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STATIONARY β-MIXING FOR SUBDIAGONAL BILINEAR TIME SERIES[†]

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ABSTRACT

We consider the subdiagonal bilinear model and ARMA model with subdiagonal bilinear errors. Sufficient conditions for geometric ergodicity of associated Markov chains are derived by using results on generalized random coefficient autoregressive models and then strict stationarity and β-mixing property with exponential decay rates for given processes are obtained.

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1. INTRODUCTION

In the last two decades, nonlinear time series have gained much attention. Many classes of nonlinear time series models including self-exciting threshold models, bilinear models, ARCH-type models, Markov switching models have been developed in the literature and successfully applied in various fields such as finance and macroeconomics (Tong, 1978; Granger and Andersen, 1978; Engle, 1982; Hamilton, 1989).

Among many other nonlinear time series models, we are interested in bilinear models introduced by Granger and Anderson (1978) and Subba Rao (1981) and later studied in, for example, Tong (1981), Pham (1985, 1986), Weiss (1986), Liu and Brockwell (1988), Chanda (1992), Liu (1992), Francq (1999), Terdik (1999), Bibi and Oyet (2002). In those papers, probabilistic as well as statistical properties such as stationarity, ergodicity, invertibility, existence of higher order moments, central limit theorem, estimation problems including model identification and finding suitable white noise are examined.

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General bilinear process of order p, q, m, l is defined by

$$X_t = \sum_{i=1}^{p} a_i X_{t-i} + e_t + \sum_{j=1}^{q} b_j e_{t-j} + \sum_{i=1}^{m}\sum_{j=1}^{l} b_{ij} X_{t-i} e_{t-j}, \tag{1.1}$$

where e_t is a sequence of independent and identically distributed (iid) random variables.

As shown by Subba Rao and Gabr (1984), the bilinear model is particularly attractive in modelling processes with sample paths of occasional sharp spikes and when interactions between $\{X_t\}$ and $\{e_t\}$ are significant.

In this paper, we shall restrict ourselves to the so-called subdiagonal models for which $q = 0$ and $b_{ij} = 0$ if $i < j$. We aim at providing some simple easy-to-verify conditions that simultaneously imply stationarity and exponential β-mixing.

The question of the existence of a stationary solution for some classes of bilinear models is discussed in details by Rao *et al.* (1983). β-mixing with a geometric convergence rate of the bilinear models can be derived from geometric ergodicity of the Markov process of its representation.

Pham (1985, 1986) gives a necessary and sufficient condition for existence of a stationary solution, geometric ergodicity and β-mixing for the bilinear model. But as shown in those papers, writing a general bilinear model into a bilinear Markovian form is not an easy task and matrices involved in the representation are in general quite complicated. Also, evaluating irreducibility for such a model is far from easy task to check.

We begin by showing that subdiagonal bilinear model can be rewritten as a case of generalized polynomial random coefficient autoregressive model (GRCA). One of the advantages of such a technique is that it enables us to use the results on GRCA models (see, Doukhan, 1994; Carrasco and Chen, 2002). We give sufficient conditions for geometric ergodicity of the auxiliary Markov chain and then derive strict stationarity and β-mixing with exponential decay rate for given process.

This paper is organized as follows. Section 2 gives terminologies and previous results. Section 3 provides geometric ergodicity and β-mixing. In Section 4, ARMA model with bilinear innovations is considered.

2. PRELIMINARIES

Let $\{X_t : t = 0, 1, 2 \ldots\}$ be a discrete time Markov chain defined on R^k, $k \geq 1$ with time homogeneous n-step transition probabilities

$$P^{(n)}(x, A) = P(X_n \in A \mid X_0 = x), \ x \in R^k, \quad A \in \mathcal{B}(R^k),$$

where $\mathcal{B}(R^k)$ is a Borel σ-field on R^k.

DEFINITION 1. *A Markov chain $\{X_t\}$ is geometrically ergodic if there exists some probability measure π on $\mathcal{B}(R^k)$ and a positive real number $r < 1$ such that, for every x,*

$$r^{-n}||P^{(n)}(x, \cdot) - \pi(\cdot)|| \to 0 \quad as \quad n \to \infty,$$

where $||\cdot||$ denotes the total variation norm.

DEFINITION 2. *Let $\{X_t\}$ is a Markov process with invariant initial distribution π. Then $\{X_t\}$ is said to be stationary β-mixing (or absolutely regular) with exponential decay rate if there exist $0 < r < 1$ and $c > 0$ such that*

$$\int ||P^{(n)}(x, \cdot) - \pi(\cdot)||\pi(dx) \leq cr^n, \quad n = 1, 2, \ldots$$

REMARK 1. According to above Definitions 1 and 2, exponential β-mixing and geometric ergodicity are equivalent for Markov processes.

REMARK 2. Recall that β-mixing is stronger than strong mixing. Therefore geometrically ergodic Markov process accommodates limiting theorems such as functional central limit theorem and the law of iterated logarithm for β-mixing process and/or strong mixing process.

One of the most well-known condition used in establishing stationarity or (geometric) ergodicity of a Markov chain is the Lyapounov-Foster drift condition which is developed in a series papers by Tweedie and his associates (see, *e.g.*, Meyn and Tweedie, 1993 and references therein).

Drift condition. There exists an extended real valued nonnegative measurable function g with $g(x) < \infty$ for at least one x, such that for some constants $b < \infty$, $0 < \lambda < 1$ and a compact set K in $\mathcal{B}(R^k)$,

$$\int g(z)P(x, dz) \leq \lambda g(x) + bI_K(x), \tag{2.1}$$

where I_K denotes the indicator function of K.

For the following Theorem 1, we suppose that X_t is generated by

$$X_t = A(e_t)X_{t-1} + B(e_t), \tag{2.2}$$

where each of $A(e_t)$ and $B(e_t)$ are matrix valued polynomial function and vector valued polynomial function, respectively. Assume that the marginal distribution of e_t is absolutely continuous with respect to Lebesgue measure and the support of e_t contains an open set and zero. Also, e_t is independent of $\sigma(X_1, \ldots, X_{t-1})$. Let $\rho(A)$ denote the spectral radius of a matrix A. We shall need the following theorem due to Doukhan (1994) (see also Carrasco and Chen, 2002) to obtain our main results.

THEOREM 1. *Consider the process X_t in (2.2). Assume that $\rho(A(0)) < 1$, the series $\sum_{k=1}^{\infty}[\Pi_{j=0}^{k-1}A(e_{t-j})]B(e_{t-k})$ converges almost surely and the sequence $\Pi_{j=0}^{k-1}A(e_{t-j})x$ converges (as $k \to \infty$) to the 0 matrix almost surely for any x. If the drift condition (2.1) holds, the process X_t defined by (2.2) is Markov geometrically ergodic and $E[g(X_t)] < \infty$. Moreover if X_0 is initialized from the invariant distribution, then $\{X_t\}$ is strictly stationary and β-mixing with exponential decay.*

For further terminologies and results in Markov chain theory, we refer to Meyn and Tweedie (1993).

3. GEOMETRIC ERGODICITY AND β-MIXING

Consider the subdiagonal bilinear time series model given by

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + e_t + \sum_{i=1}^{m}\sum_{j=1}^{l} b_{ij}X_{t-i}e_{t-j}, \quad t = 1, 2, \ldots, \tag{3.1}$$

where $p \geq l$, $b_{ij} = 0$ for $i < j$ and $\{e_t\}$ is an iid sequence of random variables with finite second moment.

Without loss of generality, we assume that $p = m$ by taking $\phi_i = 0$ for $i > p$ and $b_{ij} = 0$ for $i > m$. Define

$$Y_t = (\mathbf{Z}_{t,t-p+1}, \mathbf{Z}_{t,t-p+1}e_t, \mathbf{Z}_{t-1,t-p+1}e_{t-1}, \ldots, \mathbf{Z}_{t-l+1,t-p+1}e_{t-l+1})', \tag{3.2}$$

where $\mathbf{Z}_{t-i,t-p+1} = (X_{t-i}, X_{t-i-1}, \ldots, X_{t-p+1})$, $i = 0, 1, \ldots, l-1$.

Let

$$A(e_t) = \begin{pmatrix} \Phi & \widetilde{b_1} & \widetilde{b_2} & \widetilde{b_3} & \dots & \widetilde{b_{l-1}} & \widetilde{b_l} \\ J_{(p-1)} & 0 & 0 & 0 & \dots & 0 & 0 \\ \Phi e_t & \widetilde{b_1} e_t & \widetilde{b_2} e_t & \widetilde{b_3} e_t & \dots & \widetilde{b_{l-1}} e_t & \widetilde{b_l} e_t \\ J_{(p-1)} e_t & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & J_{(p-1)} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & J_{(p-2)} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & J_{(p-3)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & J_{(p-l+1)} & 0 \end{pmatrix},$$

where

$$\begin{aligned} \Phi &= (\phi_1, \phi_2, \dots, \phi_p) \in R^p, \\ \widetilde{b_1} &= (b_{11}, b_{21}, \dots, b_{p1}) \in R^p, \\ \widetilde{b_2} &= (b_{22}, b_{32}, \dots, b_{p2}) \in R^{p-1}, \\ &\vdots \\ \widetilde{b_l} &= (b_{ll}, b_{l+1l}, \dots, b_{pl}) \in R^{p-l+1}, \end{aligned} \tag{3.3}$$

$I_{(k)}$ is the $k \times k$ identity matrix and $J_{(p-j)}$ is a $(p-j) \times (p-j+1)$ matrix defined by

$$J_{(p-j)} = \begin{pmatrix} I_{(p-j)} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{pmatrix}, \quad j = 1, 2, \dots, l. \tag{3.4}$$

$B(e_t) = (e_t, 0, \dots, 0, e_t^2, 0, \dots, 0)'$ is an $r \times 1$ vector all of whose components are zero except for the first and $(p+1)^{th}$ which are e_t and e_t^2, respectively. Here $r = (l+1)p - l(l-1)/2$ and $A(e_t)$ is an $r \times r$ matrix valued polynomial function. Then Y_t in (3.2) can be rewritten as

$$Y_t = A(e_t) Y_{t-1} + B(e_t).$$

We make the following assumptions.

Condition C1. $\{e_t\}$ is a sequence of *iid* random variables with finite variance. The probability distribution of e_t is absolutely continuous with respect to Lebesgue measure. The support of e_t is defined by its strictly positive density and contains an open set and zero.

Condition C2. $\sum_{i=1}^{p}|\phi_i| + \mu\sum_{i=1}^{m}\sum_{j=1}^{l}|b_{ij}| < 1$ with $\mu = E|e_t|$.
For any matrix $A = (a_{ij})$, $|A|$ denotes the matrix $(|a_{ij}|)$.

LEMMA 1. *Under the condition C2, $\rho(A(0)) < 1$ and $\rho(E|A(e_t)|) < 1$.*

PROOF. (1) Characteristic polynomial of $A(0)$ is $\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_p = 0$. Hence if $\sum_{i=1}^{p}|\phi_i| < 1$, then $\rho(A(0)) < 1$.
(2) By direct calculation of the characteristic polynomial of $E|A(e_t)|$, we can show that $\rho(E|A(e_t)|) < 1$ if $\sum_{i=1}^{p}|\phi_i| + \mu\sum_{i=1}^{m}\sum_{j=1}^{l}|b_{ij}| < 1$. □

LEMMA 2. *Under the condition C2, $\sum_{k=1}^{\infty}[\Pi_{j=0}^{k-1}A(e_{t-j})]B(e_{t-k})$ converges almost surely and the sequence $\Pi_{j=0}^{k-1}A(e_{t-j})x$ converges to the 0 matrix (as $k \to \infty$) almost surely for any $x \in R^r$.*

PROOF. By definition, both the sequence of random matrices $\{A(e_t)\}$ and the sequence of random vectors $\{B(e_t)\}$ are independent and $A(e_{t-j})$ is independent of $B(e_{t-k})$ for $k \neq j$. Let $A = E|A(e_t)|, B = E|B(e_t)|$. By above Lemma 1, we have that $\rho(A) < 1$ and hence $\sum_{k=1}^{\infty}A^k < \infty$. Therefore $\sum_{k=1}^{\infty}E[\Pi_{j=0}^{k-1}A(e_{t-j})]$ $B(e_{t-k}) = \sum_{k=1}^{\infty}A^kB < \infty$ and hence it follows that $\sum_{k=1}^{\infty}\Pi_{j=0}^{k-1}A(e_{t-j})B(e_{t-k}) < \infty$ almost surely and $\Pi_{j=0}^{k-1}A(e_{t-j})x$ converges almost surely to the zero matrix (see Chen and An, 1998). □

THEOREM 2. *Consider the process X_t of (3.1). Suppose that conditions C1 and C2 hold. Then Y_t in (3.2) is geometrically ergodic and if Y_t is starting with the stationary distribution, $\{X_t\}$ in (3.1) is strictly stationary and β-mixing with exponential decay rates.*

PROOF. The main part of the proof is to define a test function which satisfies the drift condition. For simplicity of notation, we assume that $p = m = 3$ and $\phi_i \geq 0,\ b_{ij} \geq 0,\ 1 \leq i, j, l \leq 3$. The case $p > 3$ is entirely analogous, but involves messier notation. In the case $p = m = l = 3$, define

$$Y_t = (X_t, X_{t-1}, X_{t-2}, X_te_t, X_{t-1}e_t, X_{t-2}e_t, X_{t-1}e_{t-1}, X_{t-2}e_{t-1}, X_{t-2}e_{t-2})'.$$

□

Define a test function $g : R^9 \to R$ by

$$g(x_1, \ldots, x_9) = \sum_{i=1}^{9}\gamma_i|x_i| + 1,$$

where nonnegative constants γ_i, $i = 1, 2, \ldots, 9$ are to be defined later. For $Y_{t-1} = y = (x_1, x_2, x_3, u_1, \ldots, u_6)'$, Y_t, given that $Y_{t-1} = y$, is defined by

$$[Y_t \mid Y_{t-1} = y] = (X_t, x_1, x_2, X_t e_t, x_1 e_t, x_2 e_t, u_1, u_2, u_4)'$$

with

$$X_t = \sum_{i=1}^{3} \phi_i x_i + b_{11} u_1 + b_{21} u_2 + b_{31} u_3 + b_{22} u_4 + b_{32} u_5 + b_{33} u_6 + e_t,$$

and hence we have that

$$\begin{aligned}
&E[g(Y_t) \mid Y_{t-1} = y] \\
&\le \gamma_1 (\sum_{i=1}^{3} \phi_i |x_i| + b_{11}|u_1| + \cdots + b_{33}|u_6|) + \gamma_2 |x_1| + \gamma_3 |x_2| \\
&\quad + \gamma_4 \mu (\sum_{i=1}^{3} \phi_i |x_i| + b_{11}|u_1| + \cdots + b_{33}|u_6|) \\
&\quad + \gamma_5 \mu |x_1| + \gamma_6 \mu |x_2| + \gamma_7 |u_1| + \gamma_8 |u_2| + \gamma_9 |u_4| + \gamma_1 \mu + \gamma_4 E(e_t^2) + 1 \\
&\le \sum_{i=1}^{3} \gamma_i^* |x_i| + \sum_{i=1}^{6} \gamma_{i+3}^* |u_i| + \gamma_1 \mu + \gamma_4 E(e_t^2) + 1, \qquad (3.5)
\end{aligned}$$

where

$$\gamma_1^* = \gamma_1 \phi_1 + \gamma_2 + \gamma_4 \phi_1 \mu + \gamma_5 \mu, \qquad (3.6)$$
$$\gamma_2^* = \gamma_1 \phi_2 + \gamma_3 + \gamma_4 \phi_2 \mu + \gamma_6 \mu, \qquad (3.7)$$
$$\gamma_3^* = \gamma_1 \phi_3 + \gamma_4 \phi_3 \mu, \qquad (3.8)$$
$$\gamma_4^* = \gamma_1 b_{11} + \gamma_4 b_{11} \mu + \gamma_7, \qquad (3.9)$$
$$\gamma_5^* = \gamma_1 b_{21} + \gamma_4 b_{21} \mu + \gamma_8, \qquad (3.10)$$
$$\gamma_6^* = \gamma_1 b_{31} + \gamma_4 b_{31} \mu, \qquad (3.11)$$
$$\gamma_7^* = \gamma_1 b_{22} + \gamma_4 b_{22} \mu + \gamma_9, \qquad (3.12)$$
$$\gamma_8^* = \gamma_1 b_{32} + \gamma_4 b_{32} \mu, \qquad (3.13)$$
$$\gamma_9^* = \gamma_1 b_{33} + \gamma_4 b_{33} \mu. \qquad (3.14)$$

Now from the condition C2, we can choose a nonnegative constant $\rho < 1$ so that

$$\sum_{i=1}^{3} \phi_i + \mu \sum_{i=1}^{3} \sum_{j=1}^{3} b_{ij} < \rho^3 < \rho^2 < \rho < 1. \qquad (3.15)$$

Choose $\gamma_1 > 0$ arbitrarily. Let $\gamma_i^* = \rho\gamma_i,\ 2 \le i \le 9$ and solve the equation (3.7)–(3.14). From simple calculation, we have that

$$\gamma_2 = a(\frac{1}{\rho}\phi_2 + \frac{1}{\rho^2}\phi_3 + \frac{1}{\rho^2}b_{31}\mu), \tag{3.16}$$
$$\gamma_3 = \frac{a}{\rho}\phi_3, \tag{3.17}$$
$$\gamma_4 = ak, \tag{3.18}$$
$$\gamma_5 = a(\frac{1}{\rho}b_{21} + \frac{1}{\rho^2}b_{32}), \tag{3.19}$$
$$\gamma_6 = \frac{a}{\rho}b_{31}, \tag{3.20}$$
$$\gamma_7 = a(\frac{1}{\rho}b_{22} + \frac{1}{\rho^2}b_{33}), \tag{3.21}$$
$$\gamma_8 = \frac{a}{\rho}b_{32}, \tag{3.22}$$
$$\gamma_9 = \frac{a}{\rho}b_{33}, \tag{3.23}$$

where

$$a = \gamma_1 + \mu\gamma_4 = \frac{\gamma_1}{1-\mu k} \tag{3.24}$$

and $k = b_{11}/\rho + b_{22}/\rho^2 + b_{33}/\rho^3$. Note that $\gamma_i > 0,\ 2 \le i \le 9$ if $1-\mu k > 0$, that is, $\mu(b_{11}/\rho + b_{22}/\rho^2 + b_{33}/\rho^3) < 1$. From (3.6), (3.15), (3.16), (3.19) and (3.24), we can drive that

$$\begin{aligned}\gamma_1^* &= \gamma_1\phi_1 + \gamma_2 + \gamma_4\phi_1\mu + \gamma_5\mu \\ &= \frac{\gamma_1}{1-\mu k}(\phi_1 + \frac{1}{\rho}\phi_2 + \frac{1}{\rho^2}\phi_3 + \frac{1}{\rho}b_{21}\mu + \frac{1}{\rho^2}b_{31}\mu + \frac{1}{\rho^2}b_{32}\mu) \\ &< \rho\gamma_1.\end{aligned} \tag{3.25}$$

Therefore, from (3.5)–(3.15) and (3.25),

$$E[g(Y_t) \mid Y_{t-1} = y] \le \rho g(y) + c, \quad c = 1 + \gamma_1\mu + \gamma_4 E(e_t^2) < \infty.$$

For the general case that $p = m \ge 4$, we may find positive $\gamma_i,\ 1 \le i \le (p^2+3p)/2$ if

$$\begin{aligned}\phi_1\rho^{p-1} &+ \phi_2\rho^{p-2} + \cdots + \phi_p + \mu\{\rho^{p-1}b_{11} + \rho^{p-2}(b_{21}+b_{22}) + \rho^{p-3}(b_{31}+b_{32}+b_{33}) \\ &+ \rho^{p-4}(b_{41}+b_{42}+b_{43}+b_{44}) + \cdots + (b_{p1}+b_{p2}+\cdots+b_{pp})\} \\ &< \rho^p,\end{aligned}$$

which follows from choice of $\rho > 0$ such that $\sum \phi_i + \mu \sum \sum b_{ij} < \rho^p < \rho < 1$.

By equivalence of norms in R^k, for any given $\epsilon > 0$, there exist nonnegative constants ρ', $\rho < \rho' < 1$ and $M < \infty$ such that

$$E[g(Y_t) \mid Y_{t-1} = y] \leq \rho' g(y) - \epsilon \quad \text{if } y \in B_M^c$$

and

$$\sup_{y \in B_M^c} E[g(Y_t) \mid Y_{t-1} = y] < \infty \quad \text{if } y \in B_M,$$

where $B_M = \{y \mid ||y|| \leq M\}$.

Applying Lemmas 1, 2 and Theorem 1 yields the geometric ergodicity of Y_t. If Y_0 is initialized from the invariant distribution, $\{X_t\}$ is strictly stationary and β-mixing with exponential decay.

REMARK. In addition to the assumption (C1) and $\rho(A(0)) < 1$, we assume that $E[\rho(A(e_t))]^s < 1$ and $E||B(e_t)||^s < \infty$ for some even integer $s \geq 2$. Then conclusions of Theorem 2 and $E[||X_t||^s] < \infty$ follow (see, Proposition 3 in Carrasco and Chen, 2002).

4. ARMA MODELS WITH BILINEAR ERRORS

In this section, we consider an ARMA model with bilinear innovations and derive a sufficient condition for strict stationarity and exponential β-mixing.

Consider the model defined by

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t \quad \text{and} \tag{4.1}$$

$$\epsilon_t = e_t + \sum_{i=1}^{P} \sum_{j=1}^{Q} b_{ij} \epsilon_{t-i} e_{t-j}. \tag{4.2}$$

Assume that $b_{ij} = 0$ if $i < j$.

THEOREM 3. *Consider the model defined by (4.1) and (4.2). If* $(\sum |\phi_i| + 1)(\mu \sum \sum |b_{ij}| + 1) < 2$, *then there exists a strictly stationary solution of (4.1)-(4.2) and* X_t *with a stationary initial distribution is stationary* β*-mixing.*

PROOF. We may assume without loss of generality that $p = q = P = Q$. For simplicity of notation, we assume that $\phi_i \geq 0$ and $b_{ij} \geq 0$.

We only consider the case that $p = q = P = Q = 2$. For this case, define

$$Z_t = (X_t, X_{t-1}, \epsilon_t, \epsilon_{t-1}, \epsilon_t e_t, \epsilon_{t-1} e_t, \epsilon_{t-1} e_{t-1})'.$$

Then $Z_t = C(e_t) Z_{t-1} + D(e_t)$ where $C(e_t)$ and $D(e_t)$ are given by

$$C(e_t) = \begin{pmatrix} \phi_1 & \phi_2 & \theta_1 & \theta_2 & b_{11} & b_{21} & b_{22} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{11} & b_{21} & b_{22} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{11} e_t & b_{21} e_t & b_{22} e_t \\ 0 & 0 & e_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

and $D(e_t) = (e_t, 0, e_t, 0, e_t^2, 0, 0)'$. From simple calculation, we can obtain that $\rho(E|C(e_t)|) < 1$ if $(\phi_1 + \phi_2 + 1)(\mu(b_{11} + b_{21} + b_{22}) + 1) < 2$ with $\mu = E|e_t|$.

Define a test function $g : R^7 \to R$ by

$$g(x_1, x_2, \cdots, x_7) = \sum_{i=1}^{7} \gamma_i |x_i| + 1$$

with an appropriate choice of $\gamma_i \geq 0$ for $i = 1, 2, \cdots, 7$ which is given below.

Now for given $Z_{t-1} = z = (y_1, y_2, u_1, u_2, w_1, w_2, w_3)$, we have that

$$\begin{aligned} & E[g(Z_t) \mid Z_{t-1} = z] \\ & = E[g(Y_t, y_1, \epsilon_t, u_1, \epsilon_t e_t, u_1 e_t, w_1) \mid Z_{t-1} = z] \\ & \leq (\gamma_1 \phi_1 + \gamma_2)|y_1| + \gamma_1 \phi_2 |y_2| + (\gamma_1 \theta_1 + \gamma_4 + \gamma_6 \mu)|u_1| \\ & \quad + \gamma_1 \theta_2 |u_2| + (\gamma_1 b_{11} + \gamma_3 b_{11} + \gamma_5 b_{11} \mu + \gamma_7)|w_1| \\ & \quad + (\gamma_1 b_{21} + \gamma_3 b_{21} + \gamma_5 b_{21} \mu)|w_2| + (\gamma_1 b_{22} + \gamma_3 b_{22} + \gamma_5 b_{22} \mu)|w_3|. \end{aligned} \tag{4.3}$$

Choose $\rho > 0$ so that $xy + x + y < \rho^2 < \rho < 1$ with $x = \sum |\phi_i|$ and $y = \mu \sum \sum |b_{ij}|$. Then define $\gamma_1 > 0$ arbitrarily and fix and let

$$\begin{aligned} \gamma_2 &= \frac{1}{\rho} \phi_2 \gamma_1, \quad \gamma_4 = \frac{1}{\rho} \theta_2 \gamma_1, \\ \gamma_3 &= \frac{(b_{11}\rho + b_{22})(\rho^2 + \rho\theta_1 + \theta_2)}{\rho^2(\rho^2 - b_{11}\rho\mu - b_{21}\mu - b_{22}\mu)} \gamma_1, \\ \gamma_5 &= \frac{(b_{11}\rho + b_{22})}{\rho^2 - b_{11}\rho\mu - b_{22}\mu} (\gamma_1 + \gamma_3), \\ \gamma_6 &= \frac{b_{21}}{\rho} (\gamma_1 + \gamma_3 + \gamma_5 \mu), \\ \gamma_7 &= \frac{b_{22}}{\rho} (\gamma_1 + \gamma_3 + \gamma_5 \mu). \end{aligned}$$

Then all $\gamma_i > 0,\ 1 \le i \le 7$ and we can easily derive that

$$E[g(Z_t) \mid Z_{t-1} = z] \le \rho g(z) + c', \quad c' = \gamma_1 \mu + \gamma_3 \mu + \gamma_5 E(e_t^2) < \infty,$$

and the drift condition (2.1) holds and hence the conclusion of Theorem 3 follows.

For the case that $\max\{p, q, P, Q\} \ge 3$, the bottom line of the proof is the same as above. □

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