

FUZZY (WEAK) IMPLICATIVE HYPER K -IDEALS

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ABSTRACT. Fuzzy (weak) implicative hyper K -ideals are introduced, and relations among fuzzy weak implicative hyper K -ideals, fuzzy implicative hyper K -ideals and fuzzy hyper K -ideals are discussed.

1. Introduction

The study of BCK -algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then many researches worked in this area. The hyperstructure theory (called also multialgebras) is introduced in 1934 by F. Marty [9] at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia, Japan and Iran. Hyperstructures have many applications to several sectors of both pure and applied sciences. Recently in [8] Y. B. Jun et al. introduced and studied hyper BCK -algebra which is a generalization of a BCK -algebra. In [1] and [2] R. A. Borzooei et al. constructed the hyper K -algebras, and studied (weak) implicative hyper K -ideals in hyper K -algebras. In [7] and [6] Jun and Shim studied the fuzzy (positive implicative) hyper K -ideals in hyper K -algebras. In this paper, we introduce the notion of fuzzy (weak) implicative hyper K -ideals, and investigate related properties. We give relations among fuzzy weak implicative hyper K -ideals, fuzzy implicative hyper K -ideals, and fuzzy hyper K -ideals.

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2. Preliminaries

We include some elementary aspects of hyper K -algebras that are necessary for this paper, and for more details we refer to [2] and [10]. Let H be a non-empty set endowed with a hyper operation “ \circ ”, that is, \circ is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. For two subsets A and B of H , denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$.

By a *hyper I -algebra* we mean a non-empty set H endowed with a hyper operation “ \circ ” and a constant 0 satisfying the following axioms:

$$(H1) \quad (x \circ z) \circ (y \circ z) < x \circ y,$$

$$(H2) \quad (x \circ y) \circ z = (x \circ z) \circ y,$$

$$(H3) \quad x < x,$$

$$(H4) \quad x < y \text{ and } y < x \text{ imply } x = y$$

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$ and $\exists b \in B$ such that $a < b$. If a hyper I -algebra $(H, \circ, 0)$ satisfies an additional condition:

$$(H5) \quad 0 < x \text{ for all } x \in H,$$

then $(H, \circ, 0)$ is called a *hyper K -algebra* (see [2]).

In a hyper I -algebra H , the following hold (see [2, Proposition 3.4]):

$$(a1) \quad (A \circ B) \circ C = (A \circ C) \circ B. \quad (a2) \quad x \circ (x \circ y) < y.$$

$$(a3) \quad x \circ y < z \Leftrightarrow x \circ z < y. \quad (a4) \quad A \circ B < C \Leftrightarrow A \circ C < B.$$

$$(a5) \quad (x \circ z) \circ (x \circ y) < y \circ z. \quad (a6) \quad (A \circ C) \circ (B \circ C) < A \circ B.$$

$$(a7) \quad A \circ (A \circ B) < B. \quad (a8) \quad A < A.$$

$$(a9) \quad A \subseteq B \text{ implies } A < B.$$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H .

A nonempty subset I of a hyper K -algebra H is called a *weak hyper K -ideal* of H (see [2]) if it satisfies

$$(I1) \quad 0 \in I,$$

$$(I2) \quad (\forall x, y \in H) \quad (x \circ y \subseteq I, y \in I \Rightarrow x \in I).$$

A nonempty subset I of a hyper K -algebra H is called a *hyper K -ideal* of H (see [2]) if it satisfies (I1) and

$$(I3) \quad (\forall x, y \in H) \quad (x \circ y < I, y \in I \Rightarrow x \in I).$$

We now review some fuzzy logic concepts. A *fuzzy set* in a set X is a function $\bar{A} : X \rightarrow [0, 1]$. For a fuzzy set \bar{A} in X and $\alpha \in [0, 1]$ define $U(\bar{A}; \alpha)$ to be the set

$$U(\bar{A}; \alpha) := \{x \in X \mid \bar{A}(x) \geq \alpha\},$$

which is called a *level set* of \bar{A} .

A fuzzy set \bar{A} in H is called a *fuzzy weak hyper K -ideal* of H (see [7]) if it satisfies

$$(\forall x, y \in H) (\bar{A}(0) \geq \bar{A}(x) \geq \min\left\{\inf_{a \in x \circ y} \bar{A}(a), \bar{A}(y)\right\}).$$

A fuzzy set \bar{A} in H is called a *fuzzy hyper K -ideal* of H (see [7]) if it satisfies the following conditions:

$$(F1) (\forall x, y \in H) (x < y \Rightarrow \bar{A}(x) \geq \bar{A}(y)),$$

$$(F2) (\forall x, y \in H) (\bar{A}(x) \geq \min\left\{\inf_{a \in x \circ y} \bar{A}(a), \bar{A}(y)\right\}).$$

3. Fuzzy (weak) implicative hyper K -ideals

In what follows let H denote a hyper K -algebra unless otherwise specified.

DEFINITION 3.1. [1] A nonempty subset I of H is called a *weak implicative hyper K -ideal* of H if it satisfies (I1) and

$$(\forall x, y, z \in H) ((x \circ z) \circ (y \circ x) \subseteq I, z \in I \Rightarrow x \in I).$$

DEFINITION 3.2. A fuzzy set \bar{A} in H is called a *fuzzy weak implicative hyper K -ideal* of H if it satisfies

$$(\forall x, y, z \in H) (\bar{A}(0) \geq \bar{A}(x) \geq \min\left\{\inf_{a \in (x \circ z) \circ (y \circ x)} \bar{A}(a), \bar{A}(z)\right\}).$$

EXAMPLE 3.3. Let $H = \{0, a, b\}$ be a hyper K -algebra with the following Cayley table:

\circ	0	a	b
0	{0}	{0}	{0}
a	{a}	{0, a}	{a}
b	{a, b}	{0, a}	{0, a}

Define a fuzzy set \bar{A} in H by $\bar{A}(0) = \bar{A}(b) = 0.6$ and $\bar{A}(a) = 0.3$. Then \bar{A} is a fuzzy weak implicative hyper K -ideal of H .

DEFINITION 3.4. [1] A nonempty subset I of H is called an *implicative hyper K -ideal* of H if it satisfies (I1) and

$$(\forall x, y, z \in H) ((x \circ z) \circ (y \circ x) < I, z \in I \Rightarrow x \in I).$$

DEFINITION 3.5. A fuzzy set \bar{A} in H is called a *fuzzy implicative hyper K -ideal* of H if it satisfies (F1) and

$$(\forall x, y, z \in H) (\bar{A}(x) \geq \min\left\{\inf_{a \in (x \circ z) \circ (y \circ x)} \bar{A}(a), \bar{A}(z)\right\}).$$

EXAMPLE 3.6. Let $H = \{0, a, b\}$ be a hyper K -algebra with the following Cayley table:

\circ	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, b\}$	$\{a\}$
b	$\{b\}$	$\{0, b\}$	$\{0, b\}$

Define a fuzzy set \bar{A} in H by $\bar{A}(0) = \bar{A}(b) = 0.7$ and $\bar{A}(a) = 0.07$. Then \bar{A} is a fuzzy implicative hyper K -ideal of H . But a fuzzy set \bar{B} in H given by $\bar{B}(0) = \bar{B}(a) = 0.5$ and $\bar{B}(b) = 0.05$ is not a fuzzy implicative hyper K -ideal of H since $b < a$ but $\bar{B}(b) \not\geq \bar{B}(a)$.

THEOREM 3.7. *Every fuzzy implicative hyper K -ideal is a fuzzy hyper K -ideal.*

Proof. Let \bar{A} be a fuzzy implicative hyper K -ideal of H . Since $(x \circ z) \circ (y \circ x) < x \circ y$ for any $x, y, z \in H$, $\exists s \in (x \circ z) \circ (y \circ x)$ and $\exists t \in x \circ y$ such that $s < t$. By (F1), we have $\bar{A}(s) \geq \bar{A}(t)$. Thus we get $\inf_{a \in (x \circ z) \circ (y \circ x)} \bar{A}(a) \geq \inf_{a \in x \circ y} \bar{A}(a)$. Hence we have: for all $x, y \in H$, $\bar{A}(x) \geq \min\{\inf_{a \in (x \circ 0) \circ (y \circ x)} \bar{A}(a), \bar{A}(0)\} \geq \min\{\inf_{a \in (x \circ 0) \circ (y \circ x)} \bar{A}(a), \bar{A}(y)\} \geq \min\{\inf_{a \in x \circ y} \bar{A}(a), \bar{A}(y)\}$. Therefore \bar{A} is a fuzzy hyper K -ideal of H . \square

The converse of Theorem 3.7 may not be true as seen in the following example.

EXAMPLE 3.8. Let $H = \{0, a, b, c\}$ be a hyper K -algebra with the following Cayley table:

\circ	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{0\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{0, a\}$	$\{c\}$	$\{0, a, c\}$

Define a fuzzy set \bar{A} in H by $\bar{A}(0) = 0.7$, $\bar{A}(a) = 0.2$ and $\bar{A}(b) = \bar{A}(c) = 0.5$. Then \bar{A} is a fuzzy hyper K -ideal of H , but \bar{A} is not a fuzzy implicative hyper K -ideal of H . since

$$\bar{A}(b) \not\geq \min\{\inf_{x \in (b \circ 0) \circ (a \circ b)} \bar{A}(x), \bar{A}(0)\} = \bar{A}(0).$$

We give a condition for a fuzzy hyper K -ideal to be a fuzzy implicative hyper K -ideal.

THEOREM 3.9. Let \bar{A} be a fuzzy hyper K -ideal of H . If H satisfies the following condition

$$(1) \quad (\forall x \in H) (\forall B \subseteq H) (x \in x \circ B),$$

then \bar{A} is a fuzzy implicative hyper K -ideal of H .

Proof. Assume H satisfies the (1). Then we have

$$(\forall x, y \in H) (\forall C \subseteq H) (x \in C \Rightarrow x \in C \circ y).$$

Thus

$$\begin{aligned} \bar{A}(x) &\geq \inf_{a \in x \circ (y \circ x)} \bar{A}(a) \\ &\geq \inf_{b \in (x \circ (y \circ x)) \circ z} \bar{A}(b) \\ &\geq \min\left\{ \inf_{b \in (x \circ (y \circ x)) \circ z} \bar{A}(b), \bar{A}(z) \right\} \\ &= \min\left\{ \inf_{b \in (x \circ z) \circ (y \circ x)} \bar{A}(b), \bar{A}(z) \right\}. \end{aligned}$$

Hence \bar{A} is a fuzzy implicative hyper K -ideal of H . \square

THEOREM 3.10. Every fuzzy implicative hyper K -ideal is a fuzzy weak implicative hyper K -ideal.

Proof. Let \bar{A} be a fuzzy implicative hyper K -ideal of H . Since $0 < x$ for all $x \in H$, it follows that $\bar{A}(0) \geq \bar{A}(x)$ for all $x \in H$. Thus \bar{A} be a fuzzy weak implicative hyper K -ideal of H . \square

The converse of Theorem 3.10 may not be true as seen in the following example.

EXAMPLE 3.11. In Example 3.6, \bar{B} is a fuzzy weak implicative hyper K -ideal of H .

In general, a fuzzy weak implicative hyper K -ideal may not be a fuzzy weak hyper K -ideal as seen in the following example.

EXAMPLE 3.12. Let $H = \{0, a, b\}$ be a hyper K -algebra with the following Cayley table:

\circ	0	a	b
0	$\{0, a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$
a	$\{a\}$	$\{0, a, b\}$	$\{a, b\}$
b	$\{a, b\}$	$\{0, a\}$	$\{0, a, b\}$

Define a fuzzy set \bar{A} in H by $\bar{A}(0) = 0.8$, $\bar{A}(a) = 0.7$ and $\bar{A}(b) = 0.07$. Then \bar{A} is a fuzzy weak implicative hyper K -ideal of H . But \bar{A} is not

fuzzy weak hyper K -ideal of H since $\bar{A}(b) \not\geq \min\{\inf_{x \in (b \circ a)} \bar{A}(x), \bar{A}(a)\} = \bar{A}(a)$.

In general, a fuzzy weak hyper K -ideal may not be a fuzzy weak implicative hyper K -ideal as seen in the following example.

EXAMPLE 3.13. Let $H = \{0, a, b\}$ be a hyper K -algebra with the following Cayley table:

\circ	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a, b\}$	$\{b\}$
b	$\{b\}$	$\{0, a, b\}$	$\{0, a\}$

Define a fuzzy set \bar{A} in H by $\bar{A}(0) = 0.8$, $\bar{A}(a) = 0.5$ and $\bar{A}(b) = 0.2$. Then \bar{A} is fuzzy weak hyper K -ideal of H . But \bar{A} is not a fuzzy weak implicative hyper K -ideal of H since

$$\bar{A}(b) \not\geq \min\{\inf_{x \in (b \circ 0) \circ (a \circ b)} \bar{A}(x), \bar{A}(0)\} = \bar{A}(a).$$

THEOREM 3.14. Let \bar{A} be a fuzzy weak hyper K -ideal of H . Then the following assertions are true:

- (i) If H satisfies the condition (1), then \bar{A} is a fuzzy weak implicative hyper K -ideal of H .
- (ii) If $0 \in H$ is a right scalar element and if \bar{A} is a fuzzy weak implicative hyper K -ideal of H , then $\bar{A}(x) \geq \inf_{a \in x \circ (y \circ x)} \bar{A}(a)$ for all $x, y \in H$.

Proof. (i) It is similar to the proof of Theorem 3.9.

(ii) Let $0 \in H$ be a right scalar element and \bar{A} be a fuzzy weak implicative hyper K -ideal of H . Then we have

$$\begin{aligned} \bar{A}(x) &\geq \min\{\inf_{a \in (x \circ 0) \circ (y \circ x)} \bar{A}(a), \bar{A}(0)\} \\ &= \min\{\inf_{a \in x \circ (y \circ x)} \bar{A}(a), \bar{A}(0)\} \\ &= \inf_{a \in x \circ (y \circ x)} \bar{A}(a) \end{aligned}$$

for all $x, y \in H$. This completes the proof. \square

THEOREM 3.15. If \bar{A} is a fuzzy (weak) implicative hyper K -ideal of H , then the level set

$$U(\bar{A}; \alpha) := \{x \in H \mid \bar{A}(x) \geq \alpha\}$$

is a (weak) implicative hyper K -ideal of H when $U(\bar{A}; \alpha) \neq \emptyset$ for $\alpha \in [0, 1]$.

Proof. Let \bar{A} be a fuzzy implicative hyper K -ideal of H . Assume that $U(\bar{A}; \alpha) \neq \emptyset$ for $\alpha \in [0, 1]$. Then there exists $a \in U(\bar{A}; \alpha)$ and so $\bar{A}(a) \geq \alpha$. Since $0 < x$ for all $x \in H$, it follows from (F1) that $\bar{A}(0) \geq \bar{A}(x)$. Thus $0 \in U(\bar{A}; \alpha)$. Let $x, y, z \in H$ be such that $(x \circ z) \circ (y \circ x) \in U(\bar{A}; \alpha)$ and $z \in U(\bar{A}; \alpha)$. Then $\bar{A}(z) \geq \alpha$ and $\bar{A}(b) \geq \alpha$ for all $b \in (x \circ z) \circ (y \circ x)$. Hence $\inf_{b \in (x \circ z) \circ (y \circ x)} \bar{A}(b) \geq \alpha$. Thus we have

$$\bar{A}(x) \geq \min\left\{\inf_{b \in (x \circ z) \circ (y \circ x)} \bar{A}(b), \bar{A}(z)\right\} \geq \alpha.$$

Therefore $x \in U(\bar{A}; \alpha)$ and this implies that $U(\bar{A}; \alpha)$ is a fuzzy implicative hyper K -ideal of H . Clearly, if \bar{A} is a fuzzy weak implicative hyper K -ideal of H , then $U(\bar{A}; \alpha)$ is a weak implicative hyper K -ideal of H . \square

THEOREM 3.16. *Let \bar{A} be a fuzzy set in H such that $U(\bar{A}; \alpha)$, $\alpha \in [0, 1]$, is a nonempty (weak) implicative hyper K -ideal of H . Then \bar{A} is a fuzzy (weak) implicative hyper K -ideal of H .*

Proof. Let $x, y \in H$ be such that $x < y$. Since $y \in U(\bar{A}; \bar{A}(y))$, we have $x \in U(\bar{A}; \bar{A}(y))$ because $U(\bar{A}; \bar{A}(y))$ is an implicative hyper K -ideal of H . Hence $\bar{A}(x) \geq \bar{A}(y)$, which proves (F1). Now for every $x, y, z \in H$, let $s = \min\left\{\inf_{a \in (x \circ z) \circ (y \circ x)} \bar{A}(a), \bar{A}(z)\right\}$. Then $z \in U(\bar{A}; s)$ and

$$\bar{A}(b) \geq \inf_{a \in (x \circ z) \circ (y \circ x)} \bar{A}(a) \geq s$$

for all $b \in (x \circ z) \circ (y \circ x)$. Hence $b \in U(\bar{A}; s)$, i.e., $(x \circ z) \circ (y \circ x) \subseteq U(\bar{A}; s)$ and thus $(x \circ z) \circ (y \circ x) < U(\bar{A}; s)$ by (a9). Since $U(\bar{A}; s)$ is an implicative hyper K -ideal of H , it follows from Definition 3.4 that $x \in U(\bar{A}; s)$. Therefore

$$\bar{A}(x) \geq s = \min\left\{\inf_{a \in (x \circ z) \circ (y \circ x)} \bar{A}(a), \bar{A}(z)\right\}.$$

for all $x, y, z \in H$. Clearly, if \bar{A} is a fuzzy set in H such that $U(\bar{A}; \alpha)$, $\alpha \in [0, 1]$, is a nonempty weak implicative hyper K -ideal of H , then \bar{A} is a fuzzy weak implicative hyper K -ideal of H . \square

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