping in Banach modules over a $C^*$-algebra

Throughout this section, assume that $A$ is a unital $C^*$-algebra with norm $\| \cdot \|$ and unitary group $U(A)$, and that $X$ and $Y$ are left Banach modules over $A$ with norms $\| \cdot \|$ and $\| \cdot \|$, respectively.
Given a mapping \( f : X \to Y \), we set
\[
D_uf(x_1, \ldots, x_{mn}) := mn \left( \frac{ux_1 + \cdots + ux_{mn}}{mn} \right) + m \left( \frac{ux_{mi-1} + \cdots + ux_{mi}}{m} \right) - \sum_{1 \leq i_1 < \cdots < i_k \leq mn} uf\left( \frac{x_{i_1} + \cdots + x_{i_k}}{k} \right)
\]
for all \( u \in \mathcal{U}(A) \) and all \( x_1, \ldots, x_{mn} \in X \).

**Theorem 3.1.** Let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) for which there is a function \( \varphi : X^{mn} \to [0, \infty) \) such that
\[
(3.i) \quad \bar{\psi}(x_1, \ldots, x_{mn}) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \ldots, 2^j x_{mn}) < \infty,
\]
\[
(3.ii) \quad \left\| D_uf(x_1, \ldots, x_{mn}) \right\| \leq \varphi(x_1, \ldots, x_{mn})
\]
for all \( u \in \mathcal{U}(A) \) and all \( x_1, \ldots, x_{mn} \in X \). Then there exists a unique A-linear generalized Trif’s mapping \( T : X \to Y \) such that
\[
(3.iii) \quad \left\| f(x) - T(x) \right\| \leq \frac{1}{2ms mn^{-2}C_{k-1}} \bar{\psi}(0, \ldots, 0, \underbrace{mx, \ldots, mx}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}) \]
\[
+ \frac{1}{2ms mn^{-2}C_{k-1}} \bar{\psi}(0, \ldots, 0, \underbrace{mx, \ldots, mx}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}) \]
\[
+ \frac{1}{2ms mn^{-2}C_{k-1}} \bar{\psi}(0, \ldots, 0, \underbrace{mx, \ldots, mx}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}})
\]
\[
+ \underbrace{0, \ldots, 0}_{m-q \text{ times}} \underbrace{0, \ldots, 0}_{m-2q \text{ times}} \underbrace{0, \ldots, 0}_{q \text{ times}} \underbrace{0, \ldots, 0}_{m-q \text{ times}} \underbrace{0, \ldots, 0}_{m-2q \text{ times}} \underbrace{0, \ldots, 0}_{q \text{ times}}
\]
for all \( x \in X \).

**Proof.** Let \( u = 1 \in \mathcal{U}(A) \). Putting \( x_{im-2q+1} = \cdots = x_{im-q} = x, x_{im-q+1} = \cdots = x_{im} = 0, x_{im+1} = \cdots = x_{im+q} = x \) for \( i = 1, 3, \ldots, 2s - 1 \), and \( x_j = 0 \) for other indices \( j \) in (3.ii), we have

\[
\left\| mn \frac{m_{n-2}C_{k-2}f\left(\frac{2qx}{mn}\right)}{m} + 2ms \frac{m_{n-2}C_{k-1}f\left(\frac{qx}{m}\right)}{m} \right\|
- k \sum_{1 \leq i_1 < \cdots < i_k \leq mn} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right)
\leq \varphi\left(\underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{x, \ldots, x}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{x, \ldots, x}_{q \text{ times}}\right)
- \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \underbrace{x, \ldots, x}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \underbrace{0, \ldots, 0}_{mn-2ms \text{ times}}
\tag{3.1}
\]

for all \( x \in X \). Putting \( x_{im-2q+1} = \cdots = x_{im-q} = x, x_{im-q+1} = \cdots = x_{im} = x \) for \( i = 1, 3, \ldots, 2s - 1 \), and \( x_j = 0 \) for other indices \( j \) in (3.ii), we have

\[
\left\| mn \frac{m_{n-2}C_{k-2}f\left(\frac{2qx}{mn}\right)}{m} + ms \frac{m_{n-2}C_{k-1}f\left(\frac{2qx}{m}\right)}{m} \right\|
- k \sum_{1 \leq i_1 < \cdots < i_k \leq mn} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right)
\leq \varphi\left(\underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{x, \ldots, x}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{x, \ldots, x}_{q \text{ times}}\right)
- \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \underbrace{x, \ldots, x}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \underbrace{0, \ldots, 0}_{mn-2ms \text{ times}}
\tag{3.2}
\]

for all \( x \in X \). It follows from (3.1) and (3.2) that

\[
\left\| 2ms \frac{m_{n-2}C_{k-1}f\left(\frac{qx}{m}\right)}{m} - ms \frac{m_{n-2}C_{k-1}f\left(\frac{2qx}{m}\right)}{m} \right\|
\leq \varphi\left(\underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{x, \ldots, x}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{x, \ldots, x}_{q \text{ times}}\right)
- \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \underbrace{x, \ldots, x}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \underbrace{0, \ldots, 0}_{mn-2ms \text{ times}}
\]
for all $x \in X$. So

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| 
\leq \frac{1}{2m^2 s m_n - 2C_{k-1}} \varphi\left( 0, \ldots, 0, \frac{m x}{q}, \ldots, \frac{m x}{q}, 0, \ldots, 0, \frac{m x}{q}, \ldots, \frac{m x}{q}, \frac{m x}{q}, \ldots, \frac{m x}{q}, 0, \ldots, 0 \right)$$

(3.3)

for all $x \in X$. Hence

$$\left\| \frac{1}{2^d} f(2^d x) - \frac{1}{2^{d+1}} f(2^{d+1} x) \right\|
\leq \frac{1}{2^d} \left\| f(2^d x) - \frac{1}{2} f(2 \cdot 2^d x) \right\|
\leq \frac{1}{2^{d+1} m s m_n - 2C_{k-1}} \varphi\left( 0, \ldots, 0, \frac{2^d m x}{q}, \ldots, \frac{2^d m x}{q}, 0, \ldots, 0 \right)$$
for all $x \in X$ and all positive integers $d$. By (3.3), we have

$$
\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^d} f(2^d x) \right\| 
\leq \sum_{j=l}^{d-1} \frac{1}{2^{j+1} m s_{m-2C_k-1}} \varphi\left( \underbrace{0, \ldots, 0, \underbrace{\frac{2^j m x}{q}, \ldots, \frac{2^j m x}{q}}_{q \text{ times}}} \right)
$$

(3.4)
for all $x \in X$ and all positive integers $l$ and $d$ with $l < d$. This shows that the sequence $\{\frac{1}{2^d} f(2^d x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{2^d} f(2^d x)\}$ converges for all $x \in X$. So we can define a mapping $T : X \to Y$ by

$$T(x) := \lim_{d \to \infty} \frac{1}{2^d} f(2^d x)$$

for all $x \in X$. Also, we get

$$\|D_1 T(x_1, \ldots, x_{mn})\| = \lim_{d \to \infty} \frac{1}{2^d} \|D_1 f(2^d x_1, \ldots, 2^d x_{mn})\|$$

$$\leq \lim_{d \to \infty} \frac{1}{2^d} \varphi(2^d x_1, \ldots, 2^d x_{mn})$$

$$= 0$$

for all $x_1, \ldots, x_{mn} \in X$. Thus $T$ is a generalized Trif’s mapping. By Lemma 2.1, $T$ is additive. Putting $l = 0$ and letting $d \to \infty$ in (3.4), we get (3.iii).

Now, let $L : X \to Y$ be another generalized Trif’s mapping satisfying (3.iii). Then we have

$$\|T(x) - L(x)\|$$

$$= \frac{1}{2^d} \|T(2^d x) - L(2^d x)\|$$

$$\leq \frac{1}{2^d} (\|T(2^d x) - f(2^d x)\| + \|L(2^d x) - f(2^d x)\|)$$

$$\leq \frac{2}{2^{d+1} m s_{m-2} C_{k-1}} \varphi\left(2^d m x, \overbrace{q, \ldots, q}^{q \text{ times}}, \overbrace{0, \ldots, 0}^{m-2 q \text{ times}}\right)$$
which tends to zero as \( d \to \infty \) for all \( x \in X \). So we can conclude that 
\( T(x) = L(x) \) for all \( x \in X \). This proves the uniqueness of \( T \).

By the assumption, for each \( u \in \mathcal{U}(\mathcal{A}) \), we get

\[
\|D_u T(x, 0, \ldots, 0)\| = \lim_{d \to \infty} \frac{1}{2^d} \|D_u f(2^d x, 0, \ldots, 0)\| \\
\leq \lim_{d \to \infty} \frac{1}{2^d} \varphi(2^d x, 0, \ldots, 0) = 0
\]

for all \( x \in X \). So

\[
m_n m_n^{-2} C_{k-2} T\left(\frac{ux}{m_n}\right) + m_m m_n^{-2} C_{k-1} T\left(\frac{ux}{m_n}\right) = k_m m_n^{-1} C_{k-1} u T\left(\frac{x}{k}\right)
\]

for all \( u \in \mathcal{U}(\mathcal{A}) \) and all \( x \in X \). Since \( m_n^{-2} C_{k-2} + m_n^{-2} C_{k-1} = m_n^{-1} C_{k-1} \) and \( T \) is additive,

\[
T(ux) = u T(x)
\]

for all \( u \in \mathcal{U}(\mathcal{A}) \) and all \( x \in X \).

Now let \( a \in A (a \neq 0) \) and \( M \) an integer greater than \( 4|a| \). Then \( \frac{a}{M} < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3} \). By [6, Theorem 1], there exist three elements \( u_1, u_2, u_3 \in \mathcal{U}(\mathcal{A}) \) such that \( 3 \frac{a}{M} = u_1 + u_2 + u_3 \). So by (3.5)

\[
\frac{T(ax)}{3} = T\left(\frac{M}{3} \cdot 3 \frac{a}{M} x\right) = M \cdot T\left(\frac{1}{3} \cdot 3 \frac{a}{M} x\right) = \frac{M}{3} T\left(3 \frac{a}{M} x\right)
\]
\[
\frac{M}{3} T(u_1 x + u_2 x + u_3 x) \\
= \frac{M}{3} (T(u_1 x) + T(u_2 x) + T(u_3 x)) \\
= \frac{M}{3} (u_1 + u_2 + u_3) T(x) \\
= \frac{M}{3} \cdot 3 \frac{a}{M} T(x) = a T(x)
\]

for all \( a \in A \) and all \( x \in X \). Hence

\[
T(ax + by) = T(ax) + T(by) = a T(x) + b T(y)
\]

for all \( a, b \in A(a, b \neq 0) \) and all \( x, y \in X \). And \( T(0x) = 0 = 0 T(x) \) for all \( x \in X \). So the generalized Trif's mapping \( T : A \to B \) is an \( A \)-linear mapping, as desired. \( \square \)

**Corollary 3.2.** Let \( \theta \) and \( p < 1 \) be positive real numbers. Let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) such that

\[
\| D_u f(x_1, \ldots, x_{mn}) \| \leq \theta \sum_{j=1}^{mn} \| x_j \| \| x \| ^p
\]

for all \( u \in U(A) \) and all \( x_1, \ldots, x_{mn} \in X \). Then there exists a unique \( A \)-linear generalized Trif's mapping \( T : X \to Y \) such that

\[
\| f(x) - T(x) \| \leq \frac{4m^{p-1} q^{1-p} \theta}{(2 - 2p) \| x \| ^p}
\]

for all \( x \in X \).

**Proof.** Define \( \varphi(x_1, \ldots, x_{mn}) = \theta \sum_{j=1}^{mn} \| x_j \| \| x \| ^p \), and apply Theorem 3.1. \( \square \)

**Theorem 3.3.** Let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) for which there is a function \( \varphi : X^{mn} \to [0, \infty) \) such that

(3.iv) \( \tilde{\varphi}(x_1, \ldots, x_{mn}) := \sum_{j=1}^{\infty} 2^j \varphi \left( \frac{x_1}{2^j}, \ldots, \frac{x_{mn}}{2^j} \right) < \infty \),

(3.v) \( \| D_u f(x_1, \ldots, x_{mn}) \| \leq \varphi(x_1, \ldots, x_{mn}) \)
for all \( u \in \mathcal{U}(A) \) and all \( x_1, \ldots, x_{mn} \in X \). Then there exists a unique \( A \)-linear generalized Trif’s mapping \( T : X \to Y \) such that

\[
\| f(x) - T(x) \| \\
\leq \frac{1}{2ms_{mn-2C_k-1}} \phi \left( 0, \ldots, 0, \frac{mx}{q}, \ldots, \frac{mx}{q}, 0, \ldots, 0, \frac{mx}{q}, \ldots, \frac{mx}{q}, 0, \ldots, 0, \ldots, 0, \ldots, 0 \right)
\]

\[
= \frac{1}{2ms_{mn-2C_k-1}} \phi \left( 0, \ldots, 0, \frac{mx}{q}, \ldots, \frac{mx}{q}, 0, \ldots, 0, \frac{mx}{q}, \ldots, \frac{mx}{q}, 0, \ldots, 0, \ldots, 0, \ldots, 0 \right)
\]

for all \( x \in X \).

**Proof.** Replacing \( x \) by \( \frac{x}{2} \) in (3.4), we have

\[
\| f(x) - 2f \left( \frac{x}{2} \right) \|
\leq \frac{1}{ms_{mn-2C_k-1}} \phi \left( 0, \ldots, 0, \frac{mx}{2q}, \ldots, \frac{mx}{2q}, 0, \ldots, 0, \frac{mx}{2q}, \ldots, \frac{mx}{2q}, 0, \ldots, 0, \ldots, 0, \ldots, 0 \right)
\]

\[
= \frac{1}{ms_{mn-2C_k-1}} \phi \left( 0, \ldots, 0, \frac{mx}{2q}, \ldots, \frac{mx}{2q}, 0, \ldots, 0, \frac{mx}{2q}, \ldots, \frac{mx}{2q}, 0, \ldots, 0, \ldots, 0, \ldots, 0 \right)
\]
for all $x \in X$. So

\[
\left\| 2^d f \left( \frac{x}{2^d} \right) - 2^{d+1} f \left( \frac{x}{2^{d+1}} \right) \right\| \\
= 2^d \left\| f \left( \frac{x}{2^d} \right) - 2f \left( \frac{x}{2 \cdot 2^d} \right) \right\| \\
\leq \frac{2^d}{ms_{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{2^{d+1}q}, \ldots, \frac{mx}{2^{d+1}q}, 0, \ldots, 0}_{q\text{ times}}, q\text{ times}) \\
\quad + \frac{2^d}{ms_{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{2^{d+1}q}, \ldots, \frac{mx}{2^{d+1}q}, 0, \ldots, 0}_{q\text{ times}}, q\text{ times}) \\
\quad + \frac{2^d}{ms_{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{2^{d+1}q}, \ldots, \frac{mx}{2^{d+1}q}, 0, \ldots, 0}_{q\text{ times}}, q\text{ times}) \\
\quad + \frac{2^d}{ms_{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{2^{d+1}q}, \ldots, \frac{mx}{2^{d+1}q}, 0, \ldots, 0}_{q\text{ times}}, q\text{ times})
\]
for all \( x \in X \) and all positive integers \( d \). By (3.6), we have

\[
\left\| 2^l f \left( \frac{x}{2^l} \right) - 2^d f \left( \frac{x}{2^d} \right) \right\|
\leq \sum_{j=1}^{d-1} \frac{2^j}{m s \, m n - 2 C_{k-1}} \varphi \left( 0, \ldots, 0, \frac{m x}{2^{j+1} q}, \frac{m x}{2^{j+1} q}, 0, \ldots, 0, \frac{m x}{2^{j+1} q}, \frac{m x}{2^{j+1} q}, 0, \ldots, 0, \frac{m x}{2^{j+1} q}, \frac{m x}{2^{j+1} q}, 0, \ldots, 0, \frac{m x}{2^{j+1} q}, \frac{m x}{2^{j+1} q}, 0, \ldots, 0 \right)
\]

\[
+ \sum_{j=1}^{d-1} \frac{2^j}{m s \, m n - 2 C_{k-1}} \varphi \left( 0, \ldots, 0, \frac{m x}{2^{j+1} q}, \frac{m x}{2^{j+1} q}, 0, \ldots, 0, \frac{m x}{2^{j+1} q}, \frac{m x}{2^{j+1} q}, 0, \ldots, 0, \frac{m x}{2^{j+1} q}, \frac{m x}{2^{j+1} q}, 0, \ldots, 0, \frac{m x}{2^{j+1} q}, \frac{m x}{2^{j+1} q}, 0, \ldots, 0 \right)
\]

for all \( x \in X \) and all positive integers \( l \) and \( d \) with \( l < d \). This shows that the sequence \( \{2^d f \left( \frac{x}{2^d} \right) \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{2^d f \left( \frac{x}{2^d} \right) \} \) converges for all \( x \in X \). So we can define a mapping \( T : X \to Y \) by

\[
T(x) := \lim_{d \to \infty} 2^d f \left( \frac{x}{2^d} \right)
\]

for all \( x \in X \). Also, we get

\[
\|D_1 T(x_1, \ldots, x_{mn})\| = \lim_{d \to \infty} 2^d \|D_1 f \left( \frac{x_1}{2^d}, \ldots, \frac{x_{mn}}{2^d} \right)\|
\]

\[
\leq \lim_{d \to \infty} 2^d \varphi \left( \frac{x_1}{2^d}, \ldots, \frac{x_{mn}}{2^d} \right) = 0
\]
for all $x_1, \ldots, x_{mn} \in X$. Thus $T$ is a generalized Trif's mapping. By Lemma 2.1, $T$ is additive. Putting $l = 0$ and letting $d \to \infty$ in (3.7), we get (3.vi).

The rest of the proof is similar to the proof of Theorem 3.1. ⊣

**Corollary 3.4.** Let $\theta$ and $p > 1$ be positive real numbers. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that

$$
\|D_u f(x_1, \ldots, x_{mn})\| \leq \theta \sum_{j=1}^{mn} \|x_j\|^p
$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \ldots, x_{mn} \in X$. Then there exists a unique $A$-linear generalized Trif's mapping $T : X \to Y$ such that

$$
\|f(x) - T(x)\| \leq \frac{4m^{p-1}q^{1-p}\theta}{(2p-2)mn^{-2}C_{k-1}} \|x\|^p
$$

for all $x \in X$.

**Proof.** Define $\varphi(x_1, \ldots, x_{mn}) = \theta \sum_{j=1}^{mn} \|x_j\|^p$, and apply Theorem 3.3. ⊣

4. Isomorphisms between unital $C^*$-algebras

Throughout this section, assume that $A$ is a unital $C^*$-algebra with norm $\|\cdot\|$, unit $e$ and unitary group $\mathcal{U}(A)$, and that $B$ is a unital $C^*$-algebra with norm $\|\cdot\|$.

We are going to investigate $C^*$-algebra isomorphisms between unital $C^*$-algebras.

**Theorem 4.1.** Let $h : A \to B$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^d u y) = h(2^d u) h(y)$ for all $u \in \mathcal{U}(A)$, all $y \in A$, and $d = 0, 1, 2, \ldots$, for which there is a function $\varphi : A^{mn} \to [0, \infty)$ satisfying (3.i) for all $x_1, \ldots, x_{mn} \in A$ such that

$$
\|mn_{2C_{k-2}}h \left( \frac{\mu x_1 + \cdots + \mu x_{mn}}{mn} \right) \\
+ mn_{2C_{k-1}} \sum_{i=1}^{n} h \left( \frac{\mu x_{mi-m+1} + \cdots + \mu x_{mi}}{m} \right) \\
- k \sum_{1 \leq i_1 < \cdots < i_k \leq mn} \mu h \left( \frac{x_{i_1} + \cdots + x_{i_k}}{k} \right) \|
$$
for all $u \in \mathcal{U}(A)$, all $x_1, \ldots, x_{mn} \in A$, all $\mu \in \mathbb{T} := \{ \lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and $d = 0, 1, 2, \ldots$. Assume that (4.iii) $\lim_{d \to \infty} \frac{h(2^d u^*)}{2^d}$ is invertible. Then the bijective mapping $h : A \to B$ is a $C^*$-algebra isomorphism.

**Proof.** We can consider a $C^*$-algebra as a Banach module over a unital $C^*$-algebra $\mathbb{C}$. So by Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $H : A \to B$ such that

\begin{align*}
(4.\text{iv}) \quad \|h(x) - H(x)\| &
\leq \frac{1}{2ms \frac{m-2q}{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \overbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}^{m-2q \text{ times}}, 0, \ldots, 0) \\
&+ \frac{1}{2ms \frac{m-2q}{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \overbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}^{m-2q \text{ times}}, \overbrace{0, \ldots, 0}^{q \text{ times}})
\end{align*}

for all $x \in A$. The mapping $H : A \to B$ is given by

\begin{equation}
(4.1) \quad H(x) = \lim_{d \to \infty} \frac{1}{2^d} h(2^d x)
\end{equation}

for all $x \in A$. 
By (3.i) and (4.ii), we get

\[ H(u^*) = \lim_{d \to \infty} \frac{h(2^d u^*)}{2^d} = \lim_{d \to \infty} \frac{h(2^d u)^*}{2^d} = \left( \lim_{d \to \infty} \frac{h(2^d u)}{2^d} \right)^* = H(u)^* \]

for all \( u \in \mathcal{U}(A) \). Since \( H \) is \( \mathbb{C} \)-linear and each \( x \in A \) is a finite linear combination of unitary elements (see [7, Theorem 4.1.7]), i.e.,

\( x = \sum_{j=1}^{d} \lambda_j u_j \) \( (\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(A)) \),

\[ H(x^*) = H \left( \sum_{j=1}^{d} \overline{\lambda_j u_j}^* \right) = \sum_{j=1}^{d} \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^{d} \overline{\lambda_j} H(u_j)^* \]

\[ = \left( \sum_{j=1}^{d} \lambda_j H(u_j) \right)^* \]

\[ = H \left( \sum_{j=1}^{d} \lambda_j u_j \right)^* = H(x)^* \]

for all \( x \in A \).

Since \( h(2^d uy) = h(2^d u)h(y) \) for all \( u \in \mathcal{U}(A) \), all \( y \in A \), and all \( d = 0, 1, 2, \ldots \),

\[ H(uy) = \lim_{d \to \infty} \frac{1}{2^d} h(2^d uy) = \lim_{d \to \infty} \frac{1}{2^d} h(2^d u)h(y) = H(u)h(y) \quad (4.2) \]

for all \( u \in \mathcal{U}(A) \) and all \( y \in A \). By the additivity of \( H \) and (4.2),

\[ 2^d H(uy) = H(2^d uy) = H(u(2^d y)) = H(u)h(2^d y) \]

for all \( u \in \mathcal{U}(A) \) and all \( y \in A \). Hence

\[ H(uy) = \frac{1}{2^d} H(u)h(2^d y) = H(u) \frac{1}{2^d} h(2^d y) \quad (4.3) \]

for all \( u \in \mathcal{U}(A) \) and all \( y \in A \). Taking the limit in (4.3) as \( d \to \infty \), we obtain

\[ H(uy) = H(u)H(y) \quad (4.4) \]
for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since $H$ is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^{d} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$), it follows from (4.4) that

$$H(xy) = H\left(\sum_{j=1}^{d} \lambda_j u_j y\right) = \sum_{j=1}^{d} \lambda_j H(u_j y) = \sum_{j=1}^{d} \lambda_j H(u_j) H(y)$$

$$= H\left(\sum_{j=1}^{d} \lambda_j u_j\right) H(y)$$

$$= H(x) H(y)$$

for all $x, y \in \mathcal{A}$.

By (4.2) and (4.4),

$$H(e) H(y) = H(e y) = H(e) h(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{d \to \infty} \frac{h(2^d e)}{2^d} = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

Therefore, the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a $C^*$-algebra isomorphism. \qed

**Corollary 4.2.** Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^d uy) = h(2^d u) h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $d = 0, 1, 2, \ldots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\left\|mn (mn-2) \sum_{i=1}^{n} \frac{h(\mu x_{i1} + \cdots + \mu x_{mn})}{mn} \right\|$$

$$+ m \left(\sum_{i=1}^{n} \frac{h(\mu x_{i1} + \cdots + \mu x_{mi})}{m}\right)$$

$$- k \left(\sum_{i_1 < \cdots < i_k \leq mn} \mu h\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right)\right) \leq \theta \sum_{j=1}^{mn} ||x_j||^p,$$

$$\left\|h(2^d u^*) - h(2^d u)^* \right\| \leq mn 2^d \theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $d = 0, 1, 2, \ldots$, and all $x_1, \ldots, x_{mn} \in \mathcal{A}$. Assume that $\lim_{d \to \infty} \frac{h(2^d e)}{2^d}$ is invertible. Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a $C^*$-algebra isomorphism.
Proof. Define \( \varphi(x_1, \ldots, x_{mn}) = \theta \sum_{j=1}^{mn} ||x_j||^p \), and apply Theorem 4.1.

**Theorem 4.3.** Let \( h : A \to B \) be a bijective mapping satisfying \( h(0) = 0 \) and \( h(2^d u y) = h(2^d u) h(y) \) for all \( u \in U(A) \), all \( y \in A \), and \( d = 0, 1, 2, \ldots \), for which there is a function \( \varphi : A^{mn} \to [0, \infty) \) satisfying (3.i), (4.ii), and (4.iii) such that

\[
\left\| \sum_{i=1}^{mn} h \left( \frac{\mu x_{i1} + \cdots + \mu x_{im}}{m} \right) \right\| \\
+ m \sum_{i=1}^{mn} h \left( \frac{\mu x_{i1} \cdots + \mu x_{im}}{m} \right) \\
- k \sum_{1 \leq i_1 < \cdots < i_k \leq mn} \mu h \left( \frac{x_{i1} + \cdots + x_{ik}}{k} \right) \\
\leq \varphi(x_1, \ldots, x_{mn})
\]

(4.v)

for all \( x_1, \ldots, x_{mn} \in A \) and \( \mu = 1, i \). If \( h(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then the bijective mapping \( h : A \to B \) is a C*-algebra isomorphism.

Proof. Put \( \mu = 1 \) in (4.v). By the same reasoning as in the proof of Theorem 4.1, there exists a unique generalized Trif's mapping \( H : A \to B \) satisfying (4.iv). By the same reasoning as in the proof of [12, Theorem], the additive mapping \( H : A \to B \) is \( \mathbb{R} \)-linear.

Put \( \mu = i \) in (4.v). By the same method as in the proof of Theorem 4.1, one can obtain that

\[
H(ix) = \lim_{d \to \infty} \frac{h(2^d ix)}{2^d} = \lim_{d \to \infty} \frac{i h(2^d x)}{2^d} = i H(x)
\]

for all \( x \in A \).

For each element \( \lambda \in \mathbb{C} \), \( \lambda = r + it \), where \( r, t \in \mathbb{R} \). So

\[
H(\lambda x) = H(rx + itx) = rH(x) + tH(ix) = rH(x) + iTH(x)
\]

\[
= (r + it)H(x) = \lambda H(x)
\]

for all \( \lambda \in \mathbb{C} \) and all \( x \in A \). So

\[
H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)
\]
for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in A$. Hence the additive mapping $H : A \to B$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 4.1. \(\square\)

Now we prove the Cauchy-Rassias stability of $C^*$-algebra homomorphisms in unital $C^*$-algebras.

**Theorem 4.4.** Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : A^{mn} \to [0, \infty)$ satisfying (3.i), (4.i), and (4.ii) such that

\[
\| h(2^d u \cdot 2^d v) - h(2^d u) h(2^d v) \| \leq \varphi(2^d u, 2^d v, 0, \ldots, 0 ) \tag{4.vi}
\]

for all $u, v \in \mathcal{U}(A)$ and $d = 0, 1, 2, \ldots$. Then there exists a unique $C^*$-algebra homomorphism $H : A \to B$ satisfying (4.iv).

**Proof.** By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive mapping $H : A \to B$ satisfying (4.iv).

By (4.vi),

\[
\frac{1}{2^{2d}} \| h(2^d u \cdot 2^d v) - h(2^d u) h(2^d v) \| \leq \frac{1}{2^{2d}} \varphi(2^d u, 2^d v, 0, \ldots, 0 ) \tag{4.vi}
\]

\[
\leq \frac{1}{2^d} \varphi(2^d u, 2^d v, 0, \ldots, 0 ), \tag{4.vi}
\]

which tends to zero by (3.i) as $d \to \infty$. By (4.1),

\[
H(uv) = \lim_{d \to \infty} \frac{h(2^d u \cdot 2^d v)}{2^{2d}} = \lim_{d \to \infty} \frac{h(2^d u) h(2^d v)}{2^{2d}}
\]

\[
= \lim_{d \to \infty} \frac{h(2^d u) h(2^d v)}{2^d 2^d} = H(u) H(v)
\]

for all $u, v \in \mathcal{U}(A)$. Since $H$ is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^d \lambda_j u_j \ (\lambda_j \in \mathbb{C}, u_j \in$
\( \mathcal{U}(\mathcal{A}) \),
\[
H(xv) = H \left( \sum_{j=1}^{d} \lambda_j u_j v \right) = \sum_{j=1}^{d} \lambda_j H(u_j v) = \sum_{j=1}^{d} \lambda_j H(u_j) H(v) \\
= H \left( \sum_{j=1}^{d} \lambda_j u_j \right) H(v) \\
= H(x) H(v)
\]
for all \( x \in \mathcal{A} \) and all \( v \in \mathcal{U}(\mathcal{A}) \). By the same method as given above, one can obtain that
\[
H(xy) = H(x) H(y)
\]
for all \( x, y \in \mathcal{A} \). So the mapping \( H : \mathcal{A} \rightarrow \mathcal{B} \) is a \( C^* \)-algebra homomorphism, as desired. \( \square \)

5. Homomorphisms between Lie \( JC^* \)-algebras

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [21]). Let \( \mathcal{L}(\mathcal{H}) \) be the real vector space of all bounded self-adjoint linear operators on \( \mathcal{H} \), interpreted as the (bounded) observables of the system. In 1932, Jordan observed that \( \mathcal{L}(\mathcal{H}) \) is a (nonassociative) algebra via the anticommutator product \( x \circ y := \frac{x y + y x}{2} \). A commutative algebra \( X \) with product \( x \circ y \) is called a Jordan algebra. A unital Jordan \( C^* \)-subalgebra of a \( C^* \)-algebra, endowed with the anticommutator product, is called a \( JC^* \)-algebra.

A unital \( C^* \)-algebra \( \mathcal{C} \), endowed with the Lie product \( [x, y] = \frac{x y - y x}{2} \) on \( \mathcal{C} \), is called a Lie \( C^* \)-algebra. A unital \( C^* \)-algebra \( \mathcal{C} \), endowed with the Lie product \( [\cdot, \cdot] \) and the anticommutator product \( \circ \), is called a \( JC^* \)-algebra if \( (\mathcal{C}, \circ) \) is a \( JC^* \)-algebra and \( (\mathcal{C}, [\cdot, \cdot]) \) is a Lie \( C^* \)-algebra.

Throughout this paper, let \( \mathcal{A} \) be a unital Lie \( JC^* \)-algebra with norm \( \| \cdot \| \), unit \( e \) and unitary group \( \mathcal{U}(\mathcal{A}) = \{ u \in \mathcal{A} \mid uu^* = u^* u = e \} \), and \( \mathcal{B} \) a unital Lie \( JC^* \)-algebra with norm \( \| \cdot \| \) and unit \( e' \).

**Definition 5.1.** A \( \mathbb{C} \)-linear mapping \( H : \mathcal{A} \rightarrow \mathcal{B} \) is called a Lie \( JC^* \)-algebra homomorphism if \( H : \mathcal{A} \rightarrow \mathcal{B} \) satisfies
\[
H(x \circ y) = H(x) \circ H(y), \\
H([x, y]) = [H(x), H(y)], \\
H(x^*) = H(x)^* \]
for all \( x, y \in \mathcal{A} \).

**Remark 5.1.** A \( \mathbb{C} \)-linear mapping \( H : \mathcal{A} \to \mathcal{B} \) is a \( C^* \)-algebra homomorphism if and only if the mapping \( H : \mathcal{A} \to \mathcal{B} \) is a Lie \( JC^* \)-algebra homomorphism.

Assume that \( H \) is a Lie \( JC^* \)-algebra homomorphism. Then

\[
H(xy) = H([x,y] + x \circ y) = H([x,y]) + H(x \circ y) = [H(x), H(y)] + H(x) \circ H(y) = H(x)H(y)
\]

for all \( x, y \in \mathcal{A} \). So \( H \) is a \( C^* \)-algebra homomorphism.

Assume that \( H \) is a \( C^* \)-algebra homomorphism. Then

\[
H([x,y]) = H\left(\frac{xy - yx}{2}\right) = \frac{H(x)H(y) - H(y)H(x)}{2} = [H(x), H(y)],
\]

\[
H(x \circ y) = H\left(\frac{xy + yx}{2}\right) = \frac{H(x)H(y) + H(y)H(x)}{2} = H(x) \circ H(y)
\]

for all \( x, y \in \mathcal{A} \). So \( H \) is a Lie \( JC^* \)-algebra homomorphism.

We are going to investigate Lie \( JC^* \)-algebra homomorphisms between Lie \( JC^* \)-algebras.

**Theorem 5.1.** Let \( h : \mathcal{A} \to \mathcal{B} \) be a mapping satisfying \( h(0) = 0 \) and \( h(2^d u \circ y) = h(2^d u) \circ h(y) \) for all \( y \in \mathcal{A} \), all \( u \in \mathcal{U}(\mathcal{A}) \) and \( d = 0, 1, 2, \ldots \), for which there exists a function \( \varphi : \mathcal{A}^{mn+2} \to [0, \infty) \) such that

\[
\tilde{\varphi}(x_1, \ldots, x_{mn}, z, w) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \ldots, 2^j x_{mn}, 2^j z, 2^j w)
\]

\[
< \infty,
\]

\[
\left\| mn \sum_{m=0}^{mn-2C_{k-2}} h\left(\frac{\mu x_1 + \cdots + \mu x_{mn} + \frac{[z,w]}{mn}}{mn}ight) \right\|
\]

\[
+ m \sum_{i=1}^{n} h\left(\frac{\mu x_{m_i-m+1} + \cdots + \mu x_{m_i}}{m}\right)
\]

\[
- k \sum_{1 \leq i_1 < \cdots < i_k \leq mn} \mu h\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) - [h(z), h(w)]
\]

\[
\leq \varphi(x_1, \ldots, x_{mn}, z, w),
\]
\[(5.iii) \quad \|h(2^d u^*) - h(2^d u)^*\| \leq \varphi(2^d u, \ldots, 2^d u, 0, 0)_{m n \text{ times}}\]

for all \(\mu \in \mathbb{T}^1\), all \(u \in \mathcal{U}(A)\), all \(x_1, \ldots, x_m, z, w \in A\) and \(d = 0, 1, 2, \ldots\).

Assume

\[(5.iv) \quad \lim_{d \to \infty} \frac{h(2^d e)}{2^d} = e'.\]

Then the mapping \(h : A \to B\) is a Lie \(JC^*\)-algebra homomorphism.

**Proof.** By the same reasoning as in the proof of Theorem 4.1, there exists a unique \(C\)-linear involutive mapping \(H : A \to B\) such that

\[(5.v) \quad \|h(x) - H(x)\| \leq \frac{1}{2m s \cdot mn - 2C_{k-1}} \varphi(0, \ldots, 0, \frac{mx}{q}, \ldots, \frac{mx}{q}, 0, \ldots, 0)_{m - 2q \text{ times} \quad q \text{ times}} + \frac{1}{2m s \cdot mn - 2C_{k-1}} \varphi(0, \ldots, 0, \frac{mx}{q}, \ldots, \frac{mx}{q}, 0, \ldots, 0)_{m - 2q \text{ times} \quad q \text{ times}}\]

for all \(x \in A\).

It follows from (4.1) that

\[(5.1) \quad H(x) = \lim_{d \to \infty} \frac{h(2^d x)}{2^d}\]
for all \( x \in \mathcal{A} \). Let \( x_1 = \cdots = x_{mn} = 0 \) in (5.ii). Then we get

\[
\left\| mn_{mn-2} \frac{[z, w]}{C_k-2} h \left( \frac{[z, w]}{mn_{mn-2} C_k-2} \right) - [h(z), h(w)] \right\| \leq \varphi(0, \ldots, 0, z, w)
\]

for all \( z, w \in \mathcal{A} \). So

\[
\frac{1}{2^d} \left\| mn_{mn-2} \frac{[z, w]}{C_k-2} h \left( \frac{2^d [z, w]}{mn_{mn-2} C_k-2} \right) - [h(2^d z), h(2^d w)] \right\|
\]

\[
\leq \frac{1}{2^d} \varphi(0, \ldots, 0, 2^d z, 2^d w)
\]

(5.2)

\[
\leq \frac{1}{2^d} \varphi(0, \ldots, 0, 2^d z, 2^d w)
\]

for all \( z, w \in \mathcal{A} \). By (5.i), (5.1), and (5.2),

\[
mn_{mn-2} \frac{[z, w]}{C_k-2} H \left( \frac{[z, w]}{mn_{mn-2} C_k-2} \right)
\]

\[
= \lim_{d \to \infty} \frac{mn_{mn-2} \frac{[z, w]}{C_k-2} h \left( \frac{2^d [z, w]}{mn_{mn-2} C_k-2} \right)}{2^d}
\]

\[
= \lim_{d \to \infty} \frac{mn_{mn-2} \frac{[2^d z, 2^d w]}{C_k-2} h \left( \frac{[2^d z, 2^d w]}{mn_{mn-2} C_k-2} \right)}{2^d}
\]

\[
= \lim_{d \to \infty} \frac{1}{2^d} [h(2^d z), h(2^d w)]
\]

\[
= \lim_{d \to \infty} \frac{h(2^d z)}{2^d}, \frac{h(2^d w)}{2^d}
\]

\[
= [H(z), H(w)]
\]

for all \( z, w \in \mathcal{A} \). So

\[
H([z, w]) = mn_{mn-2} \frac{[z, w]}{C_k-2} H \left( \frac{[z, w]}{mn_{mn-2} C_k-2} \right) = [H(z), H(w)]
\]

for all \( z, w \in \mathcal{A} \).
Since \( h(2^d u \circ y) = h(2^d u) \circ h(y) \) for all \( y \in \mathcal{A} \), all \( u \in \mathcal{U}(\mathcal{A}) \) and \( d = 0, 1, 2, \ldots \),

\[
H(u \circ y) = \lim_{d \to \infty} \frac{1}{2^d} h(2^d u \circ y) = \lim_{d \to \infty} \frac{1}{2^d} h(2^d u) \circ h(y) = H(u) \circ h(y)
\]

(5.3)

for all \( y \in \mathcal{A} \) and all \( u \in \mathcal{U}(\mathcal{A}) \). By the additivity of \( H \) and (5.3),

\[
2^d H(u \circ y) = H(2^d u \circ y) = H(u \circ (2^d y)) = H(u) \circ h(2^d y)
\]

for all \( y \in \mathcal{A} \) and all \( u \in \mathcal{U}(\mathcal{A}) \). Hence

\[
H(u \circ y) = \frac{1}{2^d} H(u) \circ h(2^d y) = H(u) \circ \frac{1}{2^d} h(2^d y)
\]

(5.4)

for all \( y \in \mathcal{A} \) and all \( u \in \mathcal{U}(\mathcal{A}) \). Taking the limit in (5.4) as \( d \to \infty \), we obtain

\[
H(u \circ y) = H(u) \circ H(y)
\]

(5.5)

for all \( y \in \mathcal{A} \) and all \( u \in \mathcal{U}(\mathcal{A}) \). Since \( H \) is \( \mathbb{C} \)-linear and each \( x \in \mathcal{A} \) is a finite linear combination of unitary elements i.e., \( x = \sum_{j=1}^{d} \lambda_j u_j \) (\( \lambda_j \in \mathbb{C} \), \( u_j \in \mathcal{U}(\mathcal{A}) \)),

\[
H(x \circ y) = H\left( \sum_{j=1}^{d} \lambda_j u_j \circ y \right) = \sum_{j=1}^{d} \lambda_j H(u_j \circ y) = \sum_{j=1}^{d} \lambda_j H(u_j) \circ H(y)
\]

\[
= H\left( \sum_{j=1}^{d} \lambda_j u_j \right) \circ H(y) = H(x) \circ H(y)
\]

for all \( x, y \in \mathcal{A} \).

By (5.iv), (5.3), and (5.5),

\[
H(y) = H(e \circ y) = H(e) \circ h(y) = e' \circ h(y) = h(y)
\]

for all \( y \in \mathcal{A} \). So

\[
H(y) = h(y)
\]

for all \( y \in \mathcal{A} \).

Therefore, the mapping \( h : \mathcal{A} \to \mathcal{B} \) is a Lie \( JC^* \)-algebra homomorphism. \( \square \)
Theorem 5.2. Let \( h : \mathcal{A} \to \mathcal{B} \) be a mapping satisfying \( h(2x) = 2h(x) \) for all \( x \in \mathcal{A} \) for which there exists a function \( \varphi : \mathcal{A}^{mn+2} \to [0, \infty) \) satisfying (5.ii), (5.iii), (5.iii), and (5.iv) such that

\[
(5.vi) \quad \|h(2^d u \circ y) - h(2^d u) \circ h(y)\| \leq \varphi(u, y, 0, \ldots, 0)
\]

for all \( y \in \mathcal{A}, \) all \( u \in \mathcal{U}(\mathcal{A}) \) and \( d = 0, 1, 2, \ldots \). Then the mapping \( h : \mathcal{A} \to \mathcal{B} \) is a Lie \( JC^* \)-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique \( \mathbb{C} \)-linear mapping \( H : \mathcal{A} \to \mathcal{B} \) satisfying (5.v).

By (5.vi) and the assumption that \( h(2x) = 2h(x) \) for all \( x \in \mathcal{A}, \)

\[
\|h(2^d u \circ y) - h(2^d u) \circ h(y)\|
\]

\[
= \frac{1}{2^l} \|h(2^l \cdot 2^d u \circ 2^l y) - h(2^l \cdot 2^d u) \circ h(2^l y)\|
\]

\[
\leq \frac{1}{2^l} \varphi(2^l u, 2^l y, 0, \ldots, 0)
\]

\[
\leq \frac{1}{2^l} \varphi(2^l u, 2^l y, 0, \ldots, 0),
\]

which tends to zero as \( l \to \infty \) by (5.i). So

\[
h(2^d u \circ y) = h(2^d u) \circ h(y)
\]

for all \( y \in \mathcal{A}, \) all \( u \in \mathcal{U}(\mathcal{A}) \) and \( d = 0, 1, 2, \ldots \). But by (4.1),

\[
H(x) = \lim_{d \to \infty} \frac{1}{2^d} h(2^d x) = h(x)
\]

for all \( x \in \mathcal{A}. \)

The rest of the proof is similar to the proof of Theorem 5.1. \( \square \)

We are going to show the Cauchy-Rassias stability of Lie \( JC^* \)-algebra homomorphisms in Lie \( JC^* \)-algebras.

Theorem 5.3. Let \( h : \mathcal{A} \to \mathcal{B} \) be a mapping satisfying \( h(0) = 0 \) for which there exists a function \( \varphi : \mathcal{A}^{mn+4} \to [0, \infty) \) such that

\[
(5.vii) \quad \varphi(x_1, \ldots, x_{mn}, z, w, a, b)
\]

\[
= \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \ldots, 2^j x_{mn}, 2^j z, 2^j w, 2^j a, 2^j b) < \infty,
\]
\[
\left\| mn \left( m_{n-2} C_{k-2} h \left( \frac{\mu x_1 + \cdots + \mu x_{mn} + \frac{[z,w] + a \circ b}{mn}}{mn} \right) \right) + m \left( m_{n-2} C_{k-1} \sum_{i=1}^{n} h \left( \frac{\mu x_{mi-m+1} + \cdots + \mu x_{mi}}{m} \right) \right) \\
- k \sum_{1 \leq i_1 < \cdots < i_k \leq mn} \mu h \left( \frac{x_{i_1} + \cdots + x_{i_k}}{k} \right) - \left[ h(z), h(w) \right] \\
- h(a) \circ h(b) \right\| \leq \varphi(x_1, \ldots, x_{mn}, z, w, a, b)
\]

(5.8)

\[
\| h(2^d u^*) - h(2^d u^*) \| \leq \varphi(2^d u, \ldots, 2^d u, 0, 0, 0, 0)
\]

for all \( \mu \in \mathbb{T} \), all \( u \in \mathcal{U}(A) \), \( d = 0, 1, 2, \ldots \), and all \( x_1, \ldots, x_{mn}, z, w, a, b \in \mathcal{A} \). Then there exists a unique \( \mathcal{J}C^* \)-algebra homomorphism \( H : \mathcal{A} \to \mathcal{B} \) such that

(5.9)

\[
\| h(x) - H(x) \| \\
\leq \frac{1}{2ms m_{n-2} C_{k-1}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \times q \text{ times}}, 0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-q \times q \text{ times}}, \ldots, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \times q \text{ times}}, 0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-q \times q \text{ times}}, 0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2ms \times q \text{ times}}) \\
+ \frac{1}{2ms m_{n-2} C_{k-1}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \times q \text{ times}}, \ldots, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \times q \text{ times}}, 0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-q \times q \text{ times}}, \ldots, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \times q \text{ times}}, 0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-q \times q \text{ times}}, \ldots, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2ms \times q \text{ times}})
\]

for all \( x \in \mathcal{A} \).
Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive mapping $H : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (5.x). The rest of the proof is similar to the proof of Theorem 5.1. \hfill \Box


DEFINITION 6.1. A $\mathbb{C}$-linear mapping $D : \mathcal{A} \rightarrow \mathcal{A}$ is called a Lie $JC^*$-algebra derivation if $D : \mathcal{A} \rightarrow \mathcal{A}$ satisfies

\[
D(x \circ y) = (Dx) \circ y + x \circ (Dy),
\]

\[
D([x, y]) = [Dx, y] + [x, Dy],
\]

\[
D(x^*) = D(x)^*
\]

for all $x, y \in \mathcal{A}$.

REMARK 6.1. A $\mathbb{C}$-linear mapping $D : \mathcal{A} \rightarrow \mathcal{A}$ is a $C^*$-algebra derivation if and only if the mapping $D : \mathcal{A} \rightarrow \mathcal{A}$ is a Lie $JC^*$-algebra derivation.

Assume that $D$ is a Lie $JC^*$-algebra derivation. Then

\[
D(xy) = D([x, y] + x \circ y) = D([x, y]) + D(x \circ y)
\]

\[
= [Dx, y] + [x, Dy] + (Dx) \circ y + x \circ (Dy)
\]

\[
= (Dx)y + x(Dy)
\]

for all $x, y \in \mathcal{A}$. So $D$ is a $C^*$-algebra derivation.

Assume that $D$ is a $C^*$-algebra derivation. Then

\[
D([x, y]) = D\left(\frac{xy - yx}{2}\right) = \frac{(Dx)y + x(Dy) - (Dy)x - y(Dx)}{2}
\]

\[
= [Dx, y] + [x, Dy],
\]

\[
D(x \circ y) = D\left(\frac{xy + yx}{2}\right) = \frac{(Dx)y + x(Dy) + (Dy)x + y(Dx)}{2}
\]

\[
= (Dx) \circ y + x \circ (Dy)
\]

for all $x, y \in \mathcal{A}$. So $H$ is a Lie $JC^*$-algebra derivation.

We prove the Cauchy-Rassias stability of Lie $JC^*$-algebra derivations in Lie $JC^*$-algebras.
THEOREM 6.1. Let \( h : A \to A \) be a mapping satisfying \( h(0) = 0 \) for which there exists a function \( \varphi : A^{mn+4} \to [0, \infty) \) satisfying (5.vii) and (5.ix) such that

\[
\begin{align*}
\| mn_{mn-2C_{k-2}} h \left( \frac{\mu x_1 + \cdots + \mu x_m + \frac{[z,w] + ab}{mn} \mu x_{m+1} + \cdots + \mu x_{mn}}{mn} \right) \\
+ m_{mn-2C_{k-1}} \sum_{i=1}^{n} h \left( \frac{\mu x_{mi} + \cdots + \mu x_{m+1}}{m} \right) \\
- k \sum_{1 \leq i_1 < \cdots < i_k \leq mn} \mu h \left( \frac{x_{i_1} + \cdots + x_{i_k}}{k} \right) - [h(z), w] - [z, h(w)] \\
- h(a) \circ b - a \circ h(b) \| \\
\leq \varphi(x_1, \ldots, x_{mn}, z, w, a, b)
\end{align*}
\]

for all \( \mu \in \mathbb{T}^1 \) and all \( x_1, \ldots, x_{mn}, z, w, a, b \in A \). Then there exists a unique Lie \( JC^* \)-algebra derivation \( D : A \to A \) such that

\[
\| h(x) - D(x) \|
\leq \frac{1}{2ms_{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, 0, \ldots, 0)
\]

\[
+ \frac{1}{2ms_{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, 0, 0, 0, 0)
\]

\[
+ \frac{1}{2ms_{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, 0, 0, 0, 0)
\]

\[
+ \frac{1}{2ms_{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, 0, 0, 0, 0)
\]

\[
+ \frac{1}{2ms_{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, 0, 0, 0, 0)
\]

\[
+ \frac{1}{2ms_{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, 0, 0, 0, 0)
\]

\[
+ \frac{1}{2ms_{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, 0, 0, 0, 0)
\]

\[
+ \frac{1}{2ms_{mn-2C_{k-1}}} \varphi(0, \ldots, 0, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{m-2q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, 0, 0, 0, 0)
\]
for all $x \in \mathcal{A}$.

\textit{Proof.} Put $z = w = a = b = 0$ in (6.i). By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive mapping $D : \mathcal{A} \rightarrow \mathcal{A}$ satisfying (6.ii). The $\mathbb{C}$-linear mapping $D : \mathcal{A} \rightarrow \mathcal{A}$ is given by

\begin{equation}
D(x) = \lim_{d \to \infty} \frac{1}{2^d} h(2^d x)
\end{equation}

for all $x \in \mathcal{A}$.

It follows from (6.1) that

\begin{equation}
D(x) = \lim_{d \to \infty} \frac{h(2^d x)}{2^d}
\end{equation}

for all $x \in \mathcal{A}$. Let $x_1 = \cdots = x_{mn} = a = b = 0$ in (6.i). Then we get

\[
\left\| mn \frac{mn_{mn-2}k-2 h\left( \frac{[z, w]}{mn_{mn-2}k-2} \right)}{mn_{mn-2}k-2} - [h(z), w] - [z, h(w)] \right\| \\
\leq \varphi(0, \ldots, 0, z, w, 0, 0)
\]

for all $z, w \in \mathcal{A}$. So

\[
\frac{1}{2^d} \left\| mn_{mn-2}k-2 h\left( \frac{[2^d z, 2^d w]}{mn_{mn-2}k-2} \right) - [h(2^d z), 2^d w] \\
- [2^d z, h(2^d w)] \right\| \\
\leq \frac{1}{2^d} \varphi(0, \ldots, 0, 2^d z, 2^d w, 0, 0)
\]

for all $z, w \in \mathcal{A}$. By (5.vii), (6.2), and (6.3),

\[
\begin{align*}
& mn_{mn-2}k-2 D\left( \frac{[z, w]}{mn_{mn-2}k-2} \right) \\
= & \lim_{d \to \infty} \frac{mn_{mn-2}k-2 h\left( 2^d \frac{[z, w]}{mn_{mn-2}k-2} \right)}{2^d}
\end{align*}
\]
\[ mn \left[ \frac{m_{n-2}C_{k-2}h\left(\frac{2^d z, 2^d w}{mn_{n-2}C_{k-2}}\right)}{2^{2d}} \right] = \lim_{d \to \infty} \left[ \frac{h(2^d z, 2^d w)}{2^d} + \frac{2^d z, h(2^d w)}{2^d} \right] = [D(z), w] + [z, D(w)] \]

for all \( z, w \in A \). So

\[ D([z, w]) = mn_{n-2}C_{k-2}D\left(\frac{[z, w]}{mn_{n-2}C_{k-2}}\right) = [D(z), w] + [z, D(w)] \]

for all \( z, w \in A \).

Similarly, one can obtain that

\[ mn_{n-2}C_{k-2}D\left(\frac{a \circ b}{mn_{n-2}C_{k-2}}\right) = \lim_{d \to \infty} \left(\frac{2^{2d} a \circ b}{mn_{n-2}C_{k-2}}\right) = \lim_{d \to \infty} \left(\frac{h(2^d a, 2^d b)}{2^d} + \frac{2^d a \circ b}{2^d} \right) = D(a) \circ b + a \circ D(b) \]

for all \( a, b \in A \). So

\[ D(a \circ b) = mn_{n-2}C_{k-2}D\left(\frac{a \circ b}{mn_{n-2}C_{k-2}}\right) = D(a) \circ b + a \circ D(b) \]

for all \( a, b \in A \). Hence the \( \mathbb{C} \)-linear mapping \( D : A \to A \) is a Lie \( JC^* \)-algebra derivation satisfying (6.ii), as desired.

\[ \square \]

**Corollary 6.2.** Let \( h : A \to A \) be a mapping satisfying \( h(0) = 0 \)
for which there exist constants \( \theta \geq 0 \) and \( p \in [0, 1) \) such that

\[
\begin{align*}
&\left\| mn \frac{\mu x_1 + \cdots + \mu x_{mn} + \frac{[z,w]+a\circ b}{mn}}{C_{k-2}} \right\| \\
&+ m \frac{\mu x_{m+1} + \cdots + \mu x_m}{C_{k-1}} \\
&- k \sum_{1 \leq i_1 < \cdots < i_k \leq mn} \mu h\left( \frac{x_{i_1} + \cdots + x_{i_k}}{k} \right) - [h(z), w] - [z, h(w)] \\
&- h(a) \circ b - a \circ h(b) \right\| \\
&\leq \theta \sum_{j=1}^{mn} \|x_j\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p,
\end{align*}
\]

\[
\|h(2^d u^*) - h(2^d u)^*\| \leq mn2^{dp} \theta
\]

for all \( \mu \in \mathbb{T}^1 \), all \( u \in \mathcal{U}(A) \), \( d = 0, 1, 2, \ldots \), and all \( x_1, \ldots, x_{mn}, z, w, a, b \) \( \in A \). Then there exists a unique Lie \( JC^* \)-algebra derivation \( D : A \to A \) such that

\[
\|h(x) - D(x)\| \leq \frac{4m^{p-1}q^{1-p} \theta}{(2-2^p)mnC_{k-1}} \|x\|^p
\]

for all \( x \in A \).

**Proof.** Define \( \varphi(x_1, \ldots, x_{mn}, z, w, a, b) = \theta(\sum_{j=1}^{mn} \|x_j\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p) \), and apply Theorem 6.1. \( \square \)

**References**


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