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FREE NODES AND BUSY NODES OF A FUZZY GRAPH

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ABSTRACT. In networking, it is essential to analyse the nodes along with their neighbours. The properties of fuzzy finite graphs are very much useful in obtaining solutions to many problems like traffic congestion problem. In this paper, free nodes and busy nodes are defined and the properties of nodes are examined at various levels of their degrees.

1. Introduction

A fuzzy subset [3] of a nonempty set S is a mapping $\sigma : S \to [0,1]$. A fuzzy relation on S is a fuzzy subset of $S \times S$. If μ and ν are fuzzy relations, then $\mu \circ \nu(u,w) = sup\{\mu(u,v) \wedge \nu(v,w) : v \in S\}$ and $\mu^k = sup\,\{\mu(u,u_1) \wedge \mu(u_1,u_2) \wedge \mu(u_2,u_3) \wedge \ldots \wedge \mu(u_{k-1},v) : u_1, u_2, \ldots u_{k-1} \in S\}$, where $\wedge$ stands for minimum. A fuzzy graph is a pair of functions $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$, where for all $u, v \in V$, we have $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$. The degree of a vertex v is defined by $d(v) = \sum\limits_{u \neq v} \mu(v,u)$.

A path ρ in a fuzzy graph is a sequence of distinct nodes $v_0, v_1, v_2, \ldots, v_n$ such that $\mu(v_{i-1}, v_i) > 0, 1 \leq i \leq n$; here $n \geq 0$ is called the length of the path ρ. The consecutive pairs (v_{i-1}, v_i) are called the arcs of the path. The strength of a path is defined as $\bigwedge\limits_{i=1}^{n} \mu(v_{i-1}, v_i)$. In other words, the strength of a path is defined to be the weight of the weakest arc of the path. If a path has length zero, then its strength is defined

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to be $\sigma(v_0)$. Also, $sup\{\mu^k(u,v) : k = 1,2,3,..\}$ gives the strength of the connectedness between any two nodes u and v, which is denoted by $\mu^\infty(u,v)$. A fuzzy graph $G : (\sigma,\mu)$ is connected if $\mu^\infty(u,v) > 0$ for all $u,v \in V$. The underlying crisp graph [2] of the fuzzy graph $G : (\sigma,\mu)$ is denoted by $G^* : (\sigma^*,\mu^*)$ where $\sigma^* = \{u \in S : \sigma(u) > 0\}$ and $\mu^* = \{(u,v) \in V \times V : \mu(u,v) > 0\}$. A fuzzy graph $G : (\sigma,\mu)$ is a strong fuzzy graph if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $(u,v) \in \mu^*$ and is a complete fuzzy graph if $\mu(v,u) = \sigma(u) \wedge \sigma(v)$ for all $u,v \in \sigma^*$. Two nodes u and v are said to be neighbours if $\mu(v,u) > 0$. Throughout this paper we deal only with finite connected fuzzy graphs.

2. μ-complemet

J.N.Mordeson [1] defined complement of a fuzzy graph $G : (\sigma,\mu)$ as a fuzzy graph $G^c : (\sigma^c,\mu^c)$ where $\sigma^c = \sigma$ and $\mu^c(u,v) = 0$ if $\mu(u,v) > 0$ and $\mu^c(u,v) = \sigma(u) \wedge \sigma(v)$ otherwise. It follows from the definition that G^c is a fuzzy graph even if G is not a fuzzy graph and $(G^c)^c = G$ is false in general. Later M.S.Sunitha and A.Vijayakumar[4] have modified the notion of complement of a fuzzy graph as follows. The complement of a fuzzy graph $G : (\sigma,\mu)$ is a fuzzy graph $\bar{G}:(\bar{\sigma},\bar{\mu})$ where $\bar{\sigma} = \sigma$ and $\bar{\mu}(u,v) = \sigma(u) \wedge \sigma(v) - \mu(u,v)$. This definition holds almost all properties similar to those in crisp graph. But our aim requires some different definition for complement of a fuzzy graph. Hence, we introduce μ-complement of a fuzzy graph, with a little deviation from the above two, keeping the following fact in view. In crisp graph, if the degrees of two nodes are equal then we can infer that they have same number of neighbours. In the case of fuzzy graph, it is not so. For example, let $V = \{v_1,v_2,v_3,v_4\}$ and define σ on V and μ on $V \times V$ as $\sigma(v_1) = \sigma(v_2) = \sigma(v_3) = \sigma(v_4) = 1$ and $\mu(v_1,v_2) = 0.5$, $\mu(v_2,v_3) = 0.3$, $\mu(v_2,v_4) = 0.2$ and $\mu(v_3,v_4) = 0.2$. Then by the definition of degree of a node, $d(v_1) = 0.5$, $d(v_2) = 1$, $d(v_3) = 0.5$, $d(v_4) = 0.4$. Here, we see that degree of v_1 and degree of v_3 are equal; but the node v_1 has only one neighbour v_2, whereas the node v_3 has two neighbours v_2 and v_4. Hence, from the degree of a node alone, we can't understand its relationship with other nodes. Thus a deeper study on the degree of a node is imperative.

DEFINITION 2.1. *The μ-complement of a fuzzy graph $G : (\sigma, \mu)$ is denoted by $G^\mu : (\sigma^\mu, \mu^\mu)$ and defined as $\sigma^\mu = \sigma$ and $\mu^\mu(u, v) = 0$ if $\mu(u, v) = 0$ and $\mu^\mu(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v)$ if $\mu(u, v) > 0$.*

EXAMPLE 2.1. *Let $G : (\sigma, \mu)$ be a fuzzy graph such that $V = \{v_1, v_2, v_3, v_4\}$ and σ on V and μ on $V \times V$ be defined as $\sigma(v_1) = 0.6$, $\sigma(v_2) = 0.8$, $\sigma(v_3) = 0.5$, $\sigma(v_4) = 0.7$ and $\mu(v_1, v_2) = 0.5$, $\mu(v_1, v_3) = 0.3$, $\mu(v_1, v_4) = 0.4$, $\mu(v_2, v_3) = 0.1$ and $\mu(v_2, v_4) = 0.7$. Then $G^\mu : (\sigma^\mu, \mu^\mu)$ is the fuzzy graph with same four nodes such that $\sigma^\mu(v_1) = 0.6$, $\sigma^\mu(v_2) = 0.8$, $\sigma^\mu(v_3) = 0.5$, $\sigma^\mu(v_4) = 0.7$, $\mu^\mu(v_1, v_2) = 0.1$, $\mu^\mu(v_1, v_3) = 0.2$, $\mu^\mu(v_1, v_4) = 0.2$, $\mu^\mu(v_2, v_3) = 0.4$ and $\mu^\mu(v_2, v_4) = 0$.*

REMARK 2.1. *If $G : (\sigma, \mu)$ is a complete fuzzy graph then $G^\mu : (\sigma^\mu, \mu^\mu)$ contains only isolated nodes. (i.e.) $\mu^\mu(u, v) = 0$ for all $u, v \in V$.*

DEFINITION 2.2. *The busy value of a node v of a fuzzy graph $G : (\sigma, \mu)$ is defined to be $D(v) = \sum_i \sigma(v) \wedge \sigma(v_i)$ where v_i are neighbours of v and the busy value of a fuzzy graph G is defined to be the sum of the busy values of all nodes of G. (i.e.) $D(G) = \sum_i D(v_i)$ where v_i are nodes of G.*

EXAMPLE 2.2. *For the fuzzy graph $G : (\sigma, \mu)$ given in example 2.1*

$$D(v_1) = \sigma(v_1) \wedge \sigma(v_2) + \sigma(v_1) \wedge \sigma(v_3) + \sigma(v_1) \wedge \sigma(v_4)$$
$$= 0.6 + 0.5 + 0.6 = 1.7$$
$$D(v_2) = \sigma(v_2) \wedge \sigma(v1) + \sigma(v_2) \wedge \sigma(v_3) + \sigma(v_2) \wedge \sigma(v_4)$$
$$= 0.6 + 0.5 + 0.7 = 1.8$$
$$D(v_3) = \sigma(v_3) \wedge \sigma(v_1) + \sigma(v_3) \wedge \sigma(v_2)$$
$$= 0.5 + 0.5 = 1$$
$$D(v_4) = \sigma(v_4) \wedge \sigma(v_1) + \sigma(v_4) \wedge \sigma(v_2)$$
$$= 0.6 + 0.7 = 1.3$$
$$D(G) = D(v_1) + D(v_2) + D(v_3) + D(v_4)$$
$$= 1.7 + 1.8 + 1 + 1.3 = 5.8$$

PROPOSITION 2.1. *Let $G : (\sigma, \mu)$ be a fuzzy graph and $v \in V$. Then $d^{\mu}(v) = D(v) - d(v)$.*

Proof. $d^{\mu}(v) = \sum_i \mu^{\mu}(v, v_i)$
$= \sum_i (\sigma(v) \wedge \sigma(v_i) - \mu(v, v_i)), v_i$s are neighbours of v.
$= \sum_i (\sigma(v) \wedge \sigma(v_i) - \sum_i \mu(v, v_i))$
$= D(v) - d(v)$ □

REMARK 2.2. *From the definition of a complete fuzzy graph it follows that for each node v, $d(v) = D(v)$.*

3. Free node and Busy node

DEFINITION 3.1. *A node v of a fuzzy graph $G : (\sigma, \mu)$ is said to be a busy node if $\sigma(v) \le d(v)$; otherwise it is called a free node.*

EXAMPLE 3.1. *In example 2.1, the nodes v_1, v_2 and v_4 are busy nodes and v_3 is a free node.*

DEFINITION 3.2. [2] *Let $G : (\sigma, \mu)$ be a fuzzy graph. Then the order of G is defined to be $O(G) = \sum_{u \in V} \sigma(v)$. The size of G is defined to be $S(G) = \sum_{u \neq v} \mu(v, u)$.*

DEFINITION 3.3. [2] *An edge (u, v) of a fuzzy graph $G : (\sigma, \mu)$ is said to be an effective edge if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$.*

EXAMPLE 3.2. *In example 2.1, the edge (v_2, v_4) is an effective edge.*

DEFINITION 3.4. *A node v of a fuzzy graph $G : (\sigma, \mu)$ is said to be a*

- i *partial free node if it is a free node in both G and G^{μ}.*
- ii *fully free node if it is a free node in G but it is a busy node in G^{μ}.*
- iii *partial busy node if it is a busy node in both G and G^{μ}.*
- iv *fully busy node if it is a busy node in G but it is a free node in G^{μ}.*

EXAMPLE 3.3. *Let* $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ *and define* σ *on* V *and* μ *on* $V \times V$ *as* $\sigma(v_1) = 0.3$, $\sigma(v_2) = 0.2$, $\sigma(v_3) = 0.8$, $\sigma(v_4) = 1$, $\sigma(v_5) = 0.5$, $\sigma(v_6) = 0.5$, $\mu(v_1, v_2) = 0.1$, $\mu(v_2, v_3) = 0.1$, $\mu(v_2, v_4) = 0.1$, $\mu(v_2, v_5) = 0.1$, $\mu(v_3, v_4) = 0.3$, $\mu(v_3, v_6) = 0.3$, $\mu(v_4, v_5) = 0.3$ *and* $\mu(v_5, v_6) = 0.4$.

In the above example v_1 is a partial free node; v_2 is a partial busy node; v_3 is a fully free node; v_6 is a fully busy node.

4. Main Results

PROPOSITION 4.1. *If a fuzzy graph has effective edges then it has at least one busy node.*

Proof. Let $G : (\sigma, \mu)$ be a fuzzy graph, which has some effective edges. Let (u, v) be an effective edge of G. Without loss of generality let us assume that $\sigma(u) \leq \sigma(v)$. Then clearly $d(u) \geq \sigma(u)$, which means that u is a busy node. $\square$

EXAMPLE 4.1. *Let* $G : (\sigma, \mu)$ *be a fuzzy graph such that* $V = \{v_1, v_2, v_3, v_4, v_5\}$ *and* σ *on* V *and* μ *on* $V \times V$ *be defined as* $\sigma(v_1) = 0.5$, $\sigma(v_2) = 0.8$, $\sigma(v_3) = 0.8$, $\sigma(v_4) = 0.8$, $\sigma(v_5) = 0.8$ *and* $\mu(v_1, v_2) = 0.5$, $\mu(v_1, v_3) = 0.5$, $\mu(v_1, v_4) = 0.5$ *and* $\mu(v_1, v_5) = 0.5$. *In this fuzzy graph all edges are effective edges; but it has exactly only one busy node.*

PROPOSITION 4.2. *If a fuzzy graph with* n *nodes, where* $n > 1$ *is a complete fuzzy graph then it has at least* $n - 1$ *fully busy nodes.*

Proof. Let $G : (\sigma, \mu)$ be a complete fuzzy graph with n nodes. Let us label the nodes of G as $v_1, v_2, v_3, \ldots, v_n$ such that $\sigma(v_1) \geq \sigma(v_2) \geq \sigma(v_3) \geq \ldots, \geq \sigma(v_n)$. Then

$$\begin{array}{llrcl}
(1) & \cdots & d(v_1) & = & \sigma(v_2)+\sigma(v_3)+\ldots+\sigma(v_n) \\
(2) & \cdots & d(v_2) & = & \sigma(v_2)+\sigma(v_3)+\ldots+\sigma(v_n) \\
(3) & \cdots & d(v_3) & = & 2\sigma(v_3)+\sigma(v_4)+\ldots+\sigma(v_n) \\
(4) & \cdots & d(v_4) & = & 3\sigma(v_4)+\sigma(v_5)+\ldots+\sigma(v_n) \\
 & & & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
 & & & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(k-1) & \cdots & d(v_k) & = & (k-1)\sigma(v_k)+\sigma(v_{k+1})+\ldots+\sigma(v_n) \\
 & & & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
 & & & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(n-1) & \cdots & d(v_{n-1}) & = & (n-2)\sigma(n_1)+\sigma(v_n)---(n-1) \\
(n) & \cdots & d(v_n) & = & (n-1)\sigma(v_n)--(n)
\end{array}$$

From the above n equations it follows that $d(v_i) > \sigma(v_i)$ for $i = 2, 3, \cdots .n$. Hence in a complete fuzzy graph there are at least $n-1$ busy nodes. □

REMARK 4.1. *From equation (1) we infer that if $\sigma(v_1) \leq \sigma(v_2) + \sigma(v_3) + \ldots + \sigma(v_n)$ then all the nodes are busy nodes.*

COROLLARY 4.1. *In a complete fuzzy graph if there are at least two nodes with highest σ-values then all the nodes are busy nodes.*

Proof. Suppose that there are two nodes with highest σ-values, then as in the proof given in the above proposition, $\sigma(v_1) = \sigma(v_2)$. Hence by equation (1) the proof follows. □

EXAMPLE 4.2. *Consider the complete fuzzy graph $G : (\sigma, \mu)$ with three nodes v_1, v_2 and v_3 such that $\sigma(v_1) = 1$, $\sigma(v_2) = 0.5$, $\sigma(v_3) = 0.4$. In this fuzzy graph only v_1 and v_2 are busy nodes. But if we take $\sigma(v_2) = 1$ then all the nodes become busy nodes.*

PROPOSITION 4.3. *Let $G : (\sigma, \mu)$ be a complete fuzzy graph with n nodes. If the nodes of G are labeled as $v_1, v_2, v_3, \ldots, v_n$ such that $\sigma(v_1) \geq \sigma(v_2) \geq \sigma(v_3) \geq \ldots, \geq \sigma(v_n)$, then $D(G) = \sum\limits_{i=2}^{n} 2(i-1)\sigma(v_i)$.*

Proof. Adding the n equations given in proposition 4.2 and applying the remark given in 2.2, we get the required result. □

PROPOSITION 4.4. *Let $G : (\sigma, \mu)$ be a fuzzy graph. Then a node u is a*

(a): *partial free node if and only if* $D(u) < \sigma(u) + d(u) < 2\sigma(u)$
(b): *fully free node if and only if* $d(u) < \sigma(u) \leq d^{\mu}(u)$
(c): *partial busy node if and only if* $D(u) \geq \sigma(u) + d(u) \geq 2\sigma(u)$
(d): *fully busy node if and only if* $d(u) \geq \sigma(u) > d^{\mu}(u)$

Proof. (a) u is a partial free node $\Leftrightarrow d(u) < \sigma(u)$ and $D(u) - d(u) < \sigma(u)$
$\Leftrightarrow \sigma(u) + d(u) < 2\sigma(u)$ and $D(u) < \sigma(u) + d(u)$
$\Leftrightarrow D(u) < \sigma(u) + d(u) < 2\sigma(u)$
(b) It follows from the very definition of fully free node.
(c) u is a partial busy node $d(u) \geq \sigma(u)$ and $D(u) - d(u) \geq \sigma(u)$
$\Leftrightarrow \sigma(u) + d(u) \geq 2\sigma(u)$ and $D(u) \geq \sigma(u) + d(u)$
$\Leftrightarrow D(u) \geq \sigma(u) + d(u) \geq 2\sigma(u)$
(d) It follows from the very definition of fully busy node. □

PROPOSITION 4.5. *Let* $G : (\sigma, \mu)$ *be a fuzzy graph. If*

(a): *all nodes of* G *are partial free nodes then* $D(G) < O(G) + S(G) < 2O(G)$
(b): *all nodes of* G *are fully free nodes then* $S(G) < O(G) \leq D(G) - S(G)$
(c): *all nodes of* G *are partial busy nodes then* $D(G) \geq O(G) + S(G) \geq 2O(G)$
(d): *all nodes of* G *are fully busy nodes then* $S(G) \geq O(G) > D(G) - S(G)$

Proof. Taking summation over the nodes in the respective inequalities given in proposition 4.4, we get the corresponding inequalities. The converse of these results need not be true. □

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