A CLASS OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY CONVOLUTION

ROSIIAN M. ALI, M. HUSSAIN KHAN, V. RAVICHANDRAN, AND K. G. SUBRAMANIAN

ABSTRACT. For a given \( p \)-valent analytic function \( g \) with positive coefficients in the open unit disk \( \Delta \), we study a class of functions \( f(z) = z^p - \sum_{n=m}^{\infty} a_n z^n \) \( (a_n \geq 0) \) satisfying
\[
\frac{1}{p} \Re \left( \frac{z(f * g)'(z)}{(f * g)(z)} \right) > \alpha \quad (0 \leq \alpha < 1; z \in \Delta).
\]
Coefficient inequalities, distortion and covering theorems, as well as closure theorems are determined. The results obtained extend several known results as special cases.

1. Introduction

Let \( \mathcal{A}(p, m) \) be the class of all \( p \)-valent analytic functions \( f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n \) defined on the open unit disk \( \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \) and let \( \mathcal{A} := \mathcal{A}(1, 2) \). For two functions \( f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n \) and \( g(z) = z^p + \sum_{n=m}^{\infty} b_n z^n \) in \( \mathcal{A}(p, m) \), their convolution (or Hadamard product) is defined to be the function \( (f * g)(z) := z^p + \sum_{n=m}^{\infty} a_n b_n z^n \).

Let \( T(p, m) \) be the subclass of \( \mathcal{A}(p, m) \) consisting of functions of the form
\[
(1.1) \quad f(z) = z^p - \sum_{n=m}^{\infty} a_n z^n \quad (a_n \geq 0 \text{ for } n \geq m)
\]

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and let $T := T(1,2)$. A function $f(z) \in T(p,m)$ is called a function with negative coefficients. The subclass of $T(p,m)$ consisting of multivalent starlike (convex) functions of order $\alpha$ is denoted by $TS^*(p,m,\alpha)$ ($TC(p,m,\alpha)$). The classes $TS^*(\alpha) := TS^*(1,2,\alpha)$ and $TC(\alpha) := TC(1,2,\alpha)$ were studied by Silverman [4]. In this article, we study the class $TS^*_g(p,m,\alpha)$ introduced in the following:

**Definition 1.** Let $g(z) = z^p + \sum_{n=m}^{\infty} b_n z^n$ be a fixed function in $A(p,m)$ with $b_n > 0 \ (n \geq m)$. The class $TS^*_g(p,m,\alpha)$ consists of functions $f(z)$ of the form (1.1) that satisfies

$$\frac{1}{p} \Re \left( \frac{z(f*g)'(z)}{f*g(z)} \right) > \alpha \quad (0 \leq \alpha < 1; z \in \Delta).$$

Several well-known subclasses of functions are special cases of our class for suitable choices of $g(z)$ when $p = 1$ and $m = 2$. For example, if $g(z) := z/(1-z)$, the class $TS^*_g(p,m,\alpha)$ is the class $TS^*(\alpha)$ of starlike functions with negative coefficients of order $\alpha$ introduced and studied by Silverman [4]. If $g(z) := z/(1-z)^2$, the class $TS^*_g(p,m,\alpha)$ is the class $TC(\alpha)$ of convex functions with negative coefficients of order $\alpha$ (see Silverman [4]). If $g(z) := \frac{z}{(1-z)^{\lambda+1}}, \ (\lambda > -1), \ p = 1$, the class $TS^*_g(p,m,\alpha)$ reduces to the class

$$T_\lambda(\alpha) := \left\{ f \in T : \Re \frac{z(D^\lambda f(z))'}{D^\lambda f(z)} > \alpha, \ (z \in \Delta, \lambda > -1, \alpha < 1) \right\},$$

introduced and studied by Ahuja [1] where $D^\lambda$ denotes the Ruscheweyh derivatives of order $\lambda$. When $g(z) := z + \sum_{n=2}^{\infty} n! z^n$, the class $TS^*_g(p,m,\alpha)$ is the class $TS^*_l(\alpha)$ where

$$TS^*_l(\alpha) := \left\{ f \in T : \Re \left( \frac{z(D^l f(z))'}{D^l f(z)} \right) > \alpha \right\}.$$  

(Here $D^l$ denotes the Salagean derivative of order $l$ [3]).

A function $f \in A(p,m)$ is $\beta$-Pascu convex of order $\alpha$ if

$$\frac{1}{p} \Re \left( \frac{(1-\beta)zf'(z) + \beta zf'(z)}{(1-\beta)f(z) + \beta zf'(z)} \right) > \alpha \quad (\beta \geq 0; \ 0 \leq \alpha < 1).$$

We denote by $TPC(p,m,\alpha,\beta)$ the subclass of $T(p,m)$ consisting of $\beta$-Pascu convex functions of order $\alpha$. Clearly $TS^*(\alpha)$ and $TC(\alpha)$ are special cases of $TPC(1,2,\alpha,\beta)$.

In this paper, we obtain the coefficient inequalities, distortion and covering theorems, as well as closure theorems for functions in the class
$TS_g^*(p, m, \alpha)$. Several known results are easily deduced from ours, for example, results for the classes $T_\lambda(\alpha)$ and $TS_f^*(\alpha)$. Additionally, we present results for the \( \alpha \)-Pascu convex functions that unifies corresponding results for $TS_g^*(\alpha)$ and $TC(\alpha)$.

2. The class $TS_g^*(p, m, \alpha)$

We first prove a necessary and sufficient condition for functions to be in $TS_g^*(p, m, \alpha)$ in the following:

**Theorem 1.** A function $f(z)$ given by (1.1) is in $TS_g^*(p, m, \alpha)$ if and only if

$$
\sum_{n=m}^{\infty} \frac{(n - p\alpha)a_nb_n}{p(1 - \alpha)} = p
$$

Proof. If $f \in TS_g^*(p, m, \alpha)$, then (2.1) follows from (1.2) by letting $z \to 1$ through real values. To prove the converse, assume that (2.1) holds. Then by making use of (2.1), we obtain

$$
\left| \frac{z(f * g)(z) - p(f * g)(z)}{(f * g)(z)} \right| \leq \sum_{n=m}^{\infty} \frac{(n - p)a_nb_n}{1 - \sum_{n=m}^{\infty} a_nb_n} \leq p(1 - \alpha)
$$

or $f \in TS_g^*(p, m, \alpha)$. \qed

**Corollary 1.** A function $f(z)$ given by (1.1) is in TPC($p, m, \alpha, \beta$) if and only if

$$
\sum_{n=m}^{\infty} \frac{(n - p\alpha)((1 - \beta)p + \beta n)a_n}{p^2(1 - \alpha)} \leq p
$$

As an immediate application of Theorem 1, we obtain the following:

**Theorem 2.** Let $f(z)$ be given by (1.1). If $f \in TS_g^*(p, m, \alpha)$, then

$$
a_n \leq \frac{p(1 - \alpha)}{(n - p\alpha)b_n}
$$

with equality only for functions of the form

$$
f_n(z) = z^p - \frac{p(1 - \alpha)}{(n - p\alpha)b_n}z^n.
$$

Proof. If $f \in TS_g^*(p, m, \alpha)$, then, by making use of (2.1), we obtain

$$
(n - p\alpha)a_nb_n \leq \sum_{n=m}^{\infty} (n - p\alpha)a_nb_n \leq p(1 - \alpha)
$$
or

\[ a_n \leq \frac{p(1 - \alpha)}{(n - p\alpha)b_n}. \]

Clearly for \( f_n(z) = z^p - \frac{p(1 - \alpha)}{(n - p\alpha)b_n} z^n \in TS^*_g(p, m, \alpha) \), we have

\[ a_n = \frac{p(1 - \alpha)}{(n - p\alpha)b_n}. \]

\[ \square \]

**Corollary 2.** Let \( f(z) \) be given by (1.1). If \( f \in TPC(p, m, \alpha, \beta) \), then

\[ a_n \leq \frac{p^2(1 - \alpha)}{(n - p\alpha)[(1 - \beta)p + \beta n]} \]

with equality only for functions of the form

\[ f_n(z) = z^p - \frac{p^2(1 - \alpha)}{(n - p\alpha)[(1 - \beta)p + \beta n]} z^n. \]

By making use of Theorem 1, we obtain the following growth estimate for functions in the class \( TS^*_g(p, m, \alpha) \).

**Theorem 3.** If \( f \in TS^*_g(p, m, \alpha) \), then

\[ r^p - \frac{p(1 - \alpha)}{(m - p\alpha)b_m} r^m \leq |f(z)| \leq r^p + \frac{p(1 - \alpha)}{(m - p\alpha)b_m} r^m, \quad |z| = r < 1, \]

provided \( b_n \geq b_m \) \((n \geq m)\). The result is sharp with equality for

**(2.2)**

\[ f(z) = z^p - \frac{p(1 - \alpha)}{(m - p\alpha)b_m} z^m \]

at \( z = r \) and \( z = re^{i\pi (2k + 1) / m - p} \) \((k \in \mathbb{Z})\).

**Proof.** Let \( |z| = r \). Since \( f(z) = z^p - \sum_{n=m}^\infty a_n z^n \), we have

\[ |f(z)| \leq r^p + \sum_{n=m}^\infty a_n r^n \]

\[ \leq r^p + r^m \sum_{n=m}^\infty a_n. \]

**(2.3)**

Since for \( n \geq m \),

\[(m - p\alpha)b_m \leq (n - p\alpha)b_n,

\]
using (2.1) yields
\[b_m(m - p\alpha) \sum_{n=m}^{\infty} a_n \leq \sum_{n=m}^{\infty} (n - p\alpha) a_n b_n \leq p(1 - \alpha)\]
or
\[\sum_{n=m}^{\infty} a_n \leq \frac{p(1 - \alpha)}{(m - p\alpha)b_m}.\]  
(2.4)

This together with (2.3) shows that
\[|f(z)| \leq r^p + r^m \frac{p(1 - \alpha)}{(m - p\alpha)b_m}\]
and similarly we have
\[|f(z)| \geq r^p - r^m \frac{p(1 - \alpha)}{(m - p\alpha)b_m}.\]

\[\square\]

Let \(\frac{p(1 - \alpha)}{(m - p\alpha)b_m} < 1\). By letting \(r \to 1\) in Theorem 3, we see that functions \(f \in TS_g^*(p, m, \alpha)\) map the unit disk \(\Delta\) onto regions that contained the disk \(|w| < 1 - \frac{p(1 - \alpha)}{(m - p\alpha)b_m}\).

**Corollary 3.** If \(f \in TPC(p, m, \alpha, \beta)\), then
\[r^p - \frac{p^2(1 - \alpha)}{(m - p\alpha)((1 - \beta)p + \beta m)}r^m \leq |f(z)| \leq r^p + \frac{p^2(1 - \alpha)}{(m - p\alpha)((1 - \beta)p + \beta m)}r^m, \quad |z| = r < 1.\]
The result is sharp for
\[(2.5) \quad f(z) = z^p - \frac{p^2(1 - \alpha)}{(m - p\alpha)((1 - \beta)p + \beta m)}z^m.\]

We now prove the distortion theorem for the functions in \(TS_g^*(p, m, \alpha)\) in the following:

**Theorem 4.** If \(f \in TS_g^*(p, m, \alpha)\), then
\[pr^{p-1} - \frac{mp(1 - \alpha)}{(m - p\alpha)b_m}r^{m-1} \leq |f'(z)| \leq pr^{p-1} + \frac{mp(1 - \alpha)}{(m - p\alpha)b_m}r^{m-1}, \quad |z| = r < 1,\]
provided $b_n \geq b_m$. The result is sharp for $f(z)$ given by (2.2).

**Proof.** For a function $f \in TS_g^*(p, m, \alpha)$, it follows from (2.1) and (2.4) that

\[
\sum_{n=m}^{\infty} na_n \leq \frac{mp(1-\alpha)}{(m-p\alpha)b_m}.
\]

Since the remaining part of the proof is similar to the proof of Theorem 3, we omit the details. $\square$

**Corollary 4.** If $f \in TPC(p, m, \alpha, \beta)$, then

\[
p^p m^{-1} \leq \frac{mp(1-\alpha)}{(m-p\alpha)((1-\beta)p + \beta m)}r^{m-1}
\]

\[
\leq |f'(z)|
\]

\[
\leq p^p m^{-1} + \frac{mp(1-\alpha)}{(m-p\alpha)((1-\beta)p + \beta m)}r^{m-1}
\]

where $|z| = r < 1$. The result is sharp for $f(z)$ given by (2.5).

We shall now prove the following closure theorems for the class $TS_g^*(p, m, \alpha)$.

**Theorem 5.** Let $\lambda_k \geq 0$ for $k = 1, 2, \ldots, l$ and $\sum_{k=1}^{l} \lambda_k \leq 1$. If the functions $F_k(z)$ defined by

\[
F_k(z) = z^p - \sum_{n=m}^{\infty} f_{n,k}z^n
\]

are in the class $TS_g^*(p, m, \alpha)$ for every $k = 1, 2, \ldots, l$, then the function $f(z)$ defined by

\[
f(z) = z^p - \sum_{n=m}^{\infty} \left( \sum_{k=1}^{l} \lambda_k f_{n,k} \right) z^n
\]

is in the class $TS_g^*(p, m, \alpha)$.

**Proof.** Since $F_k(z) \in TS_g^*(p, m, \alpha)$, it follows from Theorem 2.1 that

\[
\sum_{n=m}^{\infty} (n-p\alpha) f_{n,k} b_n \leq p(1-\alpha)
\]
for every $k = 1, 2, \ldots, l$. Hence
\[
\sum_{n=m}^{\infty} (n - p\alpha) \left( \sum_{k=1}^{l} \lambda_k f_{n,k} \right) b_n = \sum_{k=1}^{l} \lambda_k \left( \sum_{n=m}^{\infty} (n - p\alpha) f_{n,k} b_n \right) \\
\leq p(1 - \alpha) \sum_{k=1}^{l} \lambda_k \\
\leq p(1 - \alpha).
\]
By Theorem 1, it follows that $f(z) \in TS_g^*(p, m, \alpha)$.
\[\square\]

**Corollary 5.** The class $TS_g^*(p, m, \alpha)$ is closed under convex linear combinations.

**Theorem 6.** Let $F_p(z) := z^p$ and $F_n(z) := z^p - \frac{p(1 - \alpha)}{(n - p\alpha) b_n} z^n$ for $n = m, m + 1, \ldots$. The function $f(z) \in TS_g^*(p, m, \alpha)$ if and only if $f(z)$ can be expressed in the form
\[
(2.8) \quad f(z) = \lambda_p z^p + \sum_{n=m}^{\infty} \lambda_n F_n(z)
\]
where $\lambda_n \geq 0$ for $n = p, m, m + 1, \ldots$ and $\lambda_p + \sum_{n=m}^{\infty} \lambda_n = 1$.

**Proof.** If the function $f(z)$ is expressed in the form given by (2.8), then
\[
f(z) = z^p - \sum_{n=m}^{\infty} \frac{\lambda_n p(1 - \alpha)}{(n - p\alpha) b_n} z^n
\]
and for this function, we have
\[
\sum_{n=m}^{\infty} (n - p\alpha) \frac{\lambda_n p(1 - \alpha) b_n}{(n - p\alpha) b_n} = \sum_{n=m}^{\infty} p(1 - \alpha) \lambda_n = p(1 - \alpha)(1 - \lambda_p) \leq p(1 - \alpha).
\]
By Theorem 1, we have $f(z) \in TS_g^*(p, m, \alpha)$.

Conversely, let $f(z) \in TS_g^*(p, m, \alpha)$. From Theorem 2, we have
\[
a_n \leq \frac{p(1 - \alpha)}{(n - p\alpha) b_n} \quad \text{for} \quad n = m, m + 1, \ldots.
\]
Therefore by taking
\[
\lambda_n := \frac{(n - p\alpha) a_n b_n}{p(1 - \alpha)} \quad \text{for} \quad n = m, m + 1, \ldots
\]
and
\[
\lambda_p := 1 - \sum_{n=m}^{\infty} \lambda_n,
\]
we see that \( f(z) \) is of the form given by (2.8). 

**Theorem 7.** Let \( h(z) = z^p + \sum_{n=m}^{\infty} h_n z^n \) with \( h_n > 0 \).

(i) Let \( (n - p\alpha)b_n \geq (1 - \alpha)nh_n \) and

\[
\beta := \inf_{n \geq m} \left[ \frac{(n - p\alpha)b_n - (1 - \alpha)nh_n}{(n - p\alpha)b_n - (1 - \alpha)ph_n} \right].
\]

If \( f \in TS^*_g(p, m, \alpha) \), then \( f \in TS^*_h(p, m, \beta) \).

(ii) If \( f \in TS^*_g(p, m, \alpha) \), then \( f \in TS^*_h(p, m, \beta) \) in \( |z| < r(\alpha, \beta) \), where

\[
r(\alpha, \beta) := \min \left\{ 1, \inf_{n \geq m} \left[ \frac{(n - p\alpha)(1 - \beta)b_n}{(n - p\beta)(1 - \alpha)h_n} \right]^{\frac{1}{1-\beta}} \right\}.
\]

**Proof.** (i) From the definition of \( \beta \), it follows that

\[
\beta \leq \frac{(n - p\alpha)b_n - (1 - \alpha)nh_n}{(n - p\alpha)b_n - (1 - \alpha)ph_n}
\]
or

\[
\frac{(n - p\beta)h_n}{1 - \beta} \leq \frac{(n - p\alpha)b_n}{1 - \alpha}
\]
and therefore, in view of (2.1),

\[
\sum_{n=m}^{\infty} \frac{(n - p\beta)}{p(1 - \beta)} a_n h_n \leq \sum_{n=m}^{\infty} \frac{(n - p\alpha)}{p(1 - \alpha)} a_n b_n \leq 1.
\]

This completes the proof of (i).

(ii) It is easy to see that \( f \) satisfies

\[
\frac{1}{p} \Re \left( \frac{z(f * h)'(z)}{(f * h)(z)} \right) > \beta \quad (|z| < r)
\]
if and only if

\[
\sum_{n=m}^{\infty} (n - p\beta)a_n h_n r^{n-p} \leq p(1 - \beta).
\]

From the definition of \( r(\alpha, \beta) \), we have

\[
\frac{(n - p\beta)}{p(1 - \beta)} h_n r^{n-p} \leq \frac{(n - p\alpha)}{p(1 - \alpha)} b_n
\]
and the result now follows from (2.10), (2.9) and (2.1). \( \square \)
Theorem 7 contains several results. For example, when \( p = 1, m = 2, \)
\( h(z) = z/(1 - z) \) and \( g(z) = z/(1 - z)^2, \) the class \( TS^*_g(1, 2, \alpha) \) consists
of convex functions of order \( \alpha \) in \( T. \) Theorem 7(i) yields the order
of starlikeness, i.e., \( \beta = 2/(3 - \alpha). \) Similarly, when \( p = 1, m = 2, \)
\( h(z) = z/(1 - z)^2, \) \( g(z) = z/(1 - z), \) and \( \beta = 0, \) we get the radius
of convexity for starlike functions of order \( \alpha \) in \( T. \) These results were
proved by Silverman [4].

We now prove that the class \( TS^*_g(p, m, \alpha) \) is closed under convolution
with certain functions and give an application of this result to show
that the class \( TS^*_g(p, m, \alpha) \) is closed under the familiar Bernardi integral
operator.

**Theorem 8.** Let \( h(z) = z^p + \sum_{n=m}^{\infty} h_n z^n \) be analytic in \( \Delta \) with
\( 0 \leq h_n \leq 1. \) If \( f(z) \in TS^*_g(p, m, \alpha), \) then \( (f * h)(z) \in TS^*_g(p, m, \alpha). \)

**Proof.** The result follows by a straight forward application of Theorem 1. \( \square \)

The generalized Bernardi integral operator is defined by

\[
F(z) = \frac{c + p}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1; z \in \Delta).
\]

Since

\[
F(z) = f(z) \ast \left( z^p + \sum_{n=m}^{\infty} \frac{c + p}{c + n} z^n \right),
\]

we have the following:

**Corollary 6.** If \( f(z) \in TS^*_g(p, m, \alpha), \) then \( F(z) \) given by (2.11) is
also in \( TS^*_g(p, m, \alpha). \)

**References**


Rosihan M. Ali, School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia
E-mail: rosihan@cs.usm.my

M. Hussain Khan, Department of Mathematics, Islamiah College, Vaniambadi 635 751, India

V. Ravichandran, School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia
E-mail: vravi@cs.usm.my

K. G. Subramanian, Department of Mathematics, Madras Christian College, Tambaram, Chennai-600 059, India
E-mail: kgsmani@vsnl.net