RELATIVE INTEGRAL BASES
OVER A RAY CLASS FIELD

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Abstract. Let $K$ be a number field, $K_n$ its ray class field with conductor $n$ and $L$ a Galois extension of $K$ containing $K_n$. We prove that $L/K_n$ has a relative integral basis (RIB) under certain simple condition. Also we reduce the problem of the existence of a RIB to a quadratic extension of $K_n$ under some condition.

1. Introduction

Let $L$ be an algebraic number field, $K$ be a subfield of it. Let $\mathcal{O}_L$ and $\mathcal{O}_K$ be the rings of integers in $L$ and $K$, respectively. If $\mathcal{O}_L$ is free as $\mathcal{O}_K$-module, then we say that $L/K$ has a relative integral basis (RIB). Artin in [1] raised the problem: when does $L/K$ has a relative integral basis?

XianKe Zhang and FuHua Xu in [5] proved the existence of relative integral bases for extensions of $n$-cyclic number fields under some conditions. Mario Daberkow and Michael Pohst in [2] studied relative integral bases in relative quadratic extensions. Elena Soverchia in [4] showed the following: Let $H$ be the Hilbert class field of an algebraic number field $K$ and $L$ be a Galois extension of $K$ containing $H$. If the order of $\text{Gal}(L/H)$ is odd or if the 2-Sylow subgroups of $\text{Gal}(L/H)$ are not cyclic, then $L/H$ has a relative integral basis.

It is natural to investigate analogues of Soverchia’s work for more general class fields of $K$. Let $K$ be an algebraic number field and $K_n$ be its ray class field with conductor $n$ and with genus number 1 over $K$. Let $L/K$ be a Galois extension containing $K_n$. We suppose that $L/K$ is unramified at all primes $\mathcal{B}$ dividing $n\mathcal{O}_{K_n}$. For the convenience, we
assume that \( n \) is an integral divisor. We denote the discriminant of a field basis of \( L/K_n \) by \( \Delta \). In this paper, we will prove that \( L/K_n \) has a relative integral basis if \( h \) is an odd number or if \( \Delta \) is contained in \( K_n^2 \).

We also reduce the problem of the existence of a RIB to a quadratic extension of \( K_n \) if \( h \) is an even number and \( \Delta \) is not contained in \( K_n^2 \), where \( h \) be the class number of the field \( K_n \) (Theorem 4). We emphasize that our results (with respect to the ray class field \( K_n \) of \( K \)) generalize Sovorichia's results (with respect to the Hilbert class field \( H \) of \( K \)).

2. Relative integral basis over \( K_n \)

To prove our theorem, we need some lemmas.

We denote the relative discriminant of a field extension \( E/F \) by \( d(E|F) \).

**Lemma 1.** Let \( E/F \) an extension of number fields. Then there exists a non-zero fractional ideal \( \mathcal{B} \) in \( F \) such that \( d(E|F) = \langle \mathcal{B} \rangle^2 \), where \( \langle \mathcal{B} \rangle \) is the discriminant of a field basis of \( E/F \). Moreover, \( E/F \) has a RIB if and only if \( \mathcal{B} \) is principal.

**Proof.** See [1].

**Lemma 2.** Let \( E/K \) be a Galois extension of number fields containing \( K_n \). Suppose that \( E \) is unramified at all primes \( \mathcal{B} \) dividing \( n \mathcal{O}_{K_n} \). Then \( d(E|K_n) \) is stable under the action of \( \text{Gal}(K_n/K) \).

**Proof.** Let \( \mathcal{D}_{E/K} \) (respectively, \( \mathcal{D}_{E/K_n} \)) be the different of \( \mathcal{O}_E \) over \( K \) (respectively, \( K_n \)). Since \( \mathcal{D}_{E/K} \) (respectively, \( \mathcal{D}_{E/K_n} \)) is stable under the action of \( \text{Gal}(E/K) \) (respectively, \( \text{Gal}(E/K_n) \)), \( N_{E/K_n} \mathcal{D}_{E/K_n} = d(E|K_n) \) and \( K_n \) is unramified at any prime \( p \) in \( K \) which is below a prime dividing \( d(E|K_n) \), we have \( d(E|K_n) = p_1^{t_1} \cdots p_r^{t_r} \) for some prime ideals \( p_i \) in \( K \) and some integers \( t_i \). Hence \( d(E|K_n) \) is stable under the action of \( \text{Gal}(K_n/K) \).

**Lemma 3.** Suppose that the genus number of \( K_n \) over \( K \) is equal to 1. Then every ideal of \( K_n \) prime to \( n \) and stable under the action of \( \text{Gal}(K_n/K) \) is principal.

**Proof.** Let \( H \) be the Hilbert class field of \( K_n \) and \( G = \text{Gal}(H/K) \). Since the genus number of \( K_n \) over \( K \) is equal to 1, we have \( \text{Gal}(H/K_n) = G' \) and \( \text{Gal}(K_n/K) = G gon G' \), where \( G' \) is the commutator subgroup of \( G \).

Let \( I_n(K) \) (respectively, \( I(K_n) \)) be the ideal group generated by all fractional ideals in \( K \) prime to \( n \) (respectively, by all fractional ideals in \( K_n \)) and \( P_n,1(K) \) (respectively, \( P(K_n) \)) be the subgroup of \( I_n(K) \) generated
by the principal ideals $\beta \mathcal{O}_K$ with $\beta \in \mathcal{O}_K$ and $\beta \equiv 1 \mod n\mathcal{O}_K$ (respectively, of $I(K_n)$ generated by the principal ideals in $K_n$) where $\mathcal{O}_K$ is the ring of integers in $K$. Naturally, we obtain a chain of maps

$$G \xrightarrow{\text{natural map}} G' \xrightarrow{[,K]^{-1}} \frac{I_n(K)}{P_{n,1}(K)} \xrightarrow{\text{natural map}} \frac{I(K_n)}{P(K_n)} \xrightarrow{[,K_n]} G',$$

where $[,]_K$ and $[,]_{K_n}$ are Artin maps. A brief check of the coset representatives shows that this chain of maps is a transfer $V$ of $G$ into $G'$. By the principal ideal theorem of group theory, $V(\sigma) = 1$ for all $\sigma \in G$. This implies our assertion. \qed

**Theorem 4.** Let $K$ be an algebraic number field and $K_n$ be its ray class field with conductor $n$ and with genus number 1 over $K$. Let $L/K$ be a Galois extension containing $K_n$. We suppose that $L/K$ is unramified at all primes $\mathcal{B}$ dividing $n\mathcal{O}_K$ and that $n$ is an integral divisor. Let $h$ be the class number of the field $K_n$. Then we have the following:

1. If $h$ is an odd number or if $\Delta$ is contained in $K_n^2$, then $L/K_n$ has a RIB.

2. If $h$ is an even number and if $\Delta$ is not contained in $K_n^2$, then for the field $M = K_n(\sqrt{\Delta})$, $L/K_n$ has a RIB if and only if $M/K_n$ has a RIB.

**Proof.** Let $\mathcal{B}$ be a fractional ideal in $K_n$ such that $d(L|K_n) = \Delta \mathcal{B}^2$. Lemma 2 and Lemma 3 imply that $d(L|K_n)$ is principal. Hence if $h$ is an odd number, then $\mathcal{B}$ is principal. Suppose that $\Delta$ is contained in $K_n^2$. Then $\sqrt{\Delta} \mathcal{B}$ is stable under the action of $Gal(K_n/K)$. By Lemma 2 and Lemma 3, $\sqrt{\Delta} \mathcal{B}$ is principal. This implies that $\mathcal{B}$ is principal. Now we assume that $h$ is an even number and that $\Delta$ is not contained in $K_n^2$. We let $\mathcal{D}$ be a fractional ideal in $K_n$ such that $d(M|K_n) = 4\Delta \mathcal{D}^2$. From Lemma 2, $\mathcal{D} \mathcal{B}^{-1}$ is stable under the action of $Gal(K_n/K)$. Hence $\mathcal{D} \mathcal{B}^{-1}$ is principal. These and Lemma 1 prove the assertions. \qed

**Remark.** Replacing $H$ in [4, Lemma 2.2] by $K_n$, we obtain the following equivalent statements: the order of $Gal(L/K_n)$ is odd or the 2-Sylow subgroup of $G$ are not cyclic if and only if $\Delta$ is contained in $K_n^2$.

**Example.** For any prime $p$, let $\zeta_p = e^{2\pi i/p}$ and $K$ the rational number field. Then $K_p$ is the maximal real subfield $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ of $p$-th cyclotomic number field $\mathbb{Q}(\zeta_p)$ and satisfies the conditions in Theorem 4. Indeed, in narrow sense the field $\mathbb{Q}(\zeta_p)$ has genus number 1 over $K$ from the genus
number formula
\[ g(\mathbb{Q}(\zeta_p)) = \frac{e(p)}{[\mathbb{Q}(\zeta_p) : K]}, \]
given in [3, p.53], where \(e(p)\) denotes the ramification index of the prime \(p\) in \(\mathbb{Q}(\zeta_p)/K\). Thus the field \(K_p\) has genus number 1 over \(K\).

References


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