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SOME POINT ESTIMATES FOR THE SHAPE PARAMETERS OF EXPONENTIATED-WEIBULL FAMILY†

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ABSTRACT

Maximum product of spacings estimator is proposed in this paper as a competent alternative of maximum likelihood estimator for the parameters of exponentiated-Weibull distribution, which does work even when the maximum likelihood estimator does not exist. In addition, a Bayes type estimator known as generalized maximum likelihood estimator is also obtained for both of the shape parameters of the aforesaid distribution. Though, the closed form solutions for these proposed estimators do not exist yet these can be obtained by simple appropriate numerical techniques. The relative performances of estimators are compared on the basis of their relative risk efficiencies obtained under symmetric and asymmetric losses. An example based on simulated data is considered for illustration.

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Keywords. Maximum product of spacings estimators, generalized maximum likelihood estimators, maximum likelihood estimator, squared error loss function, LINEX loss function, relative risk efficiency, non-informative prior.

1. INTRODUCTION

The data obtained from lifetime experiments are referred to as failure time data and such failure time data, results from different lifetime experiments conducted under sophisticated controlled and complex environments, exhibit different types of failure rates, which are generally categorized as, constant, monotone

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(increasing and decreasing) and non-monotone (unimodal and bathtub) type of failure. This specific characteristic of failure time data is an added advantage that helps us to perform the quite crucial task of model specification during its inferential procedures within parametric approach. Probably, Lawless (1982) could be one of the good references for interested readers to see the relationship among failure rate, probability density and distribution function with various examples.

A number of lifetime models have been proposed in literature, which are usually used to analyze failure time data, see Martz and Waller (1982), Lawless (1982), Sinha (1986) for details. However, most of them are quite appropriate and famous for constant and monotone type of failure rates. Exponential model for constant failure rate and Weibull distribution for monotone type of failures are few examples, which are enormously used for failure data analysis due to their standard statistical inference and closed form solutions in most of the cases. Whereas, non-monotone failure rates, which are quite common in engineering, space research, biological and biomedical fields (survival analysis), are somehow accommodated by some generalized families of distributions or some of mixture models. Generalized gamma, generalized F, generalized Reyligh, and mixture of Weibull distributions could be some common names among all, however, readers are referred to Stacy (1962), Prentice (1975), Slymen and Lachenbruch (1984), Rajarshi and Rajarshi (1988) for further details. The important drawbacks of the models, used for failure time data accommodating non-monotone type of failure rates, are that the inferential procedures related to these models are non-standard. Besides, mathematical as well as computational complications become quite strenuous as they involve too many parameters, see Bain (1974), Gore *et al.* (1986), Lawless (1982), *etc.* for detail discussion.

Hence, the need for developing one such model, which can at least deal with nonmonotone type of failure rates as efficiently as exponential and Weibull distribution do for constant and monotone type of failures, has been felt since long, but, rapid advancement in science and technologies, huge cost associated with such lifetime experiment and also due to direct impact over human health and their safety, made its need, somewhat indispensable. An attempt in this direction was made by Mudholkar and Kollia (1990) and Mudholkar *et al.* (1991) who suggested an extension of Weibull family of distributions and named it, generalized Weibull family of distributions. This generalized Weibull family of distributions accommodates all most all types of failure rates but later on, Mudholkar *et al.* (1996) pointed out that the members of generalized Weibull family with bathtub failure rate are non-regular and thus caused non-standard inference. At the same time, Mudholkar and Hutson (1996) introduced another extension of Weibull family, called, exponentiated Weibull distribution (EWD). The author discussed and showed in his paper that, this new family of distributions efficiently accommodates almost all types of failure rates (monotone, unimodal and bathtub) and it consists entirely of regular distributions. Probability density function (*pdf*) and distribution function (*df*) of EWD are given by

$$f(y|\alpha,\theta) = \alpha\theta\left\{1-\exp(-y^{\alpha})\right\}^{\theta-1}\exp(-y^{\alpha})y^{\alpha-1}, \quad 0<y<\infty, \tag{1.1}$$

$$F(y) = \left\{1-\exp(-y^{\alpha})\right\}^{\theta}, \tag{1.2}$$

where $\alpha > 0$ and $\theta > 0$, both are the shape parameters in the model (1.1). The shape of *pdf* and failure (hazard) functions governed jointly through α and θ. A brief detail about these two functions is discussed hereunder.

The *pdf* and *df* of EWD turn to the *pdf* and *df* of Weibull distribution for $\theta = 1$ and also behave like constant exponential distribution when both α and θ are equal to one. The applications and structural analysis of exponentiated-Weibull family have been discussed by Mudholkar and Hutson (1996) and they showed that the shapes of *pdf* (1.1) are monotone decreasing with $\alpha\theta < 1$ and EWD also have unimodal shape for $\alpha\theta > 1$.

On the other hand, failure (hazard) function, say, $h(y) = f(y)/\{1-F(y)\}$ of EWD accommodates a broad variety of monotone shapes including unimodal and bathtub, restricted over the range of α and θ. For example, shape of $h(y)$ is, unimodal for $\alpha < 1$ and $\alpha\theta > 1$; bathtub for $\alpha > 1$ and $\alpha\theta > 1$; monotone increasing for $\alpha \geq 1$ and $\alpha\theta \geq 1$; and monotone decreasing for $\alpha \leq 1$ and $\alpha\theta \leq 1$.

Parametric inference, mainly estimation procedures under classical and Bayesian setup for EWD seemed to be initiated in detail by Singh *et al.* (1999) and thereafter Singh *et al.* (2002) who developed maximum likelihood estimator (MLE) and different Bayes estimators, under quite general conditions and compared their performances through their different risks based on extensive simulation study. Estimation and performances of two and three parameters EWD have also been studied by Singh *et al.* (2005a, b) under classical and Bayesian set-up when sample observations are incomplete.

This work is in fact, a motive to compare the MLE of parameters of EWD with proposed maximum product of spacings estimator (MPSE) which is equivalent to the MLE but works in more general conditions as compared to MLE.

This is quite general that MLE is well appreciated and most commonly used, among all classical methods of estimation, such as method of moment, method of minimum chi-square, method of least square, method of minimum variance unbiased estimator, because, it meets certain optimum properties of a good estimator. But, awkwardly, MLE may not exist in certain cases, particularly, when the probability density function is J shaped. Sometimes MLE can lead to inconsistent estimators as reported by various authors; for examples, Huzurbazar (1948), Harter and Moore (1966), Johnson and Kotz (1970), Cheng and Amin (1983) and Ranneby (1984) *etc.* In fact, the comments of non-existence of MLE by aforesaid authors somewhat made us curious to compare the MLE of EWD with MPSE. Since the papers of Singh *et al.* (1999) and Singh *et al.* (2002) compared the MLE with different Bayes estimators, which are indeed developed under non-informative priors, however, in some circumstances, risk of MLE of EWD is quite higher than some Bayes estimators. Henceforth, we propose here to use the maximum product of spacings (MPS) method for EWD and compared with MLE, somewhat to check their trend of risks, numerically.

MPSE was introduced by Cheng and Amin (1983) as an alternative of MLE. They discussed and showed that MPSE also possesses the optimum properties of good estimators like MLE, such as, it provides consistent estimators with asymptotic efficiency not only when MLE exists, but also when MLE fails to exist. Further mentioned that in some situations, MPSE can be a function of sufficient statistics, whereas, MLE is not. Apart from this, MPSE also provides consistent estimator of parameters in those cases, where MLE is to be noted inconsistent. With such properties and characteristics, MPSE seems to be quite competent for its use whether MLE does exist or not. For mathematical and further details, see Cheng and Amin (1983) and Ranneby (1984).

The MPSE procedure is based on the maximization of geometric mean of spacings within parametric space, whereas, spacings imply for the (uniform) spacing of the ordered random sample after transforming it into unit interval. For reference, see Cheng and Amin (1983), who had discussed about the method of estimation thoroughly. Some more details about the use of MPSE in case of parametric models can be had from the papers, such as Shah and Gokhale (1993), Hossain and Nath (1997).

A Bayesian counterpart of MLE is generalized maximum likelihood estimator (GMLE) and it is defined as the value of parameter for which, the marginal posterior of the parameter attains maximum value. If a non-informative type of prior is considered for the parameter then GMLE may, therefore, be interpreted as the most probable value of the parameter in the light of the given sample. Hence, GMLE may also be considered as a competitive estimator to MPSE and MLE.

The next question that arises at this stage is, how to compare the performances of various estimators obtained by using the procedures mentioned above. In classical inference, the mean square error, which is nothing but the risk (average loss over sample space) under square error loss, is often used to study the performance of the estimators for their long-term use. No doubt, the use of squared error loss function (SELF) has been justified in statistical literature on various grounds and hence, it is considered as most general and commonly used loss function. However, SELF is symmetric loss function, which gives equal importance to under estimation and over estimation of equal magnitude. It is generally agreed upon that in life testing and reliability context, overestimation and underestimation may not be of equal importance. Thus, use of asymmetric loss function seems to be more justified in such circumstances. The most popular asymmetric loss function in literature is LINEX loss function among various others. LINEX loss is direct extension of SELF, which is introduced by Varian (1975) and popularized by Zellner (1986). The LINEX loss function (LLF) is defined as:

$$L(\Delta) = b(e^{a\Delta} - a\Delta - 1), \quad a \neq 0, \quad b > 0, \tag{1.3}$$

where $\Delta = (\overline{\Theta} - \Theta)$ denotes the scalar estimation error in using $\overline{\Theta}$ to estimate Θ, a and b are shape and scale parameters of the loss function, respectively. For small values of a (near to zero), LLF is same as SELF and for choice of negative/positive values of a, it gives more weight to overestimation/underestimation. For further details, see Zellner (1986). Various authors, including Rojo (1987), Khatree (1992) have used LINEX loss function in different estimation problems.

For the comparison of the estimators, we, therefore, propose to use the criterion of risk (average loss over sample space) under both loss functions, SELF and LLF.

MPSE and GMLE for exponentiated-Weibull shape parameters are developed in the next section assuming, none of the parameters are known. MLE is also discussed in Section 2. The proposed estimators have been illustrated through a simulated data set in Section 3. The expressions for the estimators discussed in Section 2 do not reduce into nice closed forms and hence these are obtained through iterative numerical methods. Henceforth, we performed Monte Carlo simulation in Section 4. Comparative study of their performances on the basis of their risks obtained under SELF and LLF is done in Section 5. A brief conclusion is given in the last section.

2. Estimation of Parameters

Let us assume that n items following a life time distribution expressed in (1.1) are subjected to life testing and the observed lifetimes are $y_1, y_2, \ldots, y_n$.

2.1. Maximum product of spacings estimators (MPSE)

Following Cheng and Amin (1983), MPSE of α and θ are the value of these parameters, which maximize the geometric mean of spacings and expressed below after taking logarithm:

$$G(\alpha,\theta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log(G_i(\alpha,\theta)), \tag{2.1}$$

where $G_i(\alpha,\theta) = F(y_i,\alpha,\theta) - F(y_{i-1},\alpha,\theta)$, $i = 1,2,\ldots,n+1$, $y_0 = 0$, $y_{n+1} = \infty$ and $F(y,\alpha,\theta) = F(y)$ is given in (1.2). The normal equations for the parameters α and θ can be obtained by differentiating (2.1) with respect to the parameters α and θ, respectively and equating them to zero. The resulting equations are given below.

$$\theta\sum_{i=1}^{n+1}\frac{e^{-y_i^\alpha}y_i^\alpha\log y_i(1-e^{-y_i^\alpha})^{\theta-1} - e^{-y_{i-1}^\alpha}y_{i-1}^\alpha\log y_{i-1}(1-e^{-y_{i-1}^\alpha})^{\theta-1}}{(1-e^{-y_i^\alpha})^\theta - (1-e^{-y_{i-1}^\alpha})^\theta} = 0, \tag{2.2}$$

and

$$\sum_{i=1}^{n+1}\frac{(1-e^{-y_i^\alpha})^\theta\log(1-e^{-y_i^\alpha}) - (1-e^{-y_{i-1}^\alpha})^\theta\log(1-e^{-y_{i-1}^\alpha})}{(1-e^{-y_i^\alpha})^\theta - (1-e^{-y_{i-1}^\alpha})^\theta} = 0. \tag{2.3}$$

Numerical solution of (2.2) and (2.3) is obtained by using the subroutine C05NCF of NAG (1993) library which is based on Newton-Raphson type iterative procedure. The MPSE, thus obtained, for α and θ have been denoted by $\overline{\alpha}_{mps}$ and $\overline{\theta}_{mps}$, respectively.

2.2. Generalized maximum likelihood estimator (GMLE)

The likelihood function for the sample observations can simply be obtained as,

$$l(\underline{y}|\alpha,\theta) = (\alpha\theta)^n\prod_{i=1}^{n} y_i^{\alpha-1}e^{-y_i^\alpha}(1-e^{-y_i^\alpha})^{\theta-1}, \quad \alpha,\theta > 0. \tag{2.4}$$

Independent non-informative type of priors for unknown shape parameters, α and θ to be considered as,

$$g_1(\alpha) = \frac{1}{c}, \quad 0 < \alpha < c, \tag{2.5}$$

and

$$g_2(\theta) = \frac{1}{\theta}, \quad \theta > 0. \tag{2.6}$$

Hence, the joint posterior density of the parameters can be obtained with the help of (2.4), (2.5) and (2.6) as:

$$\prod(\alpha, \theta|\underline{y}) = \frac{\alpha^n \theta^{n-1} \prod_{i=1}^{n} e^{-y_i^\alpha} y_i^{\alpha-1} (1 - e^{-y_i^\alpha})^{\theta-1}}{j_1(n-1)!}, \tag{2.7}$$

where

$$j_1 = \int_0^c \frac{\alpha^n \prod_{i=1}^{n} e^{-y_i^\alpha} y_i^{\alpha-1}}{\prod_{i=1}^{n} (1 - e^{-y_i^\alpha})\{-\log \prod_{i=1}^{n} (1 - e^{-y_i^\alpha})\}^n} d\alpha. \tag{2.8}$$

Now, the marginal posterior of a parameter is obtained by integrating out the other parameter from joint posterior distribution. Hence, the marginal posterior of α and θ, after simplification reduces to

$$\prod(\alpha|\underline{y}) = \frac{\alpha^n \prod_{i=1}^{n} e^{-y_i^\alpha} y_i^{\alpha-1}}{j_1 \prod_{i=1}^{n} (1 - e^{-y_i^\alpha})\{-\log \prod_{i=1}^{n} (1 - e^{-y_i^\alpha})\}^n}, \tag{2.9}$$

and

$$\prod(\theta|\underline{y}) = \frac{\theta^{n-1} j_2}{j_1(n-1)!}, \tag{2.10}$$

respectively, where j_1 is given in (2.8) and

$$j_2 = \int_0^c \alpha^n \prod_{i=1}^{n} (1 - e^{-y_i^\alpha})^{\theta-1} e^{-y_i^\alpha} y_i^{\alpha-1} d\alpha. \tag{2.11}$$

Since the marginal posterior of the parameters given in (2.9) and (2.10) are not in closed forms, the usual method of maximization cannot be used in this situation. However, the GMLE of the parameters of EWD have been obtained with the help of following proposed algorithm.

Evaluate the posterior function $\pi(s)$ for various values of s (starting from an initial value close to zero and taking an increment h). Simultaneously, search a value s_1 such that, the successive differences $D_1 = (s_1) - (s_1 - h)$ and $D_2 = (s_1 + h) - (s_1)$ are of different sign. Thereafter, the increment in h is changed to decrement of half magnitude (*i.e.* $h/2$) and the above process is continued backward from the point $s_1 + h$ to get the next point such that the successive differences are again of different sign. The whole process is continued till increment/decrement in h attains the desired accuracy (say, 10^4). It may be noted here that the above procedure provides the first local maxima. In order to check whether it is a global maxima over the whole parametric space, the value of the posterior function is also evaluated at other points in the parametric space and compared with the value of evaluated maxima. The results reported in the paper, therefore, give the global maxima of the posterior and hence are the GMLE. The GMLE for α and θ are denoted by $\overline{\alpha}_{gml}$ and $\overline{\theta}_{gml}$, respectively.

2.3. *Maximum likelihood estimator (MLE)*

Differentiating the logarithm of likelihood function given in (2.4) with respect to α and θ and equating them to zero, we get the normal equations for the parameters α and θ. After simplification, it is obtained as:

$$\overline{\theta} = -\frac{n}{\sum_{i=1}^{n} \log(1 - e^{-y_i^\alpha})}, \tag{2.12}$$

and

$$\frac{n}{\alpha} + \sum_{i=1}^{n} \log(y_i) - \sum_{i=1}^{n} y_i^\alpha \log(y_i) - \frac{n}{\sum_{i=1}^{n} \log(1 - e^{-y_i^\alpha})} \sum_{i=1}^{n} \frac{e^{-y_i^\alpha} y_i^\alpha \log(y_i)}{(1 - e^{-y_i^\alpha})} - \sum_{i=1}^{n} \frac{e^{-y_i^\alpha} y_i^\alpha \log(y_i)}{(1 - e^{-y_i^\alpha})} = 0. \tag{2.13}$$

Since the analytical solutions of these equations do not exist, therefore, we use iterative method to obtain the MLE of the parameters. The detail about the solution of the likelihood equations may be had from Singh *et al.* (1999). The MLE for α and θ have been denoted by $\overline{\alpha}_{mle}$ and $\overline{\theta}_{mle}$, respectively.

3. ILLUSTRATIVE EXAMPLE

We have simulated a sample of size 5 from EWD for $\alpha = 2.0$ and $\theta = 0.5$ to illustrate the estimators proposed in Section 2. The generated sample values are $Y = 1.0001$; 0.2287; 0.3851; 0.2280; 1.2162. Table 3.1 summarized the results obtained for EWD parameters using these observed samples.

The present example shows that the procedures mentioned in the previous sections for obtaining the estimates from given sample can easily be implemented. So we may expect that it can be used in practical, where the data is described by

TABLE 3.1 *Estimators under various methods for $\alpha = 2.0$ and $\theta = 0.5$*

Estimates	α	θ
MLE	2.817	0.427
MPSE	1.437	0.656
GMLE	2.817	0.331

the EWD. Since it will be not quite reasonable to make any conclusion regarding the performances of the estimators, just on the basis of only one set of example and therefore we perform an extensive simulation study in the next section to study their performances in long run use.

4. SIMULATION STUDY

Performance of the above stated estimators will be studied on the basis of their relative risk efficiencies. The relative risk efficiencies of MPSE with respect to GMLE and MLE are denoted by $E_{1S} = R_{gml}/R_{mps}$ and $E_{2S} = R_{ml}/R_{mps}$ respectively, where, R_{mps}, R_{gml} and R_{ml} denote the risk of the estimators under SELF. Similarly, the relative risk efficiencies of MPSE with respect to GMLE and MLE under LLF are defined as $E_{1L} = R'_{gml}/R'_{mps}$ and $E_{2L} = R'_{ml}/R'_{mps}$ respectively; where R'_{mps}, R'_{gml} and R'_{ml} denote the risk of the estimators under LLF. As seen in Section 2, analytical expressions for the estimators and, hence, their relative risk efficiencies do not exist in close form. Therefore, the comparison of the estimators is to be made on the basis of Monte-Carlo simulation studies.

For the simulation study, we considered following values of the parameters:

$$n = 10(10)40; \ \alpha = \theta = 0.5(0.5)5; \ c = 4(2)12; \ a = -1.0, \ 0.001, \ 1.0.$$

Five thousand samples of different sizes have been generated from EWD for each combination of α and θ. Risk efficiencies of the GMLE and MLE with respect to the MPSE have been obtained for α and θ on the basis of these 5000 samples. The results, thus, obtained are plotted on graphs, where the values of the parameter are taken on x-axis and corresponding relative risk efficiencies of the estimators are shown on the y-axis. It may be note worthy that the scale on y-axis is not same for all figures. Figures 5.1 and 5.2 show the relative risk efficiencies of estimator under SELF whereas, Figures 5.3 and 5.4 show the relative risk efficiencies under LLF.

5. Comparison

Due to paucity of space and similarity in trend of the relative risk efficiencies, only important results are shown in this paper, although the comparison is based on complete results.

It is to be noted that a change in the value of hyperparameter c will effect the risk of GMLE only, whereas the risk of other estimators will remain same. The study showed that the risk of GMLE of α is more or less constant for all choices of c, while change in the magnitude of risk of GMLE of θ is not quite significant. Hence, the results shown in this paper are the case, when $c = 6$.

Similarly, E_{2S} and E_{2L} for the parameters α and θ depict, almost same trend for all considered values of n, while E_{1S} and E_{1L} exhibit a very little change in trend. Hence, only one value of n (namely $n = 10$) is selected in this paper for illustration of results.

5.1. *Comparison of estimators under SELF*

5.1.1. Estimatiors of α. It can be seen from the Figure 5.1, E_{1s} and E_{2s} are more or less parallel to the x-axis for all considered values of θ, showing that relative risk efficiencies of the estimators of α, for fixed value of θ, do not vary much for variation of the values in α. E_{1s} and E_{2s} of the estimators are to be noted greater than one for almost all the considered values of the parameter, and hence, one can conclude that MPSE performs better than other two estimators (MLE and GMLE) of α. It may also be noted that E_{1s} is always greater than E_{2s} showing that the MLE performs better than GMLE. However, the gain in using MLE against GMLE reduces for higher values of θ, see, Figure 5.1.

5.1.2. Estimators of θ. E_{2s} of θ increases as the value of the parameter θ increases for given α, whereas E_{1s} of θ decreases. It may also be noted, E_{1s} is always less than E_{2s}, see, Figure 5.2. This shows that there is greater gain in risk, using MPSE against MLE as compared to using MPSE against GMLE. For large values of α, E_{1s} is less than one, *i.e.* GMLE performs better than MPSE.

5.2. *Comparison of estimators under LLF*

5.2.1. Estimators of α. When $a = 1.0$, *i.e.* the case when overestimation is more serious than underestimation, E_{1L} of α increases as values of α increases. Whereas, E_{2L} of α shows a similar trend, except for α lying in the range 1.0 to

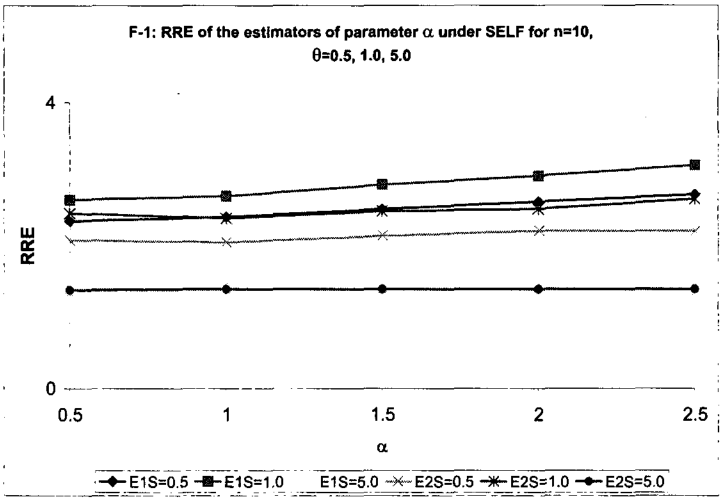

Figure 5.1 *RRE of the estimators of parameter* α *under SELF for* $n = 10, \theta = 0.5, 1.0, 5.0$.

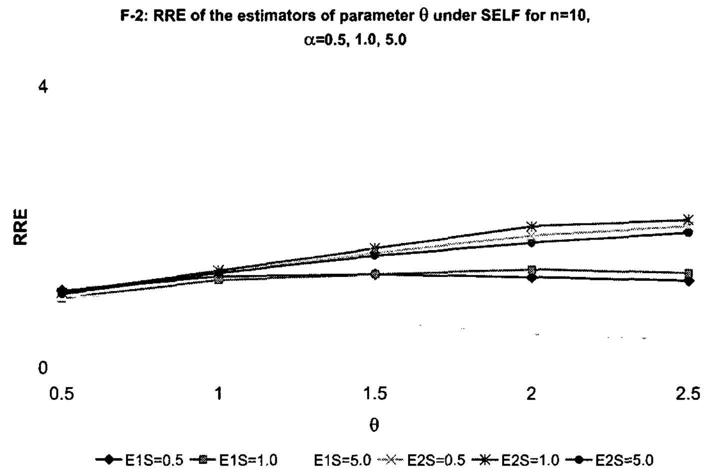

Figure 5.2 *RRE of the estimators of parameter* θ *under SELF for* $n = 10, \alpha = 0.5, 1.0, 5.0$.

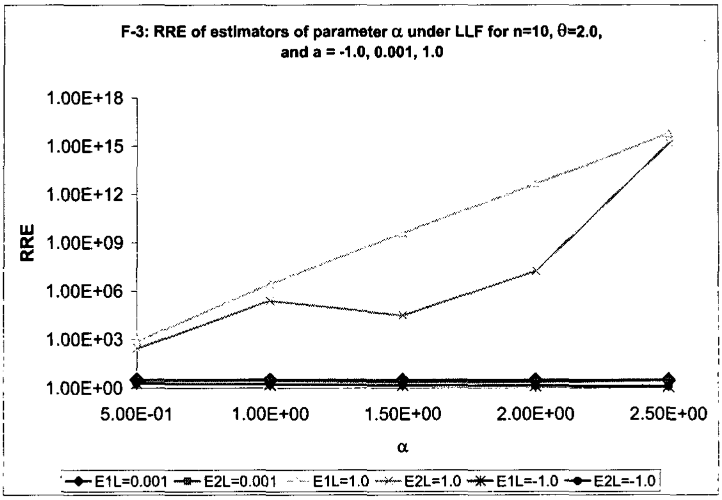

FIGURE 5.3 *RRE of the estimators of parameter α under LLF for $n = 10, \theta = 2.0$, and $a = -1.0, -0.001, 1.0$.*

1.5. Figure 5.3 exhibits these trends. This implies that MPSE has smaller risk as compared to MLE and GMLE.

For $a = -1.0$, where underestimation is more serious than overestimation, E_{1L} and E_{2L} of α are close to each other, though their magnitude are slightly higher than one. This shows that under this situation, MPSE performs slightly better than MLE and GMLE, which have more or less same magnitude of risk.

5.2.2. Estimators of θ. E_{1L} and E_{2L} of θ increases as the value of θ increases, in the case, when $a = 1.0$, which is similar to the trend obtained for the estimators of α. Further, it may be noted that for the estimators of θ, E_{2L} is greater than E_{1L}. Whereas for the estimation of α, E_{1L} was obtained to be greater than E_{2L}. This shows that use of MPSE against MLE has greater gains than use of MPSE against GMLE.

For negative value of ($a = -1.0$), it may be noted that E_{1L} and E_{2L} are quite close to each other and their magnitude are either closer to one or slightly less than one, see Figure 5.4.

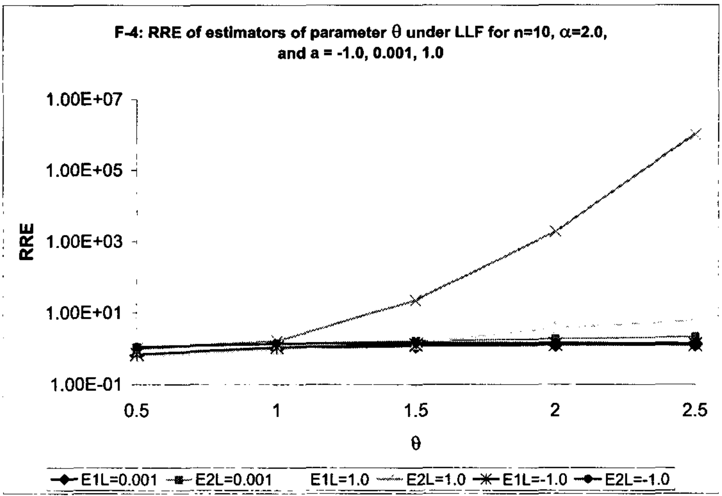

FIGURE 5.4 *RRE of estimators of parameter θ under LLF for $n = 10, \alpha = 2.0$, and $a = -1.0, 0.001, 1.0$.*

6. CONCLUSION

On the basis of above discussion, it may be concluded that the MPSE gives smaller risk in most of the situations when overestimation and underestimation are equally important, except in the case (*e.g.* $E_{1s} = 5.0$) where risk of GMLE for θ is noted to be smallest, when α is quite large. Hence MPSE could be recommended for its use in general, provided that the prior believe over parameter is not quite close to their true values, otherwise GMLE could be a good choice for the estimation of θ.

On the other hand, if overestimation is more serious than underestimation, one can safely use MPSE as it provides smaller risk than MLE and GMLE.

On the contrary, when underestimation is more serious than overestimation any one of these estimators can be used, because in this case all the estimators have risks of more or less equal magnitude.

As far as from this study, It is also worth to mention that the trend of the risks of the MLE is quite similar to the trend of risk of MPSE, whereas the risk of GMLE is shown significantly different trend, only when $a = 1.0$.

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