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COINCIDENCE POINT THEOREMS FOR SINGLE AND MULTI-VALUED CONTRACTIONS

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Abstract. In this paper two coincidence point theorems in complete metric spaces for two pairs of single and multi-valued mappings have been established.

1. Introduction

Let (X, d) be a metric space and let f and g be mappings from X into itself. In [5], Sessa defined f and g to be weakly commuting if $d(gfx, fgx) \leq d(gx, fx)$ for all $x \in X$. It can be seen that two commuting mappings are weakly commuting, but the converse is false as shown in the example of [5]. Recently Jungck [1] extended the concept of weak commutativity in the following way.

Let f and g be mappings from a metric space (X, d) into itself. The mappings f and g are said to be **compatible** if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{X_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X. It is obvious that two weakly commuting mappings are compatible, but the converse is not true, as one can see from the examples in [1].

Recently Kaneko [2] and Singh et al. [6] extended the concepts of weak commutativity and compatibility for single valued mappings to the setting of single valued and multi valued mappings, respectively. Now let (x, d) be a metric space and let $CB(X)$ denote the family of all non-empty closed and bounded subsets of X. Let H be the

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Hausdorff metric on $CB(X)$ and it is defined as

$$H(A,B) = \max\Big\{\sup_{x\varepsilon A} d(x,B), \sup_{y\varepsilon B}, d(y,A)\Big\} \quad \text{for } A, B \in CB(X),$$

Where $d(x,A) = \inf_{y \in A} d(X,y)$. It is well known that $(CB(X), H)$ is a metric space. Further if (X,d) is complete, then $(CB(X), H)$ is also complete.

The following lemma has been proved in Nadler [4].

LEMMA 1.1. *Let $A, B \in CB(X)$ and $k > 1$. Then for each $a \in A$ there exists a point $b \in B$ such that $d(a,b) \leq kH(A,B)$.*

DEFINITION 1.2. Let (X,d) be a metric space and let $f : X \to X$ and $S : X \to CB(X)$ be single valued and multi valued mappings respectively. The mappings f and S are said to be weakly commuting if for all $x \in X$, $fSx \in CB(X)$ and $H(Sfx, fSx) \leq d(fx, Sx)$, where H is the Hausdorff metric defined on $CB(X)$.

DEFINITION 1.3. The mappings f and S are said to compatible if

$$\lim_{n\to\infty} d(fy_n, Sfx_n) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = z \quad \text{for some } z \in X,$$

where $y_n \in Sx_n$ for $n = 1, 2, \ldots$.

REMARK 1.4.

(i) Definition 1.3 is slightly different from Kaneko's [2] definition.
(ii) If S is a single valued mapping on X in Definitions 1.2 and 1.3, then Definitions 1.2 and 1.3 become the definitions of weak commutativity and compatibility for single valued mapping.
(iii) If the mappings f and S are weakly commuting, then they are compatible, but the converse is not true. In fact,suppose that f and S are weakly commuting and let $\{X_n\}$ and $\{y_n\}$ be two sequences in X such that $y_n \in SX_n$ for $n = 1, 2, \ldots$ and

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = z \quad \text{for some } z \in X.$$

From $d(fx_n, Sx_n) \leq d(fx_n, y_n)$, it follows that

$$\lim_{n\to\infty} d(fx_n, Sx_n) = 0.$$

Thus, f and S are weakly commuting, we have

$$\lim_{n\to\infty} H(Sfy_n, fSx_n) = 0.$$

On the other hand, since $d(fy_n, Sfx_n) \leq H(fSx_n, Sfx_n)$, we have

$$\lim_{n\to\infty} d(fy_n, Sfx_n) = 0,$$

which means that f and S are compatible.

EXAMPLE 1.5. Let $X = [1, \infty)$ be set with the Euclidean metric d and define $fx = 2x^4 - 1$ and $Sx = [1, x^2]$ for all $X \geq 1$. Note that f and S are continuous and $S(X) = f(X) = X$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X defined by

$$x_n = y_n = 1 \quad \text{for } n = 1, 2, \ldots.$$

Then we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = 1 \in X,$$

where $y_n \in Sx_n$. On the other hand, we can show that $H(fSx_n, Sfx_n) = 2(xn^4 - 1)^2 \to 0$ if and only if $x_n \to 1$ as $n \to \infty$ and so, since $d(fy_n, Sfx_n) \leq H(fSx_n, Sfx_n)$, we have

$$\lim_{n\to\infty} d(fy_n, Sfx_n) = 0.$$

Therefore f and S are compatible, but f and S are not weakly commuting at $x = 2$.

2. Main Results

In this section we prove two coincidence point theorems and some particular cases of the same as corollaries.

THEOREM 2.1. *Let (X, d) be a complete metric space. Let f, $g : X \to X$ be a continuous mappings and $S, T : X \to CB(X)$ be H continuous mappings. Suppose $T(X) \subsetneq f(X)$, $S(X) \subsetneq g(X)$, the pair S and g are compatible mappings and*

$$(1) \qquad H(Sfx, Tgy) \leq h \max\{d(fx, Sfx), d(gy, Tgy), d(gy, Sfx), d(fx, Tgy), d(fx, gy)\}$$

for all $x, y \in X$ and $0 < h < 1$. Then S, f and T, g have a unique coincidence point.

Proof. Let $x_0 \in X$ be any arbitrary element in X. Since $S(X) \subseteq g(X)$ we have $Sfx_0 \subseteq g(X)$. This implies that there exists and element $x_1 \in X$ such that $gx_1 \in Sfx_0$. Since $T(X) \subseteq f(X)$ we have $Tgx_1 \subseteq f(X)$. Thus there exists $x_2 \in X$ such that $fx_2 \in Tgx_1$ and

$$d(gx_1, fx_2) \leq \frac{1}{p} H(Sfx_0, Tgx_1) \quad \text{where } P = \sqrt{2h} < 1.$$

Similarly, there exists $x_3 \in X$ such that $gx_3 \in Sfx_2$ and

$$d(gx_3, fx_2) \leq \frac{1}{p} H(Sfx_2, Tgx_1).$$

Now using (1), we have

$$\begin{aligned} H(Sfx_0, Tgx_1) \leq h \max\{&d(fx_0, Sfx_0), d(gx_1, Tgx_1), d(gx_1, Sfx_0), \\ &d(fx_0, Tgx_1), d(fx_0, gx_1)\} \end{aligned}$$

and so

$$\begin{aligned} \frac{1}{p} H(Sfx_0, Tgx_1) &\leq \frac{h}{p} \max\{d(fx_0, gx_1), d(gx_1, fx_2), d(gx_1, gx_1), \\ &\qquad d(fx_0, fx_2), d(fx_0, gx_1)\} \\ d(gx_1, fx_2) &\leq \frac{h}{p} \max\{d(fx_0, gx_1), d(gx_1, fx_2), d(fx_0, fx_2)\} \\ &\leq \frac{h}{p} \max(d(fx_0, gx_1), d(gx_1, fx_2), d(fx_0, gx_1), +d(gx_1, fx_2)\} \\ d(gx_1, fx_2) &\leq \frac{h}{p} [d(fx_0, gx_1) + d(gx_1, fx_2)] \\ (P - h)\ d(gx_1, fx_2) &\leq h\ d(fx_0, gx_1) \end{aligned}$$

This implies that

$$\begin{aligned} d(gx_1, fx_2) &\leq \frac{h}{P - h} d(fx_0, gx_1) \\ &= \sqrt{r}\ d(fx_0, gx_1) \quad \text{where } 0 < \sqrt{r} = \frac{h}{P - h} < 1. \end{aligned} \tag{2}$$

Also

$$\begin{aligned} d(gx_3, fx_2) &\le \frac{1}{P} H(Sfx_2, Tgx_1) \\ &\le \frac{1}{P} h \max\{d(fx_2, Sfx_2), d(gx_1, Tgx_1), d(gx_1, Sfx_2), \\ &\qquad d(fx_2, Tgx_1), d(fx_2, gx_1)\} \\ &\le \frac{1}{P} h \max\{d(fx_2, gx_3), d(gx_1, fx_2), d(gx_1, gx_3), \\ &\qquad d(fx_2, fx_2), d(fx_2, gx_1)\} \\ &\le \frac{1}{P} h \max\{d(fx_2, gx_3), d(gx_1, fx_2), d(gx_1, fx_2) + d(fx_2, fx_3)\} \end{aligned}$$

and hence

$$\begin{aligned} d(gx_3, fx_2) &\le \frac{1}{P} h\{d(fx_2, gx_3) + d(gx_1, fx_2)\}(P - h) d(gx_3, fx_2) \\ &\le hd(gx_1, fx_2) \\ d(gx_3, fx_2) &\le \frac{h}{P - h} d(gx_1, fx_2) \\ d(gx_3, fx_2) &\le \sqrt{r} d(gx_1, fx_2) \\ &\le \sqrt{r}\sqrt{r} d(fx_0, gx_1) \quad \text{(using 2.2)} \\ d(gx_3, fx_2) &\le rd(fx_0, gx_1) \end{aligned}$$

Continuing in this way, we get a sequence $\{x_n\}$ in X such that $gx_{2n+1} \in Sfx_{2n}$ and $fx_{2n} \in Tgx_{2n+1}$ for all $n \ge 1$ and so

$$d(gx_{2n+1}, fx_{2n}) \le r^n d(fx_0, gx_1) \quad \text{for } n \ge 1.$$

and

$$d(gx_{2n+1}, fx_{2n+2}) \le r^{n+1/2} d(fx_0, gx_1) \quad \text{for } n \ge 0.$$

Thus $\{gx_1, fx_2, gx_3, fx_4, \ldots, fx_{2n}, gx_{2n+1}\}$ is a Cauchy sequence, since X is complete there is a point $z \in X$ such that

$$\lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} fx_{2n} = z.$$

Now, we will prove that Z is a coincidence point of f and S. For every $n \ge 0$, we have

$$(3) \qquad d(fgx_{2n+1}, Sz) \le d(fgx_{2n+1}, Sfx_{2n}) + H(Sfx_{2n}, Sz)$$

It follows from H-continuity of S and $fx_{2n} \to z$ as $n \to \infty$ that

$$\lim_{n\to\infty} H(Sfx_{2n}, Sz) = 0 \tag{4}$$

Since f and S are compatible mappings and

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1} = z.$$

and $gx_{2n+1} \in Sfx_{2n}$ we have

$$\lim_{n\to\infty} d(fgx_{2n+1}, Sfx_{2n}) = 0 \tag{5}$$

Thus from (3), (4) and (5) we get

$$\lim_{n\to\infty} (fgx_{2n+1}, Sz) = 0$$

and so $d(fz, Sz) \le d(fz, fgx_{2n+1}) + d(fgx_{2n+1}, Sz)$.

Letting n tends to infinity, it follows that $d(fx, Sz) = 0$ this implies that $fz \in Sz$ since Sz is closed subset of X and thus z is a coincidence point of f and S. Similarly, we can prove that z is a coincidence point of g and T.

To prove the uniqueness of the coincidence point, let $z \neq y$ be another coincidence point for the pairs f, S and of g, T. Then $f(z) = g(z) = z$ and $f(y) = g(y) = y$. Also $f(z) \in S(z)$ and $g(z) \in T(z)$, $f(y) \in S(y)$ and $g(y) \in T(y)$.

Now, we have

$$\begin{aligned} H(Sfz, Tgy) &\le h \max\{d(fz, Sfz), \\ &\quad d(gy, Tgy), d(gy, Sfz), d(fz, Tgy), d(fz, gy)\} \end{aligned}$$

and so

$$H(Sz, Ty) \le h \max\{d(z, z), d(y, y), d(y, z), d(z, y), d(z, y)\}.$$

Hence

$$d(y, z) \le H(z, y) \le hd(y, z) < d(y, z),$$

which is a contradiction.
This completes the proof of the theorem. □

Letting $f = g$ as the identity mapping on X, in the above Theorem 2.1, we have the following corollary, which contains the result of Bose and Mukherjee [7].

COROLLARY 2.2. *Let (X, d) be a complete metric space and let $S, T : X \to CB(X)$ be H-continuous multi-valued mappings such that*

$$H(Sz, Ty) \leq h \max\{d(x, Sx), d(y, y), d(y, Sx), d(x, Ty), d(x, y)\}.$$

for all $x, y \in X$ and $0 \leq h < 1$, then S and T have a unique common fixed point in X.

Putting $f = g$ and $S = T$ in Theorem 2.1, we have the following corollary.

COROLLARY 2.3. *Let (X, d) be a complete metric space. Let $f : X \to X$ be a continuous mapping and let $S : X \to CB(X)$ be an H-continuous mapping such that $S(X) \subseteq f(X)$ and*

$$\begin{aligned} H(Sfx, Sfy) \quad \leq \quad & h \max\{d(fx, Sfx), \\ & d(fy, Sfy), d(fy, Sfx), d(fx, Sfy), d(fx, fy)\} \end{aligned}$$

for all $x, y \in X$ and $0 \leq h < 1$. Then f and S have a unique coincidence point.

Putting $f = g = 1$ and $T = S$ in Theorem 2.1, we have the following corollary, which includes the result of Ciric [9].

COROLLARY 2.4. *Let (X, d) be a complete metric space and let $X \to CB$ continuous mapping such that*

$$H(Sx, Sy) \leq h \max\{d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx), d(x, y)\}$$

for all $x, y \in X$ and $0 \leq h < 1$. Then S has a unique fixed point.

THEOREM 2.5. *Let (X, d) be a complete metric space. Let $f, g : X \to X$ be continuous mappings and $S, T : X \to CB(X)$ be H-continuous mappings such that $T(X) \subseteq f(X)$ and $S(X) \subseteq g(X)$; the fair S and g are compatible mappings and*

$$\begin{aligned} & H^p(Sx, Ty) \\ & \leq \max\{ad(fx, gy)d^{p-1}(fx, Sx), ad^{p-1}(gy, Ty)d(fx, gy), ad^{p-1}(gy, Ty) \\ (6) \qquad & \qquad d(fx, Sx), d(gy, Sx)[c_1 d^{p-1}(fx, Ty) + c_2 d^{p-1}(gy, Ty)]\} \end{aligned}$$

for all $x, y \in X$, integer $p \geq 2, 0 < a < 1$ and $c_1, c_2 \geq 0$, then there exists a coincidence point z of f, S and g, T. Further. if $0 < c_1 < 1$, then z is unique.

Proof. Let x_0 be an arbitrary point in X. Since $Sx_0 \subseteq g(X)$, there exists a point $x_1 \in X$ such that $gx_1 \in Sx_0$ and so there exists a point $x_2 \in X$ such that $fx_2 \in Tx_1$.

Hence by Lemma 1.1, there is $k = a^{-1/2p} > 1$ with such that $d(gx_1, fx_2) \leq kH(Sx_0, Tx_1)$. Similarly, there exists a point $x_3 \in X$ and $gx_3 \in Sx_2$ such that $d(gx_3, fx_2) \leq kH(Sx_2, Tx_1)$. Again, there exists a point $x_4 \in X, fx_4 \in Tx_3$ such that $d(gx_3, fx_4) \leq kH(Sx_2, Tx_3)$. Inductively, we can obtain a sequence $\{x_n\}$ in X such that for all $n \geq 0, fx_{2n+2} \in Tx_{2n+1}$ and $gx_{2n+1} \in Sx_{2n}$ and $d(gx_{2n+1}, fx_{2n+2}) \leq kH(Sx_{2n}, Tx_{2n+1})$.

Hence

$$\begin{aligned}
&d^p(gx_{2n+1}, fx_{2n+2})\\
&\quad\leq k^p H^p(Sx_{2n}, Tx_{2n+1})\\
&\quad\leq k^P \max\{ad(fx_{2n}, gx_{2n+1})d^{p-1}(fx_{2n}, Sx_{2n}), ad^{P-1}(gx_{2n+1}, Tx_{2n+1})\\
&\qquad d(fx_{2n}, gx_{2n+1}), ad^{P-1}(gx_{2n+1}, Tx_{2n+1})d(fx_{2n}, Sx_{2n}),\\
&\qquad d(gx_{2n+1}, Sx_{2n})[c_1 d^{P-1}(fx_{2n}, Tx_{2n+1}) + c_2 d^{P-1}(gx_{2n+1}, Tx_{2n+1})]\}\\
&\quad\leq k^P \max\{ad(fx_{2n}, gx_{2n+1})d^{P-1}(fx_{2n}, gx_{2n+1}) ad^{P-1}(gx_{2n+1}, fx_{2n+2})\\
&\qquad d(fx_{2n}, gx_{2n+1}), ad^{P-1}(gx_{2n+1}, fx_{2n+2})d(fx_{2n}, gx_{2n+1},) \text{ (using 2.1)}\\
&\qquad d(gx_{2n+1}, gx_{2n+1})[c_1 d^{P-1}(fx_{2n}, fx_{2n+2}) + c_2 d^{P-1}(gx_{2n+1}, fx_{2n+2})]\}\\
&\quad\leq k^P a \max\{d^P(fx_{2n}, gx_{2n+1}), d^{P-1}(gx_{2n+1}, fx_{2n+2})\ d(fx_{2n}, gx_{2n+1})\\
&\quad= a^{\frac{1}{2}} a \max\{d^P(fx_{2n}, gx_{2n+1}), d^{P-1}(gx_{2n+1}, fx_{2n+2})\ d(fx_{2n}, gx_{2n+1})\}\\
&d^P(gx_{2n+1}, fx_{2n+2})\\
&\quad\leq \sqrt{a} \max\{d^P(fx_{2n}, gx_{2n+1}), d^{P-1}(gx_{2n+1}, fx_{2n+2})d(fx_{2n}, gx_{2n+1})\}
\end{aligned}$$

If

$$\begin{aligned}
&\max\{d^p(fx_{2n}, gx_{2n+1}), d^{p-1}(gx_{2n+1}, fx_{2n+2})d(fx_{2n}, gx_{2n+1})\}\\
&\quad= d^{p-1}(gx_{2n+1}, fx_{2n+2})d(fx_{2n}, gx_{2n+1})
\end{aligned}$$

then

$$d^p(fx_{2n}, gx_{2n+1}) \leq d^{p-1}(gx_{2n+1}, fx_{2n+2})d(fx_{2n}, gx_{2n+1})$$

and so

$$(7) \qquad d(fx_{2n}, gx_{2n+1}) \leq d(gx_{2n+1}, fx_{2n+2})$$

Also

$$d^p(gx_{2n+1}, fx_{2n+2}) \leq \sqrt{a}\, d^{p-1}(gx_{2n+1}, fx_{2n+2}) d(fx_{2n}, gx_{2n+1})$$

$$(8) \qquad d(gx_{2n+1}, fx_{2n+2}) \leq \sqrt{a}\, d(fx_{2n}, gx_{2n+1}) < d(fx_{2n}, gx_{2n+1})$$

Hence from (7)and (8)

$$\max\{d^p(fx_{2n}, gx_{2n+1}), d^{p-1}(gx_{2n+1}, fx_{2n+2}), d(fx_{2n}, gx_{2n+1})\}$$
$$= d^p(fx_{2n}, gx_{2n+1})$$

Thus

$$d^p(gx_{2n+1}, fx_{2n+2}) \leq \sqrt{a}\, d^p(gx_{2n+1}, fx_{2n})$$

and hence

$$d(gx_{2n+1}, fx_{2n+2}) \leq \beta d(gx_{2n+1}, fx_{2n}) \quad \text{for } n \geq 0.$$

where $\beta = a^{1/2p} < 1$.

Also

$$d(gx_{2n+1}, fx_{2n+2}) \leq \beta d(gx_{2n-1}, fx_{2n}) \quad \text{for } n \geq 1$$
$$\leq \beta^n d(gx_1, fx_2) \to 0 \quad \text{as } n \to \infty \text{ (since } 0 < \beta < 1).$$

It follows that $\{gx_1, fx_2, gx_3, fx_3, fx_4, \ldots, gx_{2n-1}, fx_{2n} \ldots\}$ is a Cauchy sequence in X.

Since (X, d) is a complete metric space, there is a point z in X such that

$$\lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} fx_{2n} = z.$$

Now we will prove that z is a coincidence point of f and S. For every $n \geq 1$, we have

$$(9) \qquad d(fgx_{2n+1}, Sz) \leq d(fgx_{2n+1}, Sfx_{2n}) + H(sfx_{2n}, Sz)$$

It follows from H-continuity of S that

$$(10) \qquad \lim_{n\to\infty} H(Sfx_{2n}, Sz) = 0$$

Since $fx_{2n} \to z$ as $n \to \infty$. □

Since f and S are compatible mappings and

$$\lim_{n\to\infty} d(fgx_{2n+1}, Sfx_{2n}) = 0 \tag{11}$$

Thus from the identities (9), (10) and (11) we have $\lim_{n\to\infty} d(fgx_{2n+1}, Sz)$ $= 0$ and so $d(fz, Sz) \le (fz, fgx_{2n+1}) + d(fgx_{2n+1}, Sz)$. Letting $n \to \infty$, it follows that $d(fz, Sz) = 0$. This implies that $fz \in Sz$, since Sz is a closed subset of X. Thus z is a coincidence point of f and S. Similarly, we prove that z is a coincidence point of g and T.

Suppose $z \neq y$ is an another coincidence point for the pair f, S and g, T then $fz = gz = z$ and $fy = gy = y$. This gives that $f(z) \in S(z), g(z) \in T(z)$ and $f(y) \in S(y), g(y) \in T(y)$ and so

$$\begin{aligned}
d^p(z,y) &\le H^p(Sz, Ty) \\
&\le \max\{ad(fz, gy)d^{p-1}(fz, Sz), ad^{p-1}(gy, Ty)d(fz, gy), ad^{p-1}(gy, Ty) \\
&\qquad d(fz, Sz), d(gy, Sz)[c_1 d^{p-1}(fz, Ty) + c_2 d^{p-1}(gy, Ty)]\} \\
&= \max\{ad(z,y).0, 0.d(z,y), a.0, d(y,z)\{c_1, d^{p-1}(z,y) + c_2 x 0\} \\
&= c_1 d(y,z) d^{p-1}(y,z) \\
&< d^p(y,z) \quad (\text{since } c_1 < 1),
\end{aligned}$$

which is a contradiction.

Hence f, S and g, T have a unique coincidence point.

Allowing $c_1 = c$ and $c_2 = 0$ in Theorem 2.5, we have the following corollary.

COROLLARY 2.6 (Duran Turkoglu Orhan Ozer, and Brain Fisher [8]).

Let (X, d) be a complete metric space. Let $f, g : X \to X$ be continuous mappings and $S, T : X \to CB(X)$ be H-continuous mappings such that $T(X) \subseteq f(X)$ and $S(X) \subseteq g(X)$, the pair S and g are compatible mappings and

$$\begin{aligned}
H^p(Sx, Ty) \le \max\{&ad(fx, gy)d^{p-1}(fx, Sy), ad(fx, gy)d^{p-1}(gy, Ty), \\
&ad(fx, Sx)d^{p-1}(gy, Ty), cd^{p-1}(fx, Ty)d(gy, Sx)\}
\end{aligned}$$

for all $x, y \in X$, where $p \ge 2$ is an integer, $0 < a < 1$ and $c \le 0$. Then there exists a point $z \in X$, such that $fx \in Sz$ and $gz \in Tz$, i.e., z is a coincidence point of f, S and of g, T. Further, z is unique when $0 < c < 1$.

Letting $f = g$ as identity mapping on X, in Theorem 2.5, we have the following corollary.

COROLLARY 2.7. *Let (X, d) be a complete metric space and let $S, T : X \to CB(X)$ be H-Continuous multi-valued mappings such that*

$$H^P(Sx, Ty) \leq \max\{ad(x, y)d^{P-1}(x, Sx), ad^{p-1}(y, Ty)d(x, y), ad^{P-1}(y, Ty) \\ d(x, Sx), d(y, Sx)[c_1 d^{P-1}(x, Ty) + c_2 d^{p-1}(y, Ty)]\}$$

for all $x, y \in X$ where $p \geq 2$ is an integer $0 < a < 1$, and $c_1 + c_2 \geq 0$.

Then S and T have a common fixed-point z in X. Also S and T have a unique common fixed point z in X when $0 < c_1 < 1$.

Putting $f = g$ and $S = T$ in Theorem 2.5, we have the following corollary.

COROLLARY 2.8. *Let (X, d) be a complete metric space, let $f : X \to X$ be a continuous mapping and let $S : X \to CB(X)$ be an H-continuous mapping such that $S(X) \subseteq f(X)$ and*

$$H^P(Sx, Sy) \\ \leq \max\{ad(fx, fy)d^{P-1}(fx, Sx), ad^{p-1}(fy, Sy)d(fx, fy), \\ ad^{P-1}(fy, Sy)d(fx, Sx), d(fy, Sx)[c_1 d^{P-1}(fx, Sy) + c_2 d^{p-1}(fy, Sy)]\}$$

for all $x, y \in X$ where $p \geq 2$ is an integer, $0 < a < 1$ and $c_1 + c_2 \geq 0$. Then there exists a coincidence point z of f and S.

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