HYERS-ULAM STABILITY OF A CLOSED OPERATOR IN A HILBERT SPACE

GO HIRASAWA AND TAKESHI MIURA

Dedicated to Professor Sin-Ei Takahasi on his 60th birthday (Kanreki)

ABSTRACT. We give some necessary and sufficient conditions in order that a closed operator in a Hilbert space into another have the Hyers-Ulam stability. Moreover, we prove the existence of the stability constant for a closed operator. We also determine the stability constant in terms of the lower bound.

1. Introduction

It seems that S. M. Ulam [16, Chapter VI] first raised the stability problem of functional equations: "For what metric groups $G$ is it true that an $\varepsilon$-automorphism of $G$ is necessarily near to a strict automorphism?" An answer has been given in the following way. Let $E_1, E_2$ be two real Banach spaces and $f: E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$, the set of all real numbers, for each fixed $x \in E_1$. In 1941, D. H. Hyers [3] gave an answer to the problem above as follows. If there exists an $\varepsilon \geq 0$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in E_1$, then there exists a unique linear mapping $T: E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for every $x \in E_1$. This result is called the Hyers-Ulam stability of the additive Cauchy equation $g(x + y) = g(x) + g(y)$.

In 1978, Th. M. Rassias [9] introduced the new functional inequality and succeeded to extend the result of Hyers’ by weakening the condition for the Cauchy difference to be unbounded: If there exist an $\varepsilon \geq 0$ and
$0 \leq p < 1$ such that
\[ \|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \]
for all $x, y \in E_1$, then there exists a unique linear mapping $T : E_1 \to E_2$ such that
\[ \|f(x) - T(x)\| \leq \frac{2\theta}{|2 - 2^p|} \|x\|^p \]
for every $x \in E_1$. Since then several mathematicians were attracted to this result of Rassias and investigated a number of stability problems of functional equations. This stability phenomenon that was introduced and proved by Th. M. Rassias in his 1978 paper is called the Hyers-Ulam-Rassias stability. In 1991, Z. Gajda [1] solved the problem for $1 < p$, which was raised by Th. M. Rassias: In fact, the result of Rassias is valid for $1 < p$; Moreover, Z. Gajda gave an example that a similar stability result does not hold for $p = 1$. Another example was given by Th. M. Rassias and P. Šemrl [13, Theorem 2].

The second author, S. Miyajima and S. -E. Takahasi [7] introduced the notion of the Hyers-Ulam stability of a mapping between two normed linear spaces as follows:

**Definition 1.1.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces and $f$ be a (not necessarily linear) mapping from $X$ into $Y$. We say that $f$ has the Hyers-Ulam stability if there exists a constant $K \geq 0$ with the following property:

For any $v \in f(X)$, the range of $f$, $\varepsilon \geq 0$ and $u \in X$ with $\|f(u) - v\|_Y \leq \varepsilon$, there exists a $u_0 \in X$ such that $f(u_0) = v$ and $\|u - u_0\|_X \leq K\varepsilon$.

We call such $K \geq 0$ a HUS constant for $f$, and denote by $K_f$ the infimum of all HUS constants for $f$. If, in addition, $K_f$ becomes a HUS constant for $f$, then we call it the HUS constant for $f$.

Roughly speaking, if $f$ has the Hyers-Ulam stability, then to each "$\varepsilon$-approximate solution" $u$ of the equation $f(x) = v$ there corresponds an exact solution $u_0$ of the equation in a $K\varepsilon$-neighborhood of $u$.

In [7, 8], the second author, S. Miyajima and S. -E. Takahasi obtained some stability results for particular linear differential operators $D$: the $n$-th order linear differential operator with constant coefficients and the first order linear differential operator with a continuous function as coefficient. In fact, they gave a characterization in order that $D$ have the Hyers-Ulam stability. Among other things, for the first order linear differential operator $D$, the three authors above with H. Takagi
[8, 15] proved that the infimum $K_D$ becomes the minimum of all HUS constants: Moreover, they described $K_D$ completely.

H. Takagi, the second author and S. -E. Takahasi [14] considered a bounded linear operator $T$ from a Banach space $X$ into another Banach space $Y$. To display their result, we need some terminology. Let $\ker T$ be the kernel of $T$. Define the induced one-to-one linear operator $\hat{T}$ from the quotient Banach space $X/\ker T$ into $Y$ by

$$\hat{T}(u + \ker T) \overset{\text{def}}{=} Tu \quad \forall u \in X.$$  

Now, their result reads as follows:

**Theorem A ([14, Theorem 2]).** Let $X$ and $Y$ be two Banach spaces and $T$ be a bounded linear operator from $X$ into $Y$. Then the following statements are equivalent:

(i) $T$ has the Hyers-Ulam stability.

(ii) $T$ has closed range.

(iii) $\hat{T}^{-1}$ from $T(X)$ onto $X/\ker T$ is bounded.

Moreover, if one of (hence all of) the conditions (i), (ii), and (iii) is true, then $K_T = \|\hat{T}^{-1}\|$.

Theorem A states that $K_T = \|\hat{T}^{-1}\|$ is valid whenever $T$ has the Hyers-Ulam stability. However, the equality only means that the infimum of all HUS constants for $T$ is $\|\hat{T}^{-1}\|$. In other words, even if we restrict ourselves to a bounded linear operator $T$ between two Banach spaces, we do not know whether the minimum of all HUS constants for $T$ exists or not. O. Hatori, K. Kobayashi, H. Takagi and S. -E. Takahasi with the second author [2, Example] proved that the infimum of all HUS constants for a bounded linear operator between two Banach spaces need not be a HUS constant: That is, the minimum of all HUS constants does not exist in general.

In this paper, we are concerned with a closed operator $T$ defined on a linear subspace $\mathcal{D}(T)$ of a Hilbert space $G$ into a Hilbert space $H$. We first give some necessary and sufficient conditions in order that $T$ have the Hyers-Ulam stability: In fact, Theorem A is valid for a closed operator $T$ from $\mathcal{D}(T) \subset G$ into $H$. Moreover, we prove that $T$ has the Hyers-Ulam stability if and only if $T$ is lower semibounded. Among other things, we show that the infimum of all HUS constants for $T$ is also a HUS constant: Namely, the minimum of all HUS constants do exist. We also describe the HUS constant $K_T$ for $T$ in terms of the lower bound of $T$. 
2. Preliminaries

From now on, by an operator we shall mean a non-zero linear operator. Let $G$ and $H$ be Hilbert spaces with the norm $\| \cdot \|_G$ and $\| \cdot \|_H$, respectively. An operator $T$ with a domain $\mathcal{D}(T) \subset G$ into $H$ is said to be closed if its graph $\{(u, Tu) : u \in \mathcal{D}(T)\}$ is a closed subspace in the product Hilbert space $G \times H$. In other words, if $u_n \in \mathcal{D}(T)$ and $Tu_n \in H$ converge to $u_0 \in G$ and $v_0 \in H$, respectively, then $u_0 \in \mathcal{D}(T)$ and $v_0 = Tu_0$ holds. We remark that a bounded operator $T$ from $\mathcal{D}(T) = G$ into $H$ is a closed operator.

First, we note the notion of the Hyers-Ulam stability of a closed operator $T$. Indeed, the linearity of $T$ can make the condition simple.

**Remark 2.1.** Let $T$ be a closed operator from $\mathcal{D}(T) \subset G$ into $H$. Recall that $T$ is said to have the Hyers-Ulam stability if and only if there exists a constant $K > 0$ with the following property:

(a) For any $v \in T(\mathcal{D}(T))$, $\varepsilon \geq 0$ and $u \in \mathcal{D}(T)$ with $\|Tu - v\|_H \leq \varepsilon$ there exists a $u_0 \in \mathcal{D}(T)$ such that $Tu_0 = v$ and $\|u - u_0\|_G \leq K\varepsilon$.

We excluded the case where $K = 0$. In fact, if the condition (a) were true for $K = 0$, then taking $v = 0$, we would have $Tu = 0$ for every $u \in \mathcal{D}(T)$: This contradicts the hypothesis that an operator means non-zero. Now the linearity of $T$ implies that the condition (a) is equivalent to

(b) For any $\varepsilon \geq 0$ and $u \in \mathcal{D}(T)$ with $\|Tu\|_H \leq \varepsilon$ there exists a $u_0 \in \mathcal{D}(T)$ such that $Tu_0 = 0$ and $\|u - u_0\|_G \leq K\varepsilon$.

Put $\ker T \overset{\text{def}}{=} \{u \in \mathcal{D}(T) : Tu = 0\}$. The condition (b) is equivalent to

(c) For any $u \in \mathcal{D}(T)$ there exists a $u_0 \in \ker T$ such that $\|u - u_0\|_G \leq K\|Tu\|_H$.

Next, we define a lower semiboundedness of a closed operator.

**Definition 2.1.** Let $T$ be a closed operator from $\mathcal{D}(T) \subset G$ into $H$. We say that $T$ is lower semibounded if there exists a positive constant $\gamma > 0$ such that

\[
(2.1) \quad \|Tv\|_H \geq \gamma \|v\|_G \quad \forall v \in \mathcal{D}(T) \cap (\ker T)^\perp.
\]

Here, $(\ker T)^\perp$ stands for the orthogonal complement of the kernel $\ker T$ of $T$: More precisely, $(\ker T)^\perp$ is the set of all $x \in G$ which are orthogonal
to every $u \in \ker T$. We put
\[
\gamma(T) \overset{\text{def}}{=} \sup \{\gamma > 0 : \|Tv\|_H \geq \gamma\|v\|_G, \ \forall v \in \mathcal{D}(T) \cap (\ker T)^\perp\} \\
= \inf \{\|Tv\|_H/\|v\|_G : v \in \mathcal{D}(T) \cap (\ker T)^\perp, v \neq 0\}.
\]
(2.2)

We call $\gamma(T)$ the lower bound of $T$.

If $T$ is a closed operator from $\mathcal{D}(T) \subset G$ into $H$, then it is easy to see that $\ker T$ is a closed subspace of $G$ since $T$ is a closed operator. In particular, if $P$ is the orthogonal projection from $G$ onto $\ker T$, then $x - Px \in (\ker T)^\perp$ for every $x \in G$.

**Lemma 2.1.** Let $T$ be a closed operator from $\mathcal{D}(T) \subset G$ into $H$. If $T$ is lower semibounded, then $T$ has the Hyers-Ulam stability with a HUS constant $\gamma(T)^{-1}$.

**Proof.** Suppose that $T$ is lower semibounded with the lower bound $\gamma(T) > 0$. By definition, we have
\[
\|Tv\|_H \geq \gamma(T)\|v\|_G \quad \forall v \in \mathcal{D}(T) \cap (\ker T)^\perp.
\]
(2.3)

Let $P$ be the orthogonal projection from $G$ onto $\ker T$. Fix $u \in \mathcal{D}(T)$ arbitrarily, and put $u_0 \overset{\text{def}}{=} Pu \in \ker T$. Since $u - u_0 \in \mathcal{D}(T) \cap (\ker T)^\perp$, we have from (2.3) that
\[
\|u - u_0\|_G \leq \gamma(T)^{-1}\|T(u - u_0)\|_H = \gamma(T)^{-1}\|Tu\|_H.
\]

By Remark 2.1, this implies that $T$ has the Hyers-Ulam stability with a HUS constant $\gamma(T)^{-1}$. \hfill \Box

**Definition 2.2.** Let $T$ be a closed operator from $\mathcal{D}(T) \subset G$ into $H$. We define the induced one-to-one operator $\tilde{T}$ from $\mathcal{D}(T) \cap (\ker T)^\perp \subset G$ into $H$ by
\[
\tilde{T}v \overset{\text{def}}{=} Tv \quad \forall v \in \mathcal{D}(T) \cap (\ker T)^\perp.
\]

Since $T$ is closed, so is $\tilde{T}$.

**Remark 2.2.** Suppose that $T$ is a closed operator from $\mathcal{D}(T) \subset G$ into $H$. The induced operator $\tilde{T}$ as in Definition 2.2 is corresponding to $\tilde{T}$ as in (iii) of Theorem A.

To see this, we remark that the orthogonal complement $(\ker T)^\perp$ of $\ker T$ is isometrically isomorphic to the quotient Banach space $G/\ker T$ with the quotient norm $\| \cdot \|_q$. Indeed, $x + \ker T \mapsto Qx$ ($x \in G$) gives a
one-to-one onto correspondence between $G/\ker T$ and $(\ker T)^\perp$, where $Q$ denotes the orthogonal projection from $G$ onto $(\ker T)^\perp$; Since

$$
\|x + \ker T\|_q = \inf\{\|x + y\|_G : y \in \ker T\} \\
= \|Qx\|_G \quad \forall x \in G,
$$

(2.4)

$G/\ker T$ is isometrically isomorphic to $(\ker T)^\perp$ as a Banach space. If, in addition, we define an inner product $\langle \cdot, \cdot \rangle$ on $G/\ker T$ by

$$
\langle x + \ker T, y + \ker T \rangle \overset{\text{def}}{=} \langle Qx, Qy \rangle_G \quad \forall x, y \in G,
$$

(2.5)

then $G/\ker T$ becomes an inner product space. Here, $\langle \cdot, \cdot \rangle_G$ denotes the inner product on the Hilbert space $G$. It follows from (2.4) and (2.5) that

$$
\langle x + \ker T, x + \ker T \rangle = \langle Qx, Qx \rangle_G = \|x + \ker T\|^2_q
$$

for every $x \in G$. Consequently, $G/\ker T$ is isomorphic to $(\ker T)^\perp$ as a Hilbert space.

**Lemma 2.2.** Let $T$ be a closed operator from $\mathcal{D}(T) \subset G$ into $H$, $\tilde{T}$ be the induced operator as in Definition 2.2. Each of the following two statements implies the other:

(i) $\tilde{T}^{-1}$ is bounded.

(ii) $T$ is lower semibounded.

If, in addition, one of the conditions (i) and (ii) is true, then we have $\|\tilde{T}^{-1}\| = \gamma(T)^{-1}$.

**Proof.** Put $\tilde{H} \overset{\text{def}}{=} \{Tv \in H : v \in \mathcal{D}(T) \cap (\ker T)^\perp\}$. Note that the inverse operator $\tilde{T}^{-1}$ from $\tilde{H}$ into $G$ is well-defined since $\tilde{T}$ is an injection. If we assume that $1/0$ means $\infty$, then we obtain

$$
\sup_{w \in \tilde{H} \setminus \{0\}} \frac{\|\tilde{T}^{-1}w\|}{\|w\|} = \sup \left\{ \frac{\|v\|_G}{\|Tv\|_H} : v \in \mathcal{D}(T) \cap (\ker T)^\perp, v \neq 0 \right\} \\
= \left[ \inf \left\{ \frac{\|Tv\|_H}{\|v\|_G} : v \in \mathcal{D}(T) \cap (\ker T)^\perp, v \neq 0 \right\} \right]^{-1}.
$$

It follows that $\tilde{T}^{-1}$ is bounded if and only if $T$ is lower semibounded. In this case, the identity above with (2.2) shows that $\|\tilde{T}^{-1}\| = \gamma(T)^{-1}$. ☐
3. Main results

THEOREM 3.1. Let $T$ be a closed operator from $\mathcal{D}(T) \subset G$ into $H$, $\widetilde{T}$ be the induced operator as in Definition 2.2. The following assertions are equivalent:

(i) $T$ has the Hyers-Ulam stability.
(ii) $T$ has closed range.
(iii) $\widetilde{T}^{-1}$ is bounded.
(iv) $T$ is lower semibounded.

Moreover, if one of the conditions above is true, then $K_T = \|\widetilde{T}^{-1}\| = \gamma(T)^{-1}$.

Proof. We shall prove that (i) $\iff$ (iv) $\iff$ (iii) $\iff$ (ii).

(i) $\Rightarrow$ (iv). Suppose that $T$ has the Hyers-Ulam stability. By (c) of Remark 2.1, there exists a constant $K > 0$ with the following property: For any $u \in \mathcal{D}(T)$ there exists a $u_0 \in \ker T$ such that $\|u - u_0\|_G \leq K\|Tu\|_H$. Pick $u \in \mathcal{D}(T) \cap (\ker T)^\perp$ arbitrarily. By hypothesis, there exists a $u_0 \in \ker T$ such that $\|u - u_0\|_G \leq K\|Tu\|_H$. Since $u \in (\ker T)^\perp$ and since $u_0 \in \ker T$, we get

$$\|u\|_G^2 \leq \|u\|_G^2 + \|u_0\|_G^2 = \|u - u_0\|_G^2.$$ 

Since $u$ was arbitrary, we thus obtain

$$\|Tu\|_H \geq K^{-1}\|u\|_G \quad \forall u \in \mathcal{D}(T) \cap (\ker T)^\perp.$$ 

This implies that $T$ is lower semibounded.

(iv) $\Rightarrow$ (i) and (iv) $\iff$ (iii). These are direct consequences of Lemma 2.1 and 2.2, respectively.

Although the equivalence of (iii) and (ii) is well-known, here we give a proof.

(iii) $\Rightarrow$ (ii). Suppose $\widetilde{T}^{-1}$ is bounded. We shall show that if $Tu_n$ ($u_n \in \mathcal{D}(T)$) converges to an element, say $w \in H$, then $w = Tu_0$ for some $u_0 \in \mathcal{D}(T)$. Let $Q$ be the orthogonal projection from $G$ onto $(\ker T)^\perp$. Since $\widetilde{T}^{-1}$ is bounded,

$$\|v\|_G \leq \|\widetilde{T}^{-1}\| \|Tv\|_H \quad \forall v \in \mathcal{D}(T) \cap (\ker T)^\perp.$$ 

Note that $Qu_n \in \mathcal{D}(T)$ since $\mathcal{D}(T)$ is a linear space, which contains $\ker T$. It follows from (3.7) that

$$\|Qu_n - Qu_m\|_G \leq \|\widetilde{T}^{-1}\| \|TQ(u_n - u_m)\|_H = \|\widetilde{T}^{-1}\| \|Tu_n - Tu_m\|_H,$$
and hence \( \{Q u_n\} \) is a Cauchy sequence of \((\ker T)^\perp\). Since \((\ker T)^\perp\) is closed, \(Q u_n\) converges to an element, say \(v_0 \in (\ker T)^\perp\). Because \(T\) is a closed operator, we get \(v_0 \in \mathcal{D}(T)\) and \(w = Tv_0\).

(ii) \(\Rightarrow\) (iii). Suppose that \(T\) has closed range. That is, the range
\[
\tilde{H} = \{Tv \in H : v \in \mathcal{D}(T) \cap (\ker T)^\perp\}
\]
of \(T\) is a closed subspace of \(H\). Since \(\tilde{T}^{-1}\) is a closed operator from the Hilbert space \(\tilde{H}\) into \(G\), it follows from the closed graph theorem that \(\tilde{T}^{-1}\) is bounded.

Now, suppose that one of the conditions (i), (ii), (iii) and (iv) is true. We show that the infimum \(K_T\) of all HUS constants for \(T\) satisfies
\[
K_T = \|\tilde{T}^{-1}\| = \gamma(T)^{-1}.
\]
By Lemma 2.2, it is enough to prove that \(K_T = \gamma(T)^{-1}\). To do this, fix a HUS constant \(K_0\) for \(T\) arbitrarily. A quite similar argument to (i) \(\Rightarrow\) (iv) shows that \(\|Tu\|_H \geq K_0^{-1}\|u\|_G\) for every \(u \in \mathcal{D}(T) \cap (\ker T)^\perp\). By the definition of the lower bound, we get \(\gamma(T) \geq K_0^{-1}\). Since \(K_0\) was arbitrary, we obtain \(\gamma(T)^{-1} \leq K_T\). Recall that \(\gamma(T)^{-1}\) is a HUS constant for \(T\), by Lemma 2.1, and hence \(K_T \leq \gamma(T)^{-1}\). We now obtain \(K_T = \gamma(T)^{-1}\), and the proof is complete. \(\square\)

**Corollary 3.2.** Let \(T\) be a closed operator from \(\mathcal{D}(T) \subset G\) into \(H\). If \(T\) has the Hyers-Ulam stability, then \(K_T\) is the HUS constant for \(T\).

**Proof.** By Theorem 3.1, we see that \(K_T = \gamma(T)^{-1}\). Since \(\gamma(T)^{-1}\) is a HUS constant for \(T\), by Lemma 2.1, we conclude that \(K_T\) is the HUS constant for \(T\). \(\square\)

The authors believe that Corollary 3.2 is interesting since the infimum \(K_S\) of all HUS constants for a bounded operator \(S\) between two Banach spaces need not be a HUS constant (cf. [2, Example]). In other words, although the infimum \(K_S\) exists, \(K_S\) is not necessarily the minimum.

We recall that every closed operator \(T\) from \(\mathcal{D}(T) \subset G\) into \(H\) can be regarded as a bounded operator from a Hilbert space into \(H\). In fact, put \(G_0 \overset{\text{def}}{=} \mathcal{D}(T)\) as a set. We define
\[
< u, v >_{G_0} \overset{\text{def}}{=} < u, v >_G + < Tu, Tv >_H \quad \forall u, v \in G_0,
\]
which becomes an inner product on \(G_0\). Here \(< \cdot, \cdot >_G\) and \(< \cdot, \cdot >_H\) denote the inner product on \(G\) and \(H\), respectively. Since \(T\) is a closed operator, we see that \(G_0\) is complete with respect to the induced norm
\[
\|u\|_G_0 \overset{\text{def}}{=} \sqrt{< u, u >_{G_0}} \quad \text{for every} \ u \in G_0.
\]
Hence \(G_0\) is a Hilbert space. We now consider the operator \(T_0\) from \(G_0\) into \(H\) defined by
\[
T_0 u \overset{\text{def}}{=} Tu \quad \forall u \in G_0.
\]
Then $T_0$ is a well-defined bounded operator since
\[ \|T_0 u\|_H^2 \leq \|u\|_G^2 + \|T_0 u\|_H^2 = \|u\|_{G_0}^2 \quad \forall u \in G_0 \]
by (3.8) and (3.9).

Next, we are concerned with the Hyers-Ulam stability of $T_0$. Moreover, we describe the HUS constant $K_{T_0}$.

**Theorem 3.3.** Let $T$ be a closed operator from $\mathcal{D}(T) \subset G$ into $H$. Let $T_0$ be a bounded operator from $G_0$ into $H$ defined by (3.9). The following assertions are equivalent:

(i) $T$ has the Hyers-Ulam stability.

(ii) $T_0$ has the Hyers-Ulam stability.

Moreover, if one of the conditions (i) and (ii) is true, then the HUS constants $K_T$, $K_{T_0}$ and the lower bounds $\gamma(T)$, $\gamma(T_0)$ are connected with the following relations:

(3.10) \[ K_T = (\gamma(T))^{-1}, \quad K_{T_0} = (\gamma(T_0))^{-1} \quad \text{and} \]

(3.11) \[ K_{T_0}^2 = K_T^2 + 1. \]

*Proof.* (i) $\Rightarrow$ (ii). Suppose that $T$ has the Hyers-Ulam stability with a HUS constant $K$. We prove that for any $u \in G_0$ there exists a $u_0 \in \ker T_0$ such that $\|u - u_0\|_{G_0} \leq \sqrt{K^2 + 1} \|Tu\|_H$. To do this, pick $u \in G_0$ arbitrarily. Recall that $G_0 = \mathcal{D}(T)$, by definition, and that $\ker T = \ker T_0$. Since $T$ is assumed to have the Hyers-Ulam stability, there exists a $u_0 \in \ker T_0$ such that $\|u - u_0\|_G \leq K \|Tu\|_H$. Adding the term $\|Tu\|_H^2 = \|Tu - T_0u_0\|_H^2$ to the both sides of the last inequality, we obtain
\[ \|u - u_0\|_G^2 + \|Tu - T_0u_0\|_H^2 \leq (K^2 + 1) \|Tu\|_H^2. \]

It follows from (3.8) and (3.9) that $\|u - u_0\|_{G_0} \leq \sqrt{K^2 + 1} \|T_0u\|_H$, which implies that $T_0$ has the Hyers-Ulam stability with a HUS constant $\sqrt{K^2 + 1}$. In particular, $K_{T_0} \leq \sqrt{K^2 + 1}$, and so $K_{T_0} \leq \sqrt{K_T^2 + 1}$.

(ii) $\Rightarrow$ (i). If $T_0$ has the Hyers-Ulam stability with a HUS constant $K_0$, then to each $u \in G_0$ there corresponds a $u_0 \in \ker T_0$ such that
\[ \sqrt{\|u - u_0\|_G^2 + \|T_0u\|_H^2} = \|u - u_0\|_{G_0} \leq K_0 \|T_0u\|_H, \]
which implies that

(3.12) \[ \|u - u_0\|_G \leq \sqrt{K_0^2 - 1} \|Tu\|_H. \]
Hence $T$ has the Hyers-Ulam stability with a HUS constant $\sqrt{K_0^2 - 1}$. We especially obtain $K_T \leq \sqrt{K_0^2 - 1}$, and hence $K_T \leq \sqrt{K_{T_0}^2 - 1}$.

Suppose one of (hence both of) the conditions (i) and (ii) is true. By Theorem 3.1 and Corollary 3.2, we see that $K_T = \gamma(T)^{-1}$ and $K_{T_0} = \gamma(T_0)^{-1}$ are the HUS constants for $T$ and $T_0$, respectively. As proved above, $K_{T_0} \leq \sqrt{K_{T_0}^2 + 1}$ and $K_T \leq \sqrt{K_{T_0}^2 - 1}$. Consequently,

$$K_{T_0}^2 \leq K_T^2 + 1 \leq (K_{T_0}^2 - 1) + 1 = K_{T_0}^2,$$

and hence $K_{T_0}^2 = K_T^2 + 1$. \hfill \Box

REMARK 3.1. If we apply Theorem 3.1, then we obtain other equivalent conditions in order that $T_0$ have the Hyers-Ulam stability: Moreover, $K_{T_0}$ can be described by the induced operator $\tilde{T_0}$.

ACKNOWLEDGEMENT. The second author is partially supported by the Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science.

References


Go Hirasawa, Department of Mathematics, Nippon Institute of Technology, Miyashiro, Saitama 345-8501, Japan
E-mail: hirasawa1@muh.biglobe.ne.jp

Takeshi Miura, Department of Basic Technology, Applied Mathematics and Physics, Yamagata University, Yonezawa 992-8510, Japan
E-mail: miura@yz.yamagata-u.ac.jp