RELATIONS BETWEEN DECOMPOSITION SERIES AND TOPOLOGICAL SERIES OF CONVERGENCE SPACES

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ABSTRACT. In this paper, we will show some relations between decomposition series \( \{ \pi^\alpha q : \alpha \text{ is an ordinal} \} \) and topological series \( \{ r_\alpha q : \alpha \text{ is an ordinal} \} \) for a convergence structure \( q \) and the formula \( \pi^\beta(\tau_\alpha q) = \pi^{\omega\alpha} \beta q \), where \( \omega \) is the first limit ordinal and \( \alpha \) and \( \beta(\geq 1) \) are ordinals.

I. Introduction and Preliminaries

A convergence structure \( q \) on a set \( X \) defined by [1] in 1964 is a function from the set \( F(X) \) of all filters on \( X \) into the set \( P(X) \) of all subsets of \( X \), satisfying the following conditions:

1. \( x \in q(x) \) for all \( x \in X \);
2. \( \mathcal{F} \leq \mathcal{G} \) implies \( q(\mathcal{F}) \subseteq q(\mathcal{G}) \);
3. \( x \in q(\mathcal{F}) \) implies \( x \in q(\mathcal{F} \cap \mathcal{x}) \),

where \( \mathcal{x} \) denotes the principal ultrafilter containing \( \{ x \} \); \( \mathcal{F} \) and \( \mathcal{G} \) are in \( F(X) \). Then the pair \( (X, q) \) is called a convergence space. If \( x \in q(\mathcal{F}) \), then we say that \( \mathcal{F} \) \( q \)-converges to \( x \). The filter \( \mathcal{V}_q(x) \) obtained by intersecting all filters which \( q \)-converge to \( x \) is called the \( q \)-neighborhood filter at \( x \). If \( \mathcal{V}_q(x) \) \( q \)-converges to \( x \) for each \( x \in X \), then \( q \) is said to be pretopological and the pair \( (X, q) \) is called a pretopological convergence space.

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Let $C(X)$ be the set of all convergence structures on $X$, partially ordered as follows:

$$q_1 \leq q_2 \text{ iff } q_2(F) \subseteq q_1(F) \text{ for all } F \in F(X).$$

If $q_1 \leq q_2$, then we say that $q_1$ is coarser than $q_2$, and $q_2$ is finer than $q_1$. By [2], we know that if $q_1$ is pretopological, then

$$q_1 \leq q_2 \text{ iff } \forall q_1(x) \leq \forall q_2(x) \text{ for all } x \in X.$$ 

For any $q \in C(X)$, we define a related convergence structure $\pi(q)$, as follows:

$$x \in \pi(q)(F) \text{ iff } \forall q(x) \leq F.$$ 

In this case, $\pi(q)$ is called the pretopological modification of $q$.

In 1973, Kent and Richardson [3] introduced the associated decomposition series $\{\pi^n q : \alpha \text{ is an ordinal}\}$ defined by

$$\pi^n q(F) \overset{\alpha}{\rightarrow} x \iff \forall q^\alpha (x) \leq F, \text{ for each } F \in F(X),$$

where

$$A \in \forall q^\alpha (x) \iff x \in I_q^\alpha (A), \text{ and}$$

$$I_q^\alpha (A) = \begin{cases} I_q(I_q^{\alpha - 1}(A)), & \text{if } \alpha - 1 \text{ exists,} \\ \cap_{\beta < \alpha} I_q^\beta (A), & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

In 1996, Park [4] studied the n-th pretopological modification $\pi^n q$ and quotient map for a convergence space $q$.

In 1999, for a convergence space $(X, q)$ with a second convergence structure $p$, Wilde [5] introduced that $(X, q)$ is "$p$-topological" iff $\mathcal{F} \overset{q}{\rightarrow} x$ implies $\forall_p(\mathcal{F}) \overset{q}{\rightarrow} x$. Also they showed that there is a finest $p$-topological convergence structure $\tau_p q$ on $X$ coarser than $q$ and $\mathcal{F} \overset{\tau_p q}{\rightarrow} x$ iff there exist $G \overset{q}{\rightarrow} x$ such that $\mathcal{F} \geq \forall_p^n (G)$, for some $n \in N$. Furthermore, they induced the topological series for $q$, the descending ordinal sequence $\{\tau_\alpha q : \alpha \text{ is an ordinal}\}$ defined recursively on $X$ as follows:
Decomposition Series and Topological Series

\[ \tau_0 q = q \]
\[ \tau_1 q : F \xrightarrow{\tau_1 q} x \iff \exists G \xrightarrow{q} x \text{ and } n \in N \text{ such that } F \geq V^n_G(G) \]
\[ \tau_2 q : F \xrightarrow{\tau_2 q} x \iff \exists G \xrightarrow{q} x \text{ and } n \in N \text{ such that } F \geq V^n_{\tau_1 q}(G) \]
\[ \tau_3 q : F \xrightarrow{\tau_3 q} x \iff \exists G \xrightarrow{q} x \text{ and } n \in N \text{ such that } F \geq V^n_{\tau_2 q}(G) \]
\[ \vdots \]
\[ \tau_\alpha q : F \xrightarrow{\tau_\alpha q} x \iff \exists G \xrightarrow{q} x, n \in N \text{ and } \beta < \alpha \text{ such that } F \geq V^n_{\tau_\beta q}(G). \]

In this paper, we will show some relations between decomposition series \( \{\pi^\alpha q : \alpha \text{ is an ordinal} \} \) and topological series \( \{\tau_\alpha q : \alpha \text{ is an ordinal} \} \) for a convergence structure \( q \) and the formula \( \pi^\beta(\tau_\alpha q) = \pi^{\omega^\alpha \beta} q \), where \( \omega \) is the first limit ordinal and \( \alpha \) and \( \beta(\geq 1) \) are ordinals.

2. Decomposition Series, the Neighborhood and Interior Filter of a Filter

We shall summarize some results from [3] and other sources using more modern notation and terminology. We are mainly interested in comparing properties of decomposition series with those of the topological series, which will be introduced in [5].

Let \((X, q)\) be a convergence space. For \( A \subseteq X \), we recall that \( I_q^0(A) = A, I_q^1 = I_q(A) = \{x : A \in V_q(x)\} \).

Given an ordinal number \( \alpha \geq 1 \), let \( I_q^\alpha \) and \( cl_q^\alpha \) denote the \( \alpha \)th iterations of interior operator and closure operator for \( q \), respectively. For \( A \subseteq X \), we inductively define:

\[ I_q^\alpha(A) = \begin{cases} I_q(I_q^{\alpha-1}(A)), & \text{if } \alpha - 1 \text{ exists}, \\ \cap_{\beta<\alpha} I_q^\beta(A), & \text{if } \alpha \text{ is a limit ordinal}. \end{cases} \]

\[ cl_q^\alpha(A) = \begin{cases} cl_q(cl_q^{\alpha-1}(A)), & \text{if } \alpha - 1 \text{ exists}, \\ \cup_{\beta<\alpha} cl_q^\beta(A), & \text{if } \alpha \text{ is a limit ordinal}. \end{cases} \]
PROPOSITION 2.1. ([5]). For every ordinal $\alpha$ and $A \subseteq X$, $X \setminus cl^\alpha_q (A) = I^\alpha_q (X \setminus A)$.

If $(X, q)$ is a convergence space and $\alpha \geq 1$, let $\pi^\alpha_q$ be the pretopology on $X$ whose neighborhood filter is $\mathcal{V}^\alpha_q (x)$, that is, $\mathcal{V}^\alpha_q (x) = \mathcal{V}^\alpha_q (x)$, where $A \in \mathcal{V}^\alpha_q (x)$ iff $x \in I^\alpha_q (A)$. Since $\beta < \alpha$ implies $I^\alpha_q (A) \subseteq I^\beta_q (A)$, it follows that $\mathcal{V}^\alpha_q (x) \leq \mathcal{V}^\beta_q (x)$, and consequently $\pi^\alpha_q \leq \pi^\beta_q$.

Definition 2.2. ([3], [5]). The descending chain $\{\pi^\alpha_q: \alpha \geq 1\}$ of pretopologies on $X$ is called the decomposition series of $(X, q)$.

Clearly $\pi^1_q = \pi q$ is the pretopological modification of $q$, which is the finest pretopological convergence structure on $X$ coarser than $q$.

Definition 2.3. ([5]). For any ordinal $\alpha$, $p \in C(X)$ and $\mathcal{G} \in F(X)$, we define the neighborhood filter $\mathcal{V}^\alpha_p (\mathcal{G})$ and the interior filter $I^\alpha_p (\mathcal{G})$ of $\mathcal{G}$, respectively, as follows:

$$\mathcal{V}^\alpha_p (\mathcal{G}) = \mathcal{V}^\alpha_p (\mathcal{G}) = \{ A \subseteq X : I^\alpha_p (A) \in \mathcal{G} \}.$$ 

$$I^\alpha_p (\mathcal{G}) = I_p (\mathcal{G}), \; I^\alpha_p (\mathcal{G}) = \{ I^\alpha_p (G) : G \in \mathcal{G} \}$$ if $I_p (\mathcal{G}) \neq \emptyset$, $\forall G \in \mathcal{G}$,

where $[\mathcal{B}]$ means the filter generated by $B$ if $B$ is a filter base.

Then we know that if $\alpha < \beta$, then $\mathcal{V}^\beta_p (\mathcal{G}) \leq \mathcal{V}^\alpha_p (\mathcal{G}) \leq \mathcal{G} \leq I^\alpha_p (\mathcal{G}) \leq I^\beta_p (\mathcal{G})$.

PROPOSITION 2.4. For any ordinals $\alpha$, $\beta$, $x \in X$ and $A \subseteq X$,

(1) $I^{\alpha + \beta}_q (A) = I^\beta_q (I^\alpha_q (A))$.

(2) $\mathcal{V}^{\alpha + \beta}_q (x) = \mathcal{V}^\alpha_q (\mathcal{V}^\beta_q (x))$.

Proof. (1) Let $\alpha$ be a fixed ordinal. We use transfinite induction on $\beta$. If $\beta = 1$, $I^{\alpha + 1}_q = I^\alpha_q (I^\alpha_q (A))$ follows by definition. Next, let $\beta$ be any arbitrary ordinal.

Case 1. Assume that there exists $\bar{\beta}$ such that $\bar{\beta} + 1 = \beta$. By the induction hypothesis, $I^{\alpha + \bar{\beta}} q (A) = I^\bar{\beta}_q (I^\alpha_q (A))$, and so $I^{\alpha + \beta}_q (A) = I^{\alpha + \bar{\beta} + 1}_q (A) = I^\beta_q (I^{\alpha + \bar{\beta}} q (A)) = I^\beta_q (I^\alpha_q (I^\alpha_q (A))) = I^\beta_q (I^\alpha_q (A))$.

Case 2. Assume that $\beta$ is a limit ordinal. $I^{\alpha + \beta}_q (A) = \cap_{\gamma \leq \beta} I^{\alpha + \gamma}_q (A) = I^\beta_q (I^\alpha_q (A)) = I^\beta_q (I^\alpha_q (A))$.
(2) \( A \in \mathcal{V}_q^{\alpha+\beta}(x) \iff x \in \mathcal{I}_q^{\alpha+\beta}(A) \iff x \in \mathcal{I}_q^\beta(I_q^\alpha(A)) \iff \mathcal{I}_q^\alpha(A) \in \mathcal{V}_q^\beta(x) \iff A \in \mathcal{V}_q^\alpha(\mathcal{V}_q^\beta(x)) \). \qed

**Corollary 2.5.** For any ordinals \( \alpha, \beta, \) and \( \mathcal{F} \in \mathcal{F}(X) \),

1. \( \mathcal{I}_q^{\alpha+\beta}(\mathcal{F}) = \mathcal{I}_q^\beta(I_q^\alpha(\mathcal{F})) \) if these are filters.
2. \( \mathcal{V}_q^{\alpha+\beta}(\mathcal{F}) = \mathcal{V}_q^\alpha(\mathcal{V}_q^\beta(\mathcal{F})) \).

**3. \( p \)-Topological Convergence Spaces**

In this section, we will summary some propositions about \( p \)-topological convergence space of [5] and [6], and change two propositions, which are the following Theorem 3.4 and 3.7.

Henceforth \((X, q)\) means a convergence space equipped with a second convergence structure \( p \).

**Definition 3.1.** ([5]). A convergence space \((X, q)\) is \( p \)-topological iff \( \mathcal{F} \rightarrow^q x \) implies that there exists \( \mathcal{G} \rightarrow^q x \) such that \( \mathcal{F} \geq I_p(\mathcal{G}) \).

**Proposition 3.2.** ([5]). \((X, q)\) is \( p \)-topological, iff \( \mathcal{F} \rightarrow^q x \implies \mathcal{V}_p(\mathcal{F}) \rightarrow^q x \).

**Proposition 3.3.** ([5]). Let \((X, q)\) be a pretopological convergence. Then \((X, q)\) is \( p \)-topological iff \( \mathcal{V}_q(x) = I_p(\mathcal{V}_q(x)) \).

**Proof.** \(( \implies )\) Since \( \mathcal{V}_q(x) \rightarrow^q x \) and \((X, q)\) is \( p \)-topological, there exists \( \mathcal{G} \rightarrow^q x \) such that \( \mathcal{V}_q(x) \geq I_p(\mathcal{G}) \). Then \( \mathcal{G} \geq \mathcal{V}_q(x) \), so \( \mathcal{G} \geq I_p(\mathcal{G}) \). This implies \( \mathcal{G} = \mathcal{V}_q(x) = I_p(\mathcal{G}) = I_p(\mathcal{V}_q(x)) \).

\(( \impliedby )\) Let \( \mathcal{F} \rightarrow^q x \). Then \( \mathcal{F} \geq \mathcal{V}_q(x) = I_p(\mathcal{V}_q(x)) \). Thus, \((X, q)\) is \( p \)-topological, since \( \mathcal{V}_q(x) \rightarrow^q x \). \qed

**Theorem 3.4.** If \((X, q)\) is a pretopological and \( p \)-topological, then \( q \leq \pi^\alpha p \).
Proof. Since \((X, q)\) is a pretopological and \(p\)-topological, \(\mathcal{V}_q(x) = I_p(\mathcal{V}_q(x))\).

Claim: \(\mathcal{V}_q(x) \leq \mathcal{V}_p^\omega(x)\). Let \(V \in \mathcal{V}_q(x)\). Then \(I_p(V) \in I_p(\mathcal{V}_q(x)) = \mathcal{V}_q(x)\). By Induction, \(I_p^n(V) \in \mathcal{V}_q(x)\) for all \(n \in N\), so \(x \in I_p^n(V)\) for all \(n \in N\). Thus \(x \in \cap_{n<\omega} I_p^n(V) = I_p^\omega(V)\), and hence \(V \in \mathcal{V}_p^\omega(x)\). Thus the Claim is proved.
From \(\mathcal{V}_p^\omega(x) = \mathcal{V}_{\pi^\omega p}(x)\), we obtain \(q \leq \pi^\omega p\). □

**Proposition 3.5. ([5]).** Let \(p\) and \(q\) be topological. Then \((X, q)\) is \(p\)-topological iff \(q \leq p\).

**Proof.** Since \(q\) is topological, \(\mathcal{V}_q(x)\) has a filter base of \(q\)-open sets.

(\(\implies\)) Since \((X, q)\) is \(p\)-topological and topological, by Theorem 3.4, \(q \leq \pi^\omega p = p\).

(\(\impliedby\)) Let \(q \leq p\). Then \(I_q(A) \subseteq I_p(A) \subseteq A\). This implies that each \(q\)-open set is \(p\)-open, so \(I_p(\mathcal{V}_q(x)) = \mathcal{V}_q(x)\), by Proposition 3.3. \((X, q)\) is \(p\)-topological. □

**Proposition 3.6. ([5]).** If \((X, q)\) is \(p\)-topological and \(p < p'\), then \((X, q)\) is \(p'\)-topological.

**Proof.** It follows from \(p < p'\) implies \(I_p(\mathcal{G}) \supseteq I_{p'}(\mathcal{G})\). □

Note that for \(q \in C(X)\), \(\tau_q = \{A \subseteq X : I_q(A) = A\}\) is a topology on \(X\) and \(\tau_q\) is the convergence structure defined by

\[
\tau_q(F) \xrightarrow{q} x \iff \mathcal{V}_{\tau_q}(x) \leq F, \text{ for each } F \in F(X),
\]

where \(\mathcal{V}_{\tau_q}(x)\) is the \(\tau_q\)-neighborhood filter at \(x \in X\). Then \(\tau_q\) is the finest topological convergence structure on \(X\) coarser than \(q\).([5]).

Now, we obtain the following theorem, which is different from Corollary 2.4 of [6].

**Theorem 3.7.** If \((X, q)\) is \(p\)-topological, then:

(1) \((X, \pi q)\) is \(p\)-topological and \(\tau_q \leq \pi q \leq \pi^\omega p\).

(2) \((X, \tau_q)\) is \(p\)-topological.
**Proof.** (1) Let \( \mathcal{F} \xrightarrow{q} x \); then there exists a \( \mathcal{G} \xrightarrow{q} x \) such that \( \mathcal{F} \supseteq I_p(\mathcal{G}) \supseteq I_p(V_q(x)) \). This holds for every \( \mathcal{F} \xrightarrow{q} x \), so

\[
V_{\pi q}(x) = V_q(x) = \bigcap \{ \mathcal{F} \in F(X) : \mathcal{F} \xrightarrow{q} x \} \supseteq I_p(V_q(x)) = I_p(V_{\pi q}(x)).
\]

Thus \((X, \pi q)\) is \( p \)-topological, so the first part is proved.

It is clear that \( \tau q \leq \pi q \). Since \((X, \pi q)\) is \( p \)-topological and pre-topological, by Theorem 3.4, \( \pi q \leq \pi^\omega p \).

(2) Since \((X, \tau q)\) is \( \tau q \)-topological and \( \pi q \leq \pi^\omega p \leq p \), by Proposition 3.6, \((X, \tau q)\) is \( p \)-topological. \( \square \)

**Definition 3.8** For \( q, p \in C(X) \), \( \tau_p q \) is defined by:

\[
\mathcal{F} \xrightarrow{\tau_p q} x \iff \exists \mathcal{G} \xrightarrow{q} x \text{ and } n \in N \text{ such that } \mathcal{F} \geq V^n_p(\mathcal{G}).
\]

**Proposition 3.9.** For \( q, p \in C(X) \), \((X, \tau_p q)\) is \( p \)-topological.

**Proof.** Let \( \mathcal{F} \xrightarrow{\tau_p q} x \). Then there exists \( \mathcal{G} \xrightarrow{q} x \) and \( n \in N \) such that \( \mathcal{F} \geq V^n_p(\mathcal{G}) \), so \( V_p(\mathcal{F}) \supseteq V_p(V^n_p(\mathcal{G})) = V^{n+1}_p(\mathcal{G}), \) [5]. Thus \( V_p(\mathcal{F}) \xrightarrow{\tau_p q} x \). This means \((X, \tau_p q)\) is \( p \)-topological. \( \square \)

4. Relations between Decomposition Series and Topological Series of Convergence Spaces

In this section, we will remind "topological series" defined by [5] and show relations between decomposition series and supratopological series, the formlar \( \pi^\beta(\tau_\alpha q) = \pi^{\omega \alpha \beta} q \), where \( \omega \) is the first limit ordinal and \( \alpha \) and \( \beta (\geq 1) \) are ordinals.

Let \( q \in C(X) \) and \( \alpha \geq 0 \) ordinal number. The topological series for \( q \) is the descending ordinal sequence \( \{ \tau_\alpha q \} \) defined recursively on \( X \) as follows:

\[
\begin{align*}
\tau_0 q &= q \\
\tau_1 q & : \mathcal{F} \xrightarrow{\tau_1 q} x \iff \exists \mathcal{G} \xrightarrow{q} x \text{ and } n \in N \text{ such that } \mathcal{F} \geq V^n_q(\mathcal{G})
\end{align*}
\]
\[ \tau_2 q : \mathcal{F} \overset{\tau_2 q}{\longrightarrow} x \iff \exists G \overset{q}{\rightarrow} x \text{ and } n \in N \text{ such that } \mathcal{F} \geq V^n_{\tau_2 q}(G). \]

\[ \tau_3 q : \mathcal{F} \overset{\tau_3 q}{\longrightarrow} x \iff \exists G \overset{q}{\rightarrow} x \text{ and } n \in N \text{ such that } \mathcal{F} \geq V^n_{\tau_3 q}(G). \]

\[ \tau_{\alpha q} : \mathcal{F} \overset{\tau_{\alpha q}}{\longrightarrow} x \iff \exists G \overset{q}{\rightarrow} x, \text{ } n \in N \text{ and } \beta < \alpha \text{ such that } \mathcal{F} \geq V^n_{\tau_{\alpha q}}(G), \]

where we know that \( \tau_1 q = \tau_2 q, \tau_2 q = \tau_{\tau_1 q} q = \tau_{\tau_2 q} q, \ldots, \text{ etc.} \)

Also, we know that if there exists \( \alpha' \) such that \( \alpha = \alpha' + 1 \), then

\[ \mathcal{F} \overset{\tau_{\alpha q}}{\longrightarrow} x \iff \exists G \overset{q}{\rightarrow} x \text{ and } n \in N \text{ such that } \mathcal{F} \geq V^n_{\tau_{\alpha q}}(G), \]

**PROPOSITION 4.1.** ([5]). For \( q \in C(X) \), there exists \( \tilde{q} \) which is the finest \( q \)-topological convergence structure on \( X \), and \( \mathcal{F} \overset{\tilde{q}}{\longrightarrow} x \) iff \( \mathcal{F} \geq V^n_{\tilde{q}}(x) \) for some \( n \in N \).

**LEMMA 4.2.** If \( G \overset{q}{\rightarrow} x \), then \( V_{\tilde{q}}^{n+1}(x) \leq V^n_{\tilde{q}}(G) \).

**Proof.** \( A \in V_{\tilde{q}}^{n+1}(x) \implies x \in I_{\tilde{q}}^{n+1}(A) \implies x \in I_q(I_{\tilde{q}}^{n}(A)) \implies I_{\tilde{q}}^{n}(A) \in V_q(x) \implies I_q^n(A) \in G \), since \( G \overset{q}{\rightarrow} x \implies G \geq V_q(x) \). Thus \( A \in V^n_{\tilde{q}}(G) \). \( \square \)

**PROPOSITION 4.3.** \( \tilde{q} = \tau_1 q \).

**Proof.** We have already known \( \tilde{q} \geq \tau_1 q \), so it remain to show \( \tau_1 q \geq \tilde{q} \).

Let \( \mathcal{F} \overset{\tau_1 q}{\rightarrow} x \). Then there exists \( G \overset{q}{\rightarrow} x \) and \( n \in N \) such that \( \mathcal{F} \geq V^n_{\tilde{q}}(G) \).

By the above Lemma, \( \mathcal{F} \geq V^n_{\tilde{q}}(G) \geq V_{\tilde{q}}^{n+1}(x) \), so \( \mathcal{F} \overset{\tilde{q}}{\rightarrow} x \). \( \square \)

**PROPOSITION 4.4.** (1) \( q \geq \pi^n q \geq \tilde{q} \geq \pi^n q \). (2) \( \pi(\tau_1 q) = \pi^n q \).

**Proof.** (1) It is clear that \( q \geq \pi^n q \). Let \( n \in N \) and \( \mathcal{F} \in F(X) \).

Then \( \mathcal{F} \overset{\pi^n q}{\rightarrow} x \iff \mathcal{F} \geq V^n_{\pi q}(x) \implies \mathcal{F} \overset{\tilde{q}}{\rightarrow} x, \text{ since } x \overset{q}{\rightarrow} x \).

Thus, \( \pi^n q \geq \tilde{q} \) for each \( n \in N \).
Decomposition Series and Topological Series

Also, \( \mathcal{F} \xrightarrow{\tilde{q}} x \iff \exists n \in N \text{ such that } \mathcal{F} \geq \mathcal{V}^n_q(x) \geq \cap_n \omega \mathcal{V}^n_q(x) = \mathcal{V}^n_q(x) \), where \( \mathcal{V}^n_q(x) = \mathcal{V}^n_{\pi^\omega q}(x) \iff \mathcal{F} \xrightarrow{\pi^\omega q} x \).

(2) Since \( \tilde{q} = \tau_1 q \), by (1) \( \pi(\tau_1 q) \geq \pi(\pi^\omega q) = \pi^\omega q \). While, by Theorem 3.7, \( \pi(\tau_1 q) \leq \pi^\omega q \), since \( \tau_1 q \) is a \( q \)-topological. Thus, \( \pi(\tau_1 q) = \pi^\omega q \).

We know that for \( q \in C(X) \), the first term in the topological series for \( q \) is \( \tau_1 q = \tilde{q} \). \( \tau_1 q \) is the finest topological convergence structure on \( X \) and also the lower \( q \)-topological modification of \( q \), since \( \tau_1 q = \tilde{q} \leq \pi^\omega q \leq q \). Note that \( q \) has no upper \( q \)-topological modification unless \( q \) is a topology. We next show that that \( \tau_2 q \) is related to \( \tau_1 q \) exactly as \( \tau_1 q \) is related to \( q \). Note that the lower \( \tau_1 q \)-topological modification of \( \tau_1 q \) is \( \tau_1 \tilde{q} \) defined by:

\[
\mathcal{F} \xrightarrow{\tau_1 \tilde{q}} x \iff \exists \mathcal{G} \xrightarrow{\tau_1 q} x \text{ and } n \in N \text{ such that } \mathcal{F} \geq \mathcal{V}^n_{\tau_1 q}(\mathcal{G}).
\]

**Proposition 4.5.** For any \( q \in C(X) \), \( \tau_2 q = \tau_1 \tilde{q} \).

*Proof.* \( \mathcal{F} \xrightarrow{\tau_2 q} x \iff \exists \mathcal{G} \xrightarrow{\tau_1 q} x \) and \( n \in N \) such that \( \mathcal{F} \geq \mathcal{V}^n_{\tau_1 q}(\mathcal{G}) \). But \( \mathcal{G} \xrightarrow{\tau_1 q} x \) since \( \tau_2 q \leq q \). Thus \( \mathcal{F} \xrightarrow{\tau_1 \tilde{q}} x \).

Conversely, \( \mathcal{F} \xrightarrow{\tau_1 \tilde{q}} x \iff \exists \mathcal{G} \xrightarrow{\tau_1 q} x \) and \( n \in N \) such that \( \mathcal{F} \geq \mathcal{V}^n_{\tau_1 q}(\mathcal{G}) \). Also, \( \mathcal{G} \xrightarrow{\tau_1 q} x \iff \exists \mathcal{H} \xrightarrow{\pi^\omega q} x \) and \( m \in N \) such that \( \mathcal{G} \geq \mathcal{V}^m_q(\mathcal{H}) \). Thus \( \mathcal{F} \geq \mathcal{V}^m_{\tau_1 q}(\mathcal{V}^m_q(\mathcal{H})) \geq \mathcal{V}^m_{\tau_1 q}(\mathcal{V}^m_{\tau_1 q}(\mathcal{H})) = \mathcal{V}^{m+m}_{\tau_1 q}(\mathcal{H}) \). Thus \( \mathcal{F} \xrightarrow{\tau_2 q} x \). \( \square \)

**Proposition 4.6.** \( \pi(\tau_1 q) = \pi^\omega q \) and \( \pi(\tau_2 q) = \pi^\omega(\tau_1 q) \).

*Proof.* The first equality follows from the Proposition 4.4. The second equality follows from \( \pi(\tau_2 q) = \pi(\tau_1 \tilde{q}) = \pi^\omega(\tau_1 q) \). \( \square \)

**Proposition 4.7.** If \( \alpha \) is a limit ordinal, \( \mathcal{V}^\alpha_q(x) = \cap_{\beta < \alpha} \mathcal{V}^\beta_q(x) \).
Proof. \( A \in \mathbb{Y}_q^\alpha(x) \iff x \in I^\alpha_q(A) = \cap_{\beta < \alpha} I^\beta_q(A) \iff x \in I^\beta_q(A), \forall \beta < \alpha \iff A \in \mathbb{Y}_q^\beta(x), \forall \beta < \alpha \iff A \in \cap_{\beta < \alpha} \mathbb{Y}_q^\beta(x). \) □

Proposition 4.8. \( \mathbb{V}_{\tau_nq}(x) = \mathbb{V}_q^{\omega_n}(x) \) and \( \mathbb{V}_{\tau_{\omega}q}(x) = \mathbb{V}_q^{\omega}(x) \) for all \( x \in X. \)

Proof. As we showed in Proposition 4.6, \( \pi(\tau_2 q) = \pi^\omega(\tau_1 q). \) Thus for any \( x \in X, \mathbb{V}_{\tau_2q}(x) = \mathbb{V}_q^{\omega}(x). \) Also, by Proposition 4.4, \( \mathbb{V}_{\tau_1q}(x) = \mathbb{V}_q^\omega(x). \) By Corollary 2.5, \( \mathbb{V}_{\tau_3q}^2(x) = \mathbb{V}_{\tau_1q}(\mathbb{V}_{\tau_1q}(x)) = \mathbb{V}_q^\omega(\mathbb{V}_q^\omega(x)) = \mathbb{V}_q^{\omega+\omega}(x) = \mathbb{V}_q^{\omega^2}(x). \) Similarly, \( \mathbb{V}_{\tau_{\omega}q}^n(x) = \mathbb{V}_q^{\omega_n}(x) \). Thus \( \mathbb{V}_{\tau_{\omega}q}^n(x) = \cap_{\beta < \omega} \mathbb{V}_{\tau_1q}^\beta(x) = \mathbb{V}_q^{\omega^\omega}(x). \) Expanding the reasoning of Proposition 4.6, we have \( \mathbb{V}_{\tau_{\omega}q}(x) = \mathbb{V}_q^{\omega}(x), \) for all \( x \in X, \) since \( \pi(\tau_3 q) = \pi^\omega(\tau_2 q). \) \( \mathbb{V}_{\tau_{\omega}q}^2(x) = \mathbb{V}_{\tau_{\omega}q}(\mathbb{V}_{\tau_{\omega}q}(x)) = \mathbb{V}_q^\omega(\mathbb{V}_q^{\omega^2}(x)) = \mathbb{V}_q^{\omega^3}(x). \) Similarly, \( \mathbb{V}_{\tau_{\omega}q}^n(x) = \mathbb{V}_q^{\omega_n}(x), \) so \( \mathbb{V}_{\tau_{\omega}q}(x) = \mathbb{V}_{\tau_{\omega}q}(x) = \cap_{\beta < \omega} \mathbb{V}_{\tau_{\omega}q}^\beta(x) = \cap_{\beta < \omega} \mathbb{V}_q^{\omega_n}(x) = \mathbb{V}_q^{\omega^\omega}(x). \) Likewise, we obtain \( \mathbb{V}_{\tau_{\omega}q}^n(x) = \mathbb{V}_q^{\omega_n^\omega}(x). \) This implies that \( \mathbb{V}_{\tau_{\omega}q}(x) = \cap_{\beta < \omega} \mathbb{V}_q^{\omega^\omega_n}(x) = \mathbb{V}_q^{\omega^\omega}(x). \) □

For \( q \in C(X) \) and any ordinal \( \alpha, \) let \( \tau_\alpha q \) and \( \sigma_\alpha q \) be defined inductively by \( \tau_0 q = \sigma_0 q = q \) and:

\[
F \xrightarrow{\tau_\alpha q} x \iff \exists G \xrightarrow{n} x, n \in N \text{ and } \beta < \alpha \text{ such that } F \geq \mathbb{V}_{\beta q}^n(G),
\]

\[
F \xrightarrow{\sigma_\alpha q} x \iff \exists G \xrightarrow{\sigma_{\beta q}} x, n \in N \text{ and } \beta < \alpha \text{ such that } F \geq \mathbb{V}_{\sigma_{\beta q}}^n(G).
\]

Note that \( \tau_1 q = \sigma_1 q \) is the lower \( q \)-topological modification of \( q. \) If \( \alpha + 1 \) is any non-limit ordinal, \( \sigma_{\alpha + 1} q = \tau_1(\sigma_\alpha q); \) in other words, \( \sigma_{\alpha + 1} q \) is the lower \( \sigma_\alpha q \)-topological modification of \( \sigma_\alpha q. \) If \( \alpha \) is a limit ordinal, \( \sigma_\alpha q = \inf \{ \sigma_\beta q : \beta < \alpha \}. \) Our first goal is to prove \( \sigma_\beta q = \tau_\alpha q \) for every ordinal \( \alpha. \)

Proposition 4.9. For any ordinal \( \alpha, \) \( \tau_\alpha q \geq \sigma_\alpha q. \)
Proof. Assume that \( \tau_{\beta q} \geq \sigma_{\alpha q} \) for every ordinal \( \beta < \alpha \). Then 
\[ \mathcal{F} \xrightarrow{\tau_{\alpha q}} x \implies \exists G \xrightarrow{q} x \text{ and } \beta < \alpha \text{ such that } \mathcal{F} \geq \mathcal{V}_{\tau_{\beta q}}(G) \geq \mathcal{V}_{\sigma_{\beta q}}(G). \]
Also, since \( G \xrightarrow{q} x \), \( G \xrightarrow{\sigma_{\alpha q}} x \). Thus \( \mathcal{F} \xrightarrow{\sigma_{\alpha q}} x. \)

**Proposition 4.10.** For any ordinal \( \alpha \), \( \tau_{\alpha q} = \sigma_{\alpha q} \).

**Proof.** The result is known for \( \alpha = 1 \). Assume the equality holds for \( \beta < \alpha \). By Proposition 4.9, it remains to show that \( \mathcal{F} \xrightarrow{\sigma_{\alpha q}} x \implies \mathcal{F} \xrightarrow{\tau_{\alpha q}} x. \)

Case 1. \( \exists a' \) such that \( \alpha = \alpha' + 1 \). Let \( \mathcal{F} \xrightarrow{\sigma_{\alpha' q}} x \). Then there exists \( \mathcal{F} \xrightarrow{\sigma_{\alpha' q}} x \) and \( n \in \mathbb{N} \) such that \( \mathcal{F} \geq \mathcal{V}_{\sigma_{\alpha q}}(G) = \mathcal{V}_{\tau_{\alpha q}}(G) \). Also, by induction hypothesis, \( G \xrightarrow{\tau_{\alpha q}} x \), so there exists \( H \xrightarrow{q} x \), \( \beta < \alpha' \) and \( m \in \mathbb{N} \) such that \( G \geq \mathcal{V}_{\tau_{\beta q}}(H) \). Thus, \( \mathcal{F} \geq \mathcal{V}_{\tau_{\dot{\alpha} q}}(G) \geq \mathcal{V}_{\tau_{\dot{\alpha} q}}(\mathcal{V}_{\tau_{\beta q}}(H)) \geq \mathcal{V}_{\tau_{\dot{\alpha} q}}(\mathcal{V}_{\tau_{\beta q}}(H)) \geq \mathcal{V}_{\tau_{\dot{\alpha} q}}(H) \), and hence \( \mathcal{F} \xrightarrow{\tau_{\alpha q}} x. \)

Case 2. \( \alpha \) is a limit ordinal. Then by induction hypothesis, \( \tau_{\beta q} = \sigma_{\beta q} \) for \( \beta < \alpha \), so \( \sigma_{\alpha q} = \inf\{\sigma_{\beta q} : \beta < \alpha\} = \inf\{\tau_{\beta q} : \beta < \alpha\} = \tau_{\alpha q}. \)

**Proposition 4.11.** For any ordinal \( \alpha \), \( \tau_{\alpha q} = \tau_{\alpha+1 q}. \) Thus 
\( \mathcal{V}_{\tau_{\alpha+1 q}}(x) = \mathcal{V}_{\tau_{\alpha q}}(x) \) for all \( x \in X. \)

**Proof.** The first assertion follows by Proposition 4.10 and the note preceding Proposition 4.9. The second follows Proposition 4.6, since \( \pi(\tau_{\beta p}) = \tau_{\pi(p)} \) holds for any convergence structure \( p \), letting \( p = \tau_{\alpha q}. \)

**Proposition 4.12.** For any ordinal \( \alpha \) and \( x \in X \), \( \mathcal{V}_{\tau_{\alpha q}}(x) = \mathcal{V}_{\tau_{\alpha q}}(x). \)

**Proof.** We will use induction on \( \alpha \). For \( \alpha = 1 \), the result follows by Proposition 4.11. Assume the equality holds for every \( \beta < \alpha \).

Case 1. Assume that there exists \( \alpha' \) such that \( \alpha = \alpha' + 1 \). Then by Proposition 4.11, \( \mathcal{V}_{\tau_{\alpha q}}(x) = \mathcal{V}_{\tau_{\alpha q}}(x), \) where by induction hypothesis,
\( \forall_{\alpha,q}(x) = \forall_{q}^{\omega^{\alpha}}(x) \). Thus \( \forall_{\alpha,q}^{2}(x) = \forall_{\alpha,q}(\forall_{\alpha,q}(x)) = \forall_{q}^{\omega^{\alpha+2}}(x) \), and similarly \( \forall_{\alpha,q}^{n}(x) = \forall_{q}^{\omega^{\alpha n}}(x) \). Thus \( \forall_{\alpha,q}(x) = \forall_{\alpha,q}(x) = \cap_{n<\omega} \forall_{\alpha,q}^{n}(x) = \forall_{\alpha,q}(x) = \forall_{q}^{\omega^{\alpha n}}(x) = \forall_{q}^{\omega^{\alpha+1}}(x) = \forall_{q}^{\omega^{\alpha}}(x) \).

Case 2. Assume that \( \alpha \) is a limit ordinal. By induction hypothesis, \( \forall_{\alpha,q}(x) = \forall_{q}^{\omega^{\beta}}(x) \) for \( \beta < \alpha \). Thus \( \forall_{\alpha,q}(x) = \cap_{\beta < \alpha} \forall_{q}^{\omega^{\beta}}(x) = \forall_{q}^{\omega^{\alpha}}(x). \)

Consequently, our last result is the following Theorems.

**Theorem 4.13.** For every ordinal \( \alpha \) and \( \beta \geq 1 \) and every \( x \in X \),
\[
\begin{align*}
(1) & \quad \forall_{\alpha,q}^{\beta}(x) = \forall_{q}^{\omega^{\alpha+\beta}}(x), \\
(2) & \quad \pi^{\beta} = \pi^{\omega^{\alpha}}(x).
\end{align*}
\]

**Proof.** (1) We will use induction on \( \beta \). For \( \beta = 1 \), the result follows by Proposition 4.12. Assume the equality holds for every \( \gamma < \beta \).

Case 1. \( \exists \beta' \) such that \( \beta = \beta'+1 \). then by Corollary 2.5, \( \forall_{\alpha,q}^{\beta'}(x) = \forall_{\alpha,q}^{\beta'+1}(x) = \forall_{\alpha,q}^{\beta'}(\forall_{\alpha,q}(x)) = \forall_{q}^{\omega^{\alpha+\beta'}}(\forall_{q}^{\omega^{\alpha}}(x)) = \forall_{q}^{\omega^{\alpha+1}}(x) = \forall_{q}^{\omega^{\alpha}}(x). \)

Case 2. \( \beta \) is a limit ordinal. By induction hypothesis, \( \forall_{\alpha,q}^{\beta}(x) = \forall_{q}^{\omega^{\alpha}}(x) \) for \( \gamma < \beta \). Thus \( \forall_{\alpha,q}^{\beta}(x) = \cap_{\gamma < \beta} \forall_{\alpha,q}^{\gamma}(x) = \forall_{q}^{\omega^{\alpha}}(x). \)

(2) By (1), it is clear.

Finally, we define the lengths of decomposition series and topological series of \( q \in C(X) \), \( l_{D}q \), and \( l_{T}q \), respectively by:
\[
\begin{align*}
l_{D}q = & \inf\{ \lambda : \lambda \text{ is an ordinal such that } \pi^{\lambda} = \pi^{\lambda+1}q \}, \\
l_{T}q = & \inf\{ \lambda : \lambda \text{ is an ordinal such that } \tau^{\lambda} = \tau^{\lambda+1}q \}.
\end{align*}
\]

We know that \( l_{D}q = \inf\{ \lambda : \lambda \text{ is an ordinal s.t. } I_{q}^{\lambda}(A) = I_{q}^{\lambda+1}(A), \forall A \subseteq X \} = \inf\{ \lambda : \lambda \text{ is an ordinal such that } \pi^{\lambda} = \tau^{q} \} \).

**Proposition 4.14.** For \( q \in C(X) \) and an ordinal \( \alpha \),
\[
\begin{align*}
(1) & \quad \text{if } l_{T}q \leq \alpha, \text{ then } \tau_{\alpha}q = \tau^{q} \\
(2) & \quad \text{if } l_{T}q \leq \alpha, \text{ then } l_{D}q \leq \omega^{\alpha}.
\end{align*}
\]
Proof. (1) Let \( \lambda = l_T q \). Then \( \tau_\lambda q = \tau_{\lambda+1} q = \tau q \). Since \( \lambda \leq \alpha \), \( \tau_\lambda q \geq \tau_\alpha q \geq \tau q \). Thus \( \tau_\alpha q = \tau q \).

(2) Since \( l_T q \leq \alpha \), \( \tau_\alpha q = \tau q \). Thus \( \pi(\tau_\alpha q) = \pi(\tau q) \), so \( \pi^{\omega^\alpha} q = \tau q \).

Finally, \( l_D q \leq \omega^\alpha \). □

References


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