

# On the Distribution and Its Properties of the Sum of a Normal and a Doubly Truncated Normal

Hea-Jung Kim<sup>1)</sup>

## Abstract

This paper proposes a class of distributions which is useful in making inferences about the sum of values from a normal and a doubly truncated normal distribution. It is seen that the class is associated with the conditional distributions of truncated bivariate normal. The salient features of the class are mathematical tractability and strict inclusion of the normal and the skew-normal laws. Further it includes a shape parameter, to some extent, controls the index of skewness so that the class of distributions will prove useful in other contexts. Necessary theories involved in deriving the class of distributions are provided and some properties of the class are also studied.

*Keywords* : Doubly truncated normal; Class of distributions; Skewness.

## 1. Introduction

Azzalini (1985) and Henze (1986) worked on the skew-normal (written  $SN(\lambda)$ ) distribution, a class of distributions including the standard normal, but with an extra parameter  $(-\infty < \lambda < \infty)$  to regulate skewness. A probabilistic representation of the distribution is given in terms of normal and half normal laws:

$$\frac{1}{\sqrt{1+\lambda^2}}Z_1 + \frac{\lambda}{\sqrt{1+\lambda^2}}|Z_2| \sim SN(\lambda), \quad (1.1)$$

where  $Z_1$  and  $Z_2$  be independent standard normal variables. The distribution is useful to modeling random phenomena which have heavier tails than the normal as well as some skewness that appears in screening problems and regression problems with skewed error structure. We refer to Arnold et al. (1993), Chen, Dey and Shao (1999), and Kim (2002) for the applications of the distribution to those problems. Various extensions of the  $SN(\lambda)$  distribution have been given in the literatures: In one direction, Diccio and Monti (2004) and Ma and Genton (2004),

---

1) Professor, Department of Statistics, Dongguk University, Seoul 100-715, Korea.  
E-mail : kim3hj@dongguk.edu

among others, tried to find more flexible distributions that cope with prevalent deviations from normality. In the other direction, multivariate extensions of the distribution have been developed by Azzalini and Valle (1996) and Branco and Dey (2001), among others.

The purpose of the present paper is to extend the  $SN(\lambda)$  distribution in terms of yet other direction. This proposes a class of distributions that accounts for the sum of a normal and a doubly truncated normal distributions, that is the distributions of the form  $Z = c_1 V + c_2 W_{(a,b)}$ , where  $c_1$  and  $c_2$  are any real values,  $V \sim N(\mu_1, \tau_1^2)$ , and  $W_{(a,b)} \sim TN_{(a,b)}(\mu_2, \tau_2^2)$ , a truncated  $N(\mu_2, \tau_2^2)$  with respective lower and upper truncation points  $a$  and  $b$ . Such an extension is potentially relevant for practical applications, since in data analysis there are a few distributions available to dealing with the sum (or ratio) of values from a normal and a doubly truncated normal distribution. Moreover, the distribution of a variable  $Z$  of the form  $Z = c_1 V + c_2 W_{(a,b)}$  arises frequently in the theory of nonparametric statistics. We discuss examples of the nonparametric statistics in Section 4. Necessary theories involved in deriving the class of distributions is provided and some properties of the class is also studied.

## 2. The Class of Distributions

This section proposes a class of distributions of the sum of a normal and a doubly truncated normal distributions by use of the conditional truncated bivariate normal distribution.

Suppose  $f$  is the density function of a bivariate normal random variable with mean vector  $\theta = (\theta_1, \theta_2)'$  covariance matrix  $\Sigma = (\sigma_{ij})$ ,  $\sigma_{ii} = \sigma_i^2$ , with correlation  $\rho$ . Suppose that  $(X, Y)$  has joint density

$$g(x, y) = f(x, y) / [\Phi(u(b)) - \Phi(u(a))], \quad -\infty < x < \infty; a < y < b \quad (2.1)$$

where  $u(a) = (a - \theta_2) / \sigma_2$ ,  $u(b) = (b - \theta_2) / \sigma_2$ ,  $\Phi(\cdot)$  is the df of a standard normal variable, and  $a$  and  $b$  are real constants that are the lower and upper truncation points for  $Y$ , respectively. Clearly,  $(X, Y)$  has a doubly truncated bivariate normal distribution. By direct integration one obtains the density of  $Z \equiv X | a < Y < b$  given by

$$h_z(z) = \frac{\phi(u(z))[\Phi(\lambda_1 u(b) - \lambda u(z)) - \Phi(\lambda_1 u(a) - \lambda u(z))]}{\sigma_1 [\Phi(u(b)) - \Phi(u(a))]}, \quad -\infty < z < \infty \quad (2.2)$$

where  $u(z) = (z - \theta_1) / \sigma_1$ ,  $\lambda = \rho / \sqrt{1 - \rho^2}$ ,  $\lambda_1 = \lambda / \rho$  and  $\phi(\cdot)$  is the pdf of the standard normal variable. The following lemma gives a relation between the conditional distribution of a truncated bivariate normal and the sum of a normal

and a doubly truncated normal distributions.

Lemma 1. Let  $V \sim N(\mu_1, \tau_1^2)$  and  $W \sim N(\mu_2, \tau_2^2)$  be independent random variables and  $(X, Y)$  is a bivariate normal random variable with mean vector  $\theta_c = (\theta_1, \theta_2)$  and covariance  $\Sigma_c = (\sigma_{ij})$ ,  $\sigma_{ii} = \sigma_i^2$  with  $\theta_1 = \sum_{i=1}^2 c_i \mu_i$ ,  $\theta_2 = \mu_2$ ,  $\sigma_1^2 = \sum_{i=1}^2 c_i^2 \tau_i^2$ ,  $\sigma_2^2 = \tau_2^2$  and  $\rho = c_2 \sigma_2 / \sigma_1$ . Then for any real values  $c_1$  and  $c_2$  with  $c_1 \neq 0$ , the distribution of  $Z = c_1 V + c_2 W_{(a,b)}$  is equivalent to the conditional distribution of  $X$  given  $a < Y < b$ ;

$$Z = c_1 V + c_2 W_{(a,b)} \equiv X | a < Y < b \tag{2.3}$$

where  $W_{(a,b)} \sim TN_{(a,b)}(\mu_2, \tau_2^2)$ , a truncated  $N(\mu_2, \tau_2^2)$  with respective lower and upper truncation points  $a$  and  $b$ .

Proof. Let  $X = c_1 V + c_2 W$  and  $Y = W$ . Then  $(X, Y)$  is a bivariate normal random variable with mean vector  $\theta_c$  and covariance matrix  $\Sigma_c$ . Since  $V$  and  $W$  are independent, the distribution of  $X$  conditionally on  $a < Y < b$  equals that of  $c_1 V + c_2 W_{(a,b)}$ .

From now on, for convenience, we shall denote

$$\theta_1 = \sum_{i=1}^2 c_i \mu_i, \theta_2 = \mu_2, \sigma_1^2 = \sum_{i=1}^2 c_i^2 \tau_i^2, \sigma_2^2 = \tau_2^2, \rho = c_2 \sigma_2 / \sigma_1, \lambda = \rho / \sqrt{1 - \rho^2}, \lambda_1 = \lambda / \rho,$$

$$u(z) = (z - \theta_1) / \sigma_1, u(a) = (a - \theta_1) / \sigma_1 \text{ and } u(b) = (b - \theta_1) / \sigma_1.$$

Under the notations, Lemma 1 leads to the following definition.

Definition 1. Suppose  $V \sim N(\mu_1, \tau_1^2)$  and  $W_{(a,b)} \sim TN_{(a,b)}(\mu_2, \tau_2^2)$  are independent random variables. Then distribution of  $Z = c_1 V + c_2 W_{(a,b)}$  with the probability density function (2.2) is denoted by  $NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$ , where  $c = (c_1, c_2)$ .

The definition and Lemma 1 imply that the distribution function(df) of  $Z$  variable with  $NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$  distribution is the same as the conditional df of the form  $P(X \leq z | a < Y < b)$  where  $(X, Y)$  is a bivariate normal with the mean vector  $\theta_c$  and covariance matrix  $\Sigma_c$ . Thus the df of  $Z$  is

$$F_Z(z) = 1 - \frac{L(u(z), u(a), \rho) - L(u(z), u(b), \rho)}{\Phi(u(b)) - \Phi(u(a))}, \quad -\infty < z < \infty, \tag{2.4}$$

where  $L(c, d, \rho) = P(V_1 > c, V_2 > d)$  is the orthant probability of the standard bivariate normal variable  $(V_1, V_2)$ . Computing methods that evaluate  $L(c, d, \rho)$

have been given by Donnelly (1973) and Joe (1995), among others. Definition 1 also reveals the structure of the class of  $NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$  distributions and indicates the kind of departure from the normality. Furthermore, it provides one-for-one method of generating a random variable  $Z$  with density (2.2). For generating the truncated normal variable  $W_{(a,b)}$ , the one-for-one method by Devroye (1986) may be used. Now we will state some interesting properties for the  $NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$  distribution as well as the associate results.

Property 1. Let  $U(Z) = (Z - \theta_1)/\sigma_1$ , where  $Z \sim NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$ . Then

$$U(Z) \sim NT_{(a_1,b_1)}(c(\tau), 0, 0, 1, 1), \tag{2.5}$$

where  $c(\tau) = (c_1(\tau), c_2(\tau))$  with  $c_1(\tau) = c_1\tau_1/\sqrt{\sum_{i=1}^2 c_i^2\tau_i^2}$  and  $c_2(\tau) = c_2\tau_2/\sqrt{\sum_{i=1}^2 c_i^2\tau_i^2}$ , and  $a_1 = (a - \mu_2)/\tau_2$  and  $b_1 = (b - \mu_2)/\tau_2$ .

Property 2. For  $Z \sim NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$ , if  $c_1 = 0$  and  $c_2 = 1$   $Z \sim TN_{(a,b)}(\mu_2, \tau_2^2)$  while  $c_1 = 1$  and  $c_2 = 0$  reduces to  $Z \sim N(\mu_1, \tau_1^2)$ .

Property 3. If  $Z \sim NT_{(\theta_2,\infty)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$ , then  $Z = \theta_1 + \sigma_1 U$ , where  $U \sim SN(\lambda)$ . The pdf of  $Z$  is

$$h_Z(z) = 2\phi(u(z))\Phi(\lambda u(z))/\sigma_1 \text{ for } -\infty < z < \infty. \tag{2.6}$$

Corollary 1. Let  $V$  and  $W$  be independent standard normal random variables, and let  $c_1(\lambda) = 1/\sqrt{1+\lambda^2}$  and  $c_2(\lambda) = \lambda/\sqrt{1+\lambda^2}$ . Then

$$Z = \frac{1}{\sqrt{1+\lambda^2}} V + \frac{\lambda}{\sqrt{1+\lambda^2}} W_{(a,b)} \sim NT_{(a,b)}(c(\lambda), 0, 0, 1, 1), \tag{2.7}$$

where  $c(\lambda) = (c_1(\lambda), c_2(\lambda))$ .

Proof. Setting  $\mu_1 = \mu_2 = 0$  and  $\tau_1 = \tau_2 = 1$ , we have the result from Lemma 1.

Property 4.  $NT_{(0,\infty)}(c(\lambda), 0, 0, 1, 1)$  distribution is equivalent to  $SN(\lambda)$  distribution.

Property 5. If  $Z \sim NT_{(a,b)}(c(\lambda), 0, 0, 1, 1)$  and  $Z_1$  is an independent  $N(0, 1)$  random variable, then the distribution of  $(Z + Z_1)/\sqrt{2}$  is  $Z \sim NT_{(a,b)}(c, 0, 0, 1, 1)$ , where  $c_1 = 1/\sqrt{1+\lambda^2}$  and  $c_2 = \lambda/\sqrt{2(1+\lambda^2)}$ .

Corollary 2. If  $W$  is a  $N(0, 1)$  random variable then, for any real  $\lambda$  and  $\beta$ ,

$$E[\Phi(\lambda W_{(a,b)} + \beta)] = \frac{L(-\beta/\sqrt{1+\lambda^2}, a, \rho) - L(-\beta/\sqrt{1+\lambda^2}, b, \rho)}{\Phi(b) - \Phi(a)}, \tag{2.8}$$

where  $\rho = \lambda/\sqrt{1+\lambda^2}$ .

Proof. Let  $V \sim N(0, 1)$  independently of  $W$  and let  $W_{(a,b)} = U$ . Then

$$\begin{aligned} E[\Phi(\lambda U + \beta)] &= \frac{1}{2\pi(\Phi(b) - \Phi(a))} \int_a^b \int_{-\infty}^{\lambda u + \beta} e^{-v^2/2} e^{-u^2/2} \partial v \partial u \\ &= P(V - \lambda U \leq \beta) = 1 - P(Z \leq -\beta/\sqrt{1+\lambda^2}), \end{aligned}$$

where  $Z \sim NT_{(a,b)}(c(\lambda), 0, 0, 1, 1)$  random variable by Corollary 1. Thus the df in (2.4) gives the result.

Corollary 2 immediately gives the following results for  $W \sim N(0, 1)$  random variable: Upon setting  $a = -\infty$  and  $b = \infty$ , we have, for any real  $\lambda$  and  $\beta$ ,

$$E[\Phi(\lambda W + \beta)] = \Phi(\beta/\sqrt{1+\lambda^2}). \tag{2.9}$$

The expectation was derived by a different method (see, for instance, Zacks 1981, pp.53-54). When  $a = 0$  and  $b = \infty$ , (2.8) yields

$$E[\Phi(\lambda|W| + \beta)] = \Phi(\beta/\sqrt{1+\lambda^2}) + 2T(\beta/\sqrt{1+\lambda^2}, \lambda), \tag{2.10}$$

for any real  $\lambda$  and  $\beta$ , where  $T(c, d)$ ,  $c > 0$ ,  $d > 0$ , is the function which gives the integral of the standard bivariate density over the right side region bounded by lines  $x = c$ ,  $y = 0$ , and  $y = dx$  in the  $(x, y)$  plane(see, Azzalini 1985, for the properties of the function). When  $V$  and  $W$  are independent standard normal variables,  $T = V/|W|$  has a standard Cauchy distribution. From this fact, we see that the df of  $T$  can be expressed as  $F_T(t) = E[\Phi(t|W|)]$  by (2.10), where  $E[\Phi(t|W|)] = \Phi(0) + 2T(0, t) = 1/2 + \tan^{-1}t/\pi$ .

### 3. Moments

#### 3.1 Moment Generating Function

To compute the moments of the  $NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$  distribution, it suffices to compute the moments of  $U(Z) = (Z - \theta_1)/\sigma_1$ . From Property 1, we see that  $U(Z)$  has the density

$$g(u(z)) = \frac{\phi(u(z))[\Phi(\lambda_1 b_1 - \lambda u(z)) - \Phi(\lambda_1 a_1 - \lambda u(z))]}{[\Phi(b_1) - \Phi(a_1)]}, \quad -\infty < u(z) < \infty, \tag{3.1}$$

where  $a_1 = (a - \mu_2)/\tau_2$ ,  $b_1 = (b - \mu_2)/\tau_2$ ,  $\lambda = c_2\tau_2/c_1\tau_1$  and  $\lambda_1 = \sqrt{\sum_{i=1}^2 c_i^2 \tau_i^2} / (c_1\tau_1)$ .

Theorem 1. Let  $Z \sim NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$ , then the moment generating function of  $U(Z) = (Z - \theta_1)/\sigma_1$  is

$$M_{U(Z)}(t) = e^{t^2/2} \frac{\Phi(a_1 - \rho t) - \Phi(b_1 - \rho t)}{\Phi(b_1) - \Phi(a_1)} \quad \text{for } -\infty < t < \infty, \tag{3.2}$$

where  $\rho = \lambda / \sqrt{1 + \lambda^2} = c_2 \tau_2 / \sqrt{\sum_{i=1}^2 c_i^2 \tau_i^2}$ .

Proof. Considering the pdf (3.1), we have  $M_{U(Z)}(t)$  written as

$$\int_{-\infty}^{\infty} \frac{e^{t^2/2} [\Phi(\lambda_1 b_1 - \lambda u(z)) - \Phi(\lambda_1 a_1 - \lambda u(z))]}{\sqrt{2\pi} [\Phi(b_1) - \Phi(a_1)]} e^{-(u(z)-t)^2/2} du(z).$$

Using the transformation  $u(z) = w(z) + t$ , one finds

$$M_{U(Z)}(t) = \frac{e^{t^2/2}}{\sqrt{2\pi} [\Phi(b_1) - \Phi(a_1)]} (E[\Phi(\lambda_1 b_1 - \lambda t - \lambda W(Z))] - E[\Phi(\lambda_1 a_1 - \lambda t - \lambda W(Z))]),$$

where  $W(Z) \sim N(0, 1)$ . Thus we can derive the mgf (3.2) by applying (2.9).

### 3.2 Moments of the Distribution

Naturally, the moments of  $U(Z) = (Z - \theta_1)/\sigma_1$  can be obtained by using the moment generating function differentiation. For example:

$$E[U(Z)] = dM_{U(Z)}(t)/dt |_{t=0} = -\rho \frac{\phi(b_1) - \phi(a_1)}{\Phi(b_1) - \Phi(a_1)}.$$

Unfortunately, for higher moments this rapidly becomes tedious. An alternative procedure makes use of the fact that

$$\frac{d}{dx} [x^{k+1} \phi(x)] = (k+1)x^k \phi(x) - x^{k+2} \phi(x), \quad \text{for } k = -1, 0, 1, 2, 3, \dots, \tag{3.3}$$

yields the following result.

Under the distribution (3.1), the relation (3.3) and integrating by parts gives the following moments.

$$\begin{aligned} & E[(k+1)U(Z)^k - U(Z)^{k+2}] \\ &= \frac{\int_{-\infty}^{\infty} \{(k+1)u(z)^k - u(z)^{k+2}\} \phi(u(z)) \{\Phi(\lambda_1 b_1 - \lambda u(z)) - \Phi(\lambda_1 a_1 - \lambda u(z))\} du(z)}{\Phi(b_1) - \Phi(a_1)} \\ &= \frac{\rho \{ \phi(b_1) E[W + \lambda b_1]^{k+1} - \phi(a_1) E[W + \lambda a_1]^{k+1} \}}{\lambda_1^{k+1} \{ \Phi(b_1) - \Phi(a_1) \}} \quad \text{for } k \geq -1, \end{aligned} \tag{3.4}$$

where  $W \sim N(0, 1)$ . By setting  $k = -1, 0, 1$  and applying (2.9), we obtain three expressions, which may be solved to yield the first three moments of  $U(Z)$ . Higher

moments could be found similarly. One obtains

$$E[U(Z)] = -\rho \frac{\phi(b_1) - \phi(a_1)}{\Phi(b_1) - \Phi(a_1)},$$

$$E[U(Z)^2] = 1 - \rho^2 \frac{b_1\phi(b_1) - a_1\phi(a_1)}{\Phi(b_1) - \Phi(a_1)},$$

$$E[U(Z)^3] = \frac{\rho[(3 - \rho^2 + b_1\rho^2)\phi(b_1) - (3 - \rho^2 + a_1\rho^2)\phi(a_1)]}{\Phi(b_1) - \Phi(a_1)}.$$

By using the Binomial expansion, one can see that the general formula for the moments of  $Z \sim NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$  is

$$E[Z^k] = \sum_{j=0}^k \binom{k}{j} \theta_1^{k-j} \sigma_1^j E[U(Z)^j]. \tag{3.5}$$

When  $\theta_1 = 0$ ,  $E[Z^k] = \sigma_1^k E[U(Z)^k]$ . We recall the functions  $\alpha_3(a_1, b_1)$  and  $\alpha_4(a_1, b_1)$  studied by Sugiura and Gombi (1985). For real values of  $a_1$  and  $b_1 (a_1 < b_1)$ ,  $\alpha_3(a_1, b_1)$  and  $\alpha_4(a_1, b_1)$  give respective skewness and kurtosis of doubly truncated standard normal distribution. Here  $a_1$  and  $b_1$  denote the left and right truncation points of the standard normal distribution. Using these functions and the above moments, we have the following result.

**Theorem 2.** For  $Z \sim NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$ , the skewness(the standardized central moment) and kurtosis of the distribution are

$$\alpha_3 = \rho^3 (\beta_1/\beta_2)^{3/2} \alpha_3(a_1, b_1) \tag{3.6}$$

and 
$$\alpha_4 = \{3(1 - \rho^2)^2 + \rho^4 \beta_1^2 \alpha_4(a_1, b_1) + 6\rho^2(1 - \rho^2)\beta_1\} / \beta_2^2, \tag{3.7}$$

respectively, where  $\rho = c_3\tau_2 / \sqrt{\sum_{i=1}^2 c_i^2\tau_i^2}$ ,  $a_1 = (a - \mu_2)/\tau_2$ ,  $b_1 = (b - \mu_2)/\tau_2$ ,

$$\beta_1 = \left\{ 1 + \frac{a_1\phi(a_1) - b_1\phi(b_1)}{\Phi(a_1) - \Phi(b_1)} - \left( \frac{\phi(a_1) - \phi(b_1)}{\Phi(a_1) - \Phi(b_1)} \right)^2 \right\},$$

$$\beta_2 = 1 - \rho^2 \left\{ \frac{a_1\phi(a_1) - b_1\phi(b_1)}{\Phi(a_1) - \Phi(b_1)} + \left( \frac{\phi(a_1) - \phi(b_1)}{\Phi(a_1) - \Phi(b_1)} \right)^2 \right\}.$$

**Proof.** From Property 1, we see that  $\alpha_3$  and  $\alpha_4$  of  $Z$  distribution is the same as those of  $U(Z) = \sqrt{1 - \rho^2} V + \rho W_{(a_1, b_1)}$  distribution, where  $V$  and  $W$  are independent standard normal variables. From the above moments,  $Var(U(Z)) = \beta_2$ . Therefore, some algebra using the fact that  $Var(W_{(a_1, b_1)}) = \beta_1$  and  $E[W_{(a_1, b_1)}] = (\phi(b_1) - \phi(a_1)) / [\Phi(b_1) - \Phi(a_1)]$  (see, for example, Johnson et al. 1994, pp.156-158), gives the result.

Corollary 3. For  $Z \sim NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$ , the skewness depends on the following cases: (i) If  $|a_1| < b_1$ , the distribution of  $Z$  variable is skewed to the right(the left) when  $\rho > 0(\rho < 0)$ . (ii) If  $|a_1| > b_1$ , the distribution is skewed to the right(the left) when  $\rho < 0(\rho > 0)$ . (iii) If  $|a_1| = b_1$  for  $a_1 \neq 0$ , the distribution is symmetric, where

$$a_1 = (a - \mu_2)/\tau_2, \quad b_1 = (b - \mu_2)/\tau_2, \quad \text{and} \quad \rho = c_2\tau_2 / \sqrt{\sum_{i=1}^2 c_i^2 \tau_i^2}.$$

Proof. It is obvious that  $\alpha_3(a_1, b_1) > 0$  for  $|a_1| < b_1$ ,  $\alpha_3(a_1, b_1) < 0$  for  $|a_1| > b_1$ , and  $\alpha_3(a_1, b_1) = 0$  for  $|a_1| = b_1$ . This fact and (3.6) immediately gives the result.

The class of  $NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$  distribution includes well-known skew-normal distribution,  $SN(\theta_1, \sigma_1, \lambda)$ , by Azzalini (1985).  $SN(\theta_1, \sigma_1, \lambda)$  distributions is equivalent to  $NT_{(\theta_2, \infty)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$  as given in Property 3.

From (3.5) and (3.6), one finds the moments of  $Z \sim SN(\theta_1, \sigma_1, \lambda)$   $E[Z] = \theta_1 + \sigma_1 \rho \sqrt{2/\pi}$ ,  $Var(Z) = \sigma_1^2(1 - 2\rho^2/\pi)$ , and  $\alpha_3 = (4/\pi - 1) \sqrt{2/\pi} \rho^3(1 - 2\rho^2/\pi)^{-3/2}$ . These values of  $Z \sim SN(\theta_1, \sigma_1, \lambda)$  agree with those given in Arnold et al. (1993). See Azzalini (1985) and Henze (1986) for the other properties of the  $SN(\theta_1, \sigma_1, \lambda)$  distribution.

## 4. Applications

### 4.1. Nonparametric Statistics

The distribution  $NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$  of a variate  $Z = c_1 V + c_2 W_{(a,b)}$  arises frequently in the theory of nonparametric statistics, where  $V \sim N(\mu_1, \tau_1^2)$  and  $W_{(a,b)} \sim TN_{(a,b)}(\mu_2, \tau_2^2)$  with  $a \geq 0$ . Let  $c_1 = 1$  and  $c_2 = -c$ . Then the distribution function of the ratio of normal and truncated normal can be directly obtained from that of  $Z$  variate, i.e.  $P(Z \leq 0) = P(V/W_{(a,b)} \leq c)$ . Further assume that  $V$  and  $W$  are standard normals, and  $a = 0$  and  $b = \infty$ . Then

$$P(Z \leq 0) = 1/2 + \pi^{-1} \tan^{-1} c, \tag{4.1}$$

the distribution function of a standard Cauchy distribution(see, for instance, Johnson et al. 1994). In (4.1), if we set  $c_2 = -c = -\rho_c / \sqrt{1 - \rho_c^2}$ ,

$$P(Z \leq 0) = 2P(V_1 \leq 0, V_2 \leq 0) = 1/2 + \pi^{-1} \sin^{-1} \rho_c \tag{4.2}$$



by the df of  $NT_{(0,\infty)}(c, 0, 0, 1, 1)$  variable in (5), where  $(V_1, V_2)$  is the standard bivariate normal variable with correlation  $\rho_c$ . Thus (4.2) gives a simple derivation of the joint probability  $P(V_1 \leq 0, V_2 \leq 0)$  (see Johnson and Kotz (1972) for the direct derivation that uses a complicated double integration).

The probability  $F_Z(0) = P(Z \leq 0)$  of  $Z \sim NT_{(0,\infty)}(c, 0, 0, 1, 1)$  leads to the following applications to the theory of nonparametric statistics. For example, let  $(X_1, Y_1)$  and  $(X_2, Y_2)$  be independent identically distributed bivariate normal variables with variances  $\sigma_X^2$  and  $\sigma_Y^2$  and correlation  $\rho$ . Then Kendall's measure of association, which may be defined as

$$\tau = 2P[(X_2 - X_1)(Y_2 - Y_1) > 0] - 1,$$

can be written as  $\tau = 4P[(X_2 - X_1)/\sqrt{2\sigma_X^2} \leq 0, (Y_2 - Y_1)/\sqrt{2\sigma_Y^2} \leq 0] - 1$   
 $= 2F_{Z_\alpha}(0) - 1 = (2/\pi)\sin^{-1}\rho$

by (4.2), where  $Z_\alpha \sim NT_{(0,\infty)}(c, 0, 0, 1, 1)$  with  $c = (1, -\rho/\sqrt{1-\rho^2})$ . Thus  $\tau = 0$  if and only if  $\rho = 0$ . That is, we have the important relationship that  $\tau = 0$  if and only if  $X$  and  $Y$  are independent when sampling from a bivariate normal population. Another example occurs in the setting of the two-way layout. Hollander (1967) proposed a test statistic that is a sum of Wilcoxon signed rank statistics to detect  $H_1: F_1 \geq F_2 \geq \dots \geq F_k$ , where at least one of these inequality is strict, when the null hypothesis is  $H_0: F_j = F$  (unknown),  $j = 1, \dots, k$ . The variance of the asymptotic distribution of his statistic may be written as

$$\sigma_H^2 = 12P(X_1 - X_2 - X_3 + X_4 \leq 0, X_1 - X_5 - X_6 + X_7 \leq 0) - 3,$$

where  $X_1, \dots, X_7$  are independent random variables whose continuous df is  $F$ . If  $X_1, \dots, X_7$  represent a random sample from a univariate normal distribution with variance  $\sigma_X^2$ , then  $(X_1 - X_2 - X_3 + X_4)/(2\sigma_X)$  and  $(X_1 - X_5 - X_6 + X_7)/(2\sigma_X)$  are bivariate normally distributed with zero means, unit variances, and covariance 1/4. Consequently,  $\sigma_H^2 = 6F_{Z_\beta}(0) - 3 = (6/\pi)\sin^{-1}(1/4)$ , where  $Z_\beta \sim NT_{(0,\infty)}(c, 0, 0, 1, 1)$  with  $c = (1, -1/\sqrt{15})$ . These two examples show that, if one uses (4.2), the values of  $\tau$  and  $\sigma_H^2$  can be directly obtained, avoiding the complicated double integration.

#### 4.2. Sum of Values from a Normal and a Truncated Normal

Suppose an item which one makes has, among others, two parts which are assembled additively with regard to length. The length of both parts are normally distributed but, before assembly, one of the parts is subject to an inspection which

removes all individuals below a specified length. As an example, suppose that  $X$  comes from a normal distribution with a mean of 100 and standard deviation of 6, and  $Y$  comes from a normal distribution with a mean 50 with a standard deviation of 3, but with the restriction that  $Y \geq 44$ . How can we find the chance that  $X+Y$  is equal to or less than a given value, say 138? This problem was originally answered by Weinstein(1964) and three correspondents regarding to Weinstein(1964) have suggested other methods to get the same answer. The method presented here uses the df (2.4) of  $NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$  distribution as a simple alternative method to this problem. From definition 1, we see that  $X+Y \sim NT_{(44,\infty)}(c, 100, 50, 6, 3)$  with  $c = (1, 1)$ , so that  $\theta_1 = 150$ ,  $\theta_2 = 50$ ,  $\sigma_1 = \sqrt{45}$ ,  $\sigma_2 = 3$ , and  $\rho = 3/\sqrt{45}$ . Thus using (2.4), we see that  $F(138) = 1 - L(-1.7889, -2.4472, \rho)/\Phi(2) = .03276$ , and this probability is the same as the Weinstein's answer. When the parameters of  $X$  and  $Y$  distributions are unknown, the probability  $F(138)$  can be approximately calculated by using the maximum likelihood estimates of the parameters. See Johnson et al. (1994) and references therein for the estimation of the truncated normal parameters. When the restriction of  $Y$  in the form of  $a \leq Y \leq b$ , then  $X+Y \sim NT_{(a,b)}(c, 100, 50, 6, 3)$  with  $c = (1, 1)$ . Thus  $F(138)$  is immediate from (2.4) if values of  $a$  and  $b$  are given.

## 5. Conclusion

This paper has proposed a class of the sum of a normal and a doubly truncated normal, denoted by  $NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$ . The properties of the class is studied and two immediate applications of the class are given.

As given by (2.3), the special feature of the class is that it gives rich family of parametric density functions that allow a continuous variation from normality to nonnormality. Therefore the class of distribution is potentially relevant for practical application, especially for the analysis of skewed data as implied by Corollary 3. This, in turn, raises the estimation problem of  $NT_{(a,b)}(c, \mu_1, \mu_2, \tau_1, \tau_2)$  distribution based on a skewed data. A study pertaining to the application is an interesting research topic and it is left as a future study of interest.

## References

- [1] Arnold, B.C., Beaver, R.J., Groeneveld, R.A. and Meeker, W.Q. (1993). The nontruncated marginal of a truncated bivariate normal distribution. *Psychometrika*, Vol. 58, 471-478.

- [2] Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics*, Vol. 12, 171–178.
- [3] Azzalini, A. and Valle, A.D. (1996). The multivariate skew-normal distribution. *Biometrika*, Vol. 83, 715–726.
- [4] Branco, M.D. and Dey, D.K. (2001). A general class of multivariate skew-elliptical distributions. *Journal of Multivariate Analysis*, Vol. 79, 99–113.
- [5] Chen, M.H., Dey, D.K., and Shao, Q.M. (1999). A new skewed link model for dichotomous quantal response model. *Journal of the American Statistical Association*, Vol. 94, 1172–1186.
- [6] Devroye, L. (1986). *Non-Uniform Random Variate Generation*. New York: Springer Verlag.
- [7] DiCiccio, T.J. and Monti, A.C. (2004). Inferential aspects of the skew exponential power distribution. *Journal of the American Statistical Association*, Vol. 99, 439–450.
- [8] Donnelly, T.G. (1973). Algorithm 426: Bivariate normal distribution. *Communications of the Association for Computing Machinery*, Vol. 16, 638.
- [9] Henze, N. (1986). A probabilistic representation of the 'Skewed-normal' distribution. *Scandinavian Journal of Statistics*, Vol. 13, 271–275.
- [10] Hollander, M. (1967). Rank tests for randomized blocks when the alternative have an prior ordering. *The Annals of Mathematical Statistics*, Vol. 38, 867–887.
- [11] Joe, H. (1995). Approximation to multivariate normal rectangle probabilities based on conditional expectations. *Journal of the American Statistical Association*, Vol. 90, 957–966.
- [12] Johnson, N.L. and Kotz, S. (1972). *Distributions in Statistics: Continuous Multivariate Distributions*. New York: John Wiley.
- [13] Johnson, N.L., Kotz, S., and Balakrishnan, N. (1994). *Continuous Univariate Distributions*, Vol. 1. New York: John Wiley & Sons.
- [14] Kim, H.J. (2002). Binary regression with a class of skewed  $t$  link models. *Communications in Statistics-Theory and Methods*, Vol. 31, 1863–1886.
- [15] Ma, Y. and Genton, M.G. (2004). A flexible class of skew-symmetric distributions. *Scandinavian Journal of Statistics*, Vol. 31, 459–468.
- [16] Sugiura, N. and Gomi, A. (1985). Pearson diagrams for truncated normal and truncated Weibull distributions. *Biometrika*, Vol. 72, 219–222.
- [17] Weinstein, M.A. (1964). The sum of values from a normal and a truncated normal distribution (Answer to Query). *Technometrics*, Vol. 6, 104–105 and 469–471.

- [18] Zacks, S. (1981). *Parametric Statistical Inference*, Pregamon Press, Oxford.

[Received February 2006, Accepted April 2006]