AN OPTIMAL CONTROL FOR THE WAVE EQUATION WITH A LOCALIZED NONLINEAR DISSIPATION

YONG HAN KANG

Abstract. We consider the problem of an optimal control of the wave equation with a localized nonlinear dissipation. An optimal control is used to bring the state solutions close to a desired profile under a quadratic cost of control. We establish the existence of solutions of the underlying initial boundary value problem and of an optimal control that minimizes the cost functional. We derive an optimality system by formally differentiating the cost functional with respect to the control and evaluating the result at an optimal control.

1. Introduction

In this paper we consider the optimal control problem for the wave equations with a localized nonlinear dissipation:

\begin{equation}
\begin{aligned}
u_{tt} - \Delta u + a(x)u_t &= f \quad \text{in } \Omega_T = \Omega \times (0, T], \\
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) &= \quad \text{in } \Omega, \\
u(x, t) &= 0 \quad \text{on } \partial \Omega \times [0, T],
\end{aligned}
\end{equation}

where \( f : \Omega_T \to \mathbb{R}, \ u_0, u_1 : \Omega \to \mathbb{R} \) are given and \( a : \bar{\Omega} \to \mathbb{R}, \ u : \Omega_T \to \mathbb{R} \) are the unknown, \( u = u(x, t) \). We set for \( x_0 \in \mathbb{R}^N \),

\[ \Gamma(x_0) = \{ x \in \partial \Omega : (x - x_0) \cdot \nu(x) \geq 0 \} \]

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where \( v(x) \) denotes the outward unit normal of the boundary \( \partial \Omega \) at \( x \in \partial \Omega \) and let
\[
\omega = \left( \bigcup_{x \in \Gamma(\varepsilon_0)} B_\varepsilon(x) \right) \cap \bar{\Omega}
\]
where \( B_\varepsilon(x) = \{ y \in \mathbb{R}^N : \| x - y \| < \delta \text{ for some } \delta > 0 \} \). Given the control set:
\[
U_M = \{ a \in L^\infty(\omega) : M \geq a \geq c_0 \text{ for some } M, c_0 > 0 \}.
\]
In here, the corresponding state variable \( u = u(a) \) satisfies the state equation (1.1). We take as our objective functional:
\[
J(a) = \frac{1}{2} \int_0^T \int_\omega (u(a) - z_d)^2 \, dx \, dt + \frac{\beta}{2} \int_\Omega a^2(x) \, dx
\]
where \( z_d \in L^2(\Omega_T) \) is a given target function and \( a \in L^\infty(\bar{\Omega}), a(x) \geq 0 \). We can find \( a^* \in U_M \) such that
\[
J(a^*) = \min_{a \in U_M} J(a).
\]

For the background in control of PDEs, see Liang ([5]). Nakao ([6],[7]) developed decay of solutions of wave equation with a local degenerate dissipation. Bradley and Lenhart ([3]) treated \( \Delta^2 \) type of bilinear control for the Kirchhoff plate equation. Park et al. ([8],[9],[10]) treated for optimal control of parameters and operators.

The goal of this work is to obtain an unique optimal control in terms of the solution to the optimal system, which will consist of the original wave problem coupled with an adjoint problem. In Section 2, we show the well-posedness of our state problem in an appropriate solution space. Then we show the existence of an optimal control by a minimizing sequence argument. In Section 3, the optimality systems is derived by differentiating the objective functional with respect to the control. The solution map \( a \to u(a) \) is differentiated which is used in the differentiation of the objective functional. Then for sufficiently small time \( T \), under some boundedness assumption, we prove uniqueness of the optimal systems, which characterizes the unique optimal control.
2. Existence of an Optimal Control

The following assumptions are made throughout this part: $\Omega$ is bounded domain in $\mathbb{R}^N$ and $\partial \Omega$ is $C^2$ smooth. $\Omega_T = \Omega \times (0, T]$. $f, f_t \in L^2(\Omega_T)$. $a(x) \in L^\infty(\bar{\Omega}), a(x) = 0$ in $\bar{\Omega} - \omega, a(x) > 0$ in $\omega$ and $0 \leq a(x) \leq M^*, M \leq M^* < \infty$, $M^*$ is a constant.

We present our definition of weak solution.

**Definition.** Given $a \in L^\infty(\bar{\Omega})$ and $u_0 \in H^1_0(\Omega), u_1 \in L^2(\Omega), u \in C(0, T; H^1_0(\Omega))$ with $u_t \in C(0, T; \mathbb{L}^2(\Omega)), u_{tt} \in C(0, T; H^{-1}(\Omega))$ is a weak solution of the problem (1.1):

\begin{align}
(2.1) \quad (a) \quad &\int_0^T \langle u_{tt}, \phi \rangle dt + \int_0^T \langle \nabla u, \nabla \phi \rangle dt + \int_0^T \langle a(x)u_t, \phi \rangle dt \\
&= \int_0^T \langle f, \phi \rangle dt \quad \text{for any } \phi \in H^1_0(\Omega) \text{ and a.e. } 0 \leq t \leq T; \\
(b) \quad &u(x, 0) = u_0(x); \\
(c) \quad &u_t(x, 0) = u_1(x);
\end{align}

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

For notational convenience, we set
\[ \mathcal{H} = H^1_0(\Omega) \times \mathbb{L}^2(\Omega), \]
\[ u = u(a), \dot{u} = (u, u_t). \]

**Lemma 2.1. (Well-Posedness)(See [11])** For $\bar{u}_0 = (u_0, u_1) \in \mathcal{H}$ and $a \in L^\infty(\bar{\Omega})$, the problem (1.1) has a unique weak solution $u$.

**Proof.** We write (1.1) in the semigroup formulation
\begin{align}
(2.2) \quad &\frac{d}{dt} \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ -a u_t + f \end{pmatrix} \\
\ddot{u}(0) &= \bar{u}_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.
\end{align}
Define the operator $A : [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega) \to \mathcal{H}$ by

$$A\tilde{u} = \begin{pmatrix} 0 & I \\ \Lambda & 0 \end{pmatrix} \tilde{u}.$$ 

Its domain, $D(A) = [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega)$ is clearly dense in $\mathcal{H}$. Formulation (2.2) may be written as

$$\frac{d}{dt} \tilde{u}(t) = A\tilde{u}(t) + B_a(\tilde{u})(t)$$

where

$$B_a(\tilde{u}) = \begin{pmatrix} 0 \\ -au + f \end{pmatrix}.$$ 

Note that $A$ is skew adjoint (See [1]), i.e. $A^* = -A$

$$A^*\tilde{u} = \begin{pmatrix} 0 & -I \\ -\Lambda & 0 \end{pmatrix} \tilde{u}$$

and thus $D(A) = D(A^*)$. Thus we have that $A$ generates a unitary group on $\mathcal{H}$ (See [1]). Motivated by the formulation (2.3), we seek a solution of the form:

$$\tilde{u}(t) = e^{A^*t}\tilde{u}_0 + \int_0^t e^{A^*(t-\tau)} B_a(\tilde{u})(\cdot, \tau)d\tau$$

We will prove that the map $T_\alpha$,

$$T_\alpha \tilde{u}(t) = e^{A^*t}\tilde{u}_0 + \int_0^t e^{A^*(t-\tau)} B_a(\tilde{u})(\cdot, \tau)d\tau$$

has a unique fixed point in $C([0,T_0]; \mathcal{H})$.

**Step1.** We will prove that, if $T_0$ is small enough, there exists a unique fixed point such that

$$T_\alpha(\tilde{u}(t)) = \tilde{u}(t) \text{ in } C([0,T_0]; \mathcal{H}).$$

To use the contraction mapping theorem, we need to show that $T_\alpha$ is bounded and contractive.
Boundedness:

\[ (2.4) \quad \|T_0 \tilde{u}\|_{C([0,T_0];\mathcal{H})} \]
\[ \leq \|e^{A_0 t_0}\|_{C([0,T_0];\mathcal{H})} + \sup_{0 \leq t \leq T_0} \int_0^t \|e^{A(t-\tau)}B_0(\tilde{u})(\cdot, \tau)\|_{\mathcal{H}} d\tau \]
\[ \leq \|e^{A_0 t_0}\|_{C([0,T_0];\mathcal{H})} + \sup_{0 \leq t \leq T_0} \int_0^t \|e^{A(t-\tau)}|a(\cdot)u_0(\cdot, \tau) + f(\cdot, \tau)|\|_{L^2(\Omega)} d\tau \]

since \( A \) generates a unitary group and that \( \|e^{At}\| = 1 \), here \( \| \cdot \| \) denoting the operator norm. Since \( \|a\|_{\infty} \leq M^* \), we obtain

\[ (2.5) \quad \|T_0 \tilde{u}\|_{C([0,T_0];\mathcal{H})} \leq \|\tilde{u}_0\|_{\mathcal{H}} + M^* T_0 \|\tilde{u}\|_{C([0,T_0];\mathcal{H})} + T_0 \|f\|_{C([0,T_0];\mathcal{H})} \]

and hence \( T_0 \) is bounded.

Contractivity:

Similarly, for any \( \tilde{u}, \tilde{u} \in C([0,T_0];\mathcal{H}) \),

\[ \|T_0 \tilde{u} - T_0 \tilde{u}\|_{C([0,T_0];\mathcal{H})} \leq M^* T_0 \|\tilde{u} - \tilde{u}\|_{C([0,T_0];\mathcal{H})} \]

By choosing \( T_0 < 1/M^* \), we have \( T_0 \) is contractive for \( t \leq T_0 \). Thus, by the contraction mapping theorem we have the existence of a unique fixed point on \( C([0,T_0];\mathcal{H}) \).

**Step 2.** Extend the above result to a solution on \( [T_0, 2T_0] \) by selecting a new initial datum as \( \tilde{u}_{T_0} = \tilde{u}(T_0) \in \mathcal{H} \). By a second contraction argument, we have a unique solution on \( C([T_0, 2T_0];\mathcal{H}) \). Repeating the process a finite number of time, we obtain the existence of a unique weak solution to (1.1), with \( \tilde{u} \in C([0,T];\mathcal{H}) \).

**Lemma 2.2.** (Regularity) (See [11]). Assume that \( \Omega \) is a bounded domain, \( \partial \Omega \) is \( C^2 \) smooth, \( a \in L^\infty(\Omega) \), \( u_0 \in H^2(\Omega) \), \( u_1 \in H_0^1(\Omega) \) and \( f \in L^2(\Omega_T) \). Then the weak solution \( u = u(a) \) of (1.1) satisfies \( u \in C(0,T;H^2(\Omega)), u_t \in C(0,T;H^1(\Omega)) \) and \( u_{tt} \in C(0,T;L^2(\Omega)) \).

**Lemma 2.3.** (A priori Estimate)(See, [2]). If \( a \in U_M, u_0 \in H^2(\Omega), u_1 \in H_0^1(\Omega), f \in L^2(\Omega_T) \), \( \partial \Omega \) is \( C^2 \), then the weak solution \( u = u(a) \)
of \((1.1)\) satisfies
\[
\sup_{0 \leq t \leq T} (\|u(t)\|_{H^1_0(\Omega)} + \|u_0(t)\|_{L^2(\Omega)} + \|u_1(t)\|_{H^{-1}(\Omega)})
\leq C(\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)}),
\]
where \(C\) is a constant which depends only on \(\Omega\) and \(T\).

Proof. There exist sequence \(\{u_{0n}\} \subset H^2(\Omega), \{u_{1n}\} \subset H^1_0(\Omega)\) and \(\{a_n\} \subset U_M\) such that
\[
\begin{align*}
\lim_{n \to \infty} u_{0n} &\to u_0 \quad \text{strongly in } H^1_0(\Omega), \\
\lim_{n \to \infty} u_{1n} &\to u_1 \quad \text{strongly in } L^2(\Omega), \\
\lim_{n \to \infty} a_n &\to a \quad \text{weakly in } L^2(\omega) \subset L^2(\Omega).
\end{align*}
\]
Denote by \(u_n\) the weak solution of \((1.1)\) corresponding to the initial datum \(u_{0n}, u_{1n}\) with control \(a_n\); then \(u_n\) satisfies the regularity of Lemma 2.2. Multiplying the PDE \((1.1)\) by \((u_n)_\tau\), denoted by \((u_n)_\tau\), and integrating over \(\Omega_t = \Omega \times (0, t]\) with \(0 \leq t \leq T\), we obtain
\[
0 = \int_{\Omega_t} \frac{1}{2} \frac{d}{d\tau} (\|u_n\|^2 + |\nabla u_n|^2) + a_n ((u_n)_\tau)^2 - f(u_n)_\tau]dxd\tau
\]
First, we have
\[
\int_{\Omega \times \{t\}} [((u_n)_\tau)^2 + |\nabla u_n|^2]dx = \int_{\Omega} [(u_{1n})^2 + |\nabla u_{0n}|^2]dx + 2 \int_{\Omega_t} [-a_n ((u_n)_\tau)^2 + f(u_n)_\tau]dxd\tau
\]
\[
0 \leq \|u_{1n}\|_{L^2(\Omega)}^2 + \|\nabla u_{0n}\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega, \tau)}^2
\]
\[
\quad + (1 + M) \|\nabla u_{1n}|_{L^2(\Omega, \tau)}^2
\]
\[
\quad \leq \|u_{1n}\|_{L^2(\Omega)}^2 + \|\nabla u_{0n}\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega, \tau)}^2
\]
\[
\quad + (1 + M) \int_{\Omega_t} [((u_{1n})^2 + |\nabla u_{1n}|^2)]dxd\tau.
\]
Using Gronwall's inequality, we obtain
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An optimal control for the wave equation

(2.7) \[ \int_{\Omega \times (t)} [((u_n)_t)^2 + |\nabla u_n|^2] dx \]

\[ \leq [1 + (1 + M)T e^{(1+M)T}] \]
\[ \times (||u_{1n}||^2_{L^2(\Omega)} + ||\nabla u_{0n}||^2_{L^2(\Omega)} + ||f||^2_{L^2(\Omega_T)}) \]

Using Poincare’s Inequality, we obtain

(2.8) \[ \int_{\Omega \times (t)} [((u_n)_t)^2 + (u_n)^2] dx \]

\[ \leq C_1 \int_{\Omega_t} [((u_n)_\tau)^2 + (u_n)^2] dx d\tau \]
\[ + C_1(||u_{3n}||^2_{L^2(\Omega)} + ||\nabla u_{0n}||^2_{L^2(\Omega)} + ||f||^2_{L^2(\Omega_T)}) \]

where \( C_1 \) is independent of \( u_n \).

Combining (2.7), (2.8) and letting \( n \to \infty \), we get at time \( t \)

(2.9) \[ \|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{L^2(\Omega)} \]
\[ \leq C_2(||u_1||^2_{L^2(\Omega)} + ||\nabla u_0||^2_{L^2(\Omega_T)} + ||f||^2_{L^2(\Omega)}) \]

where \( C_2 \) depends only on \( \Omega, M \) and \( T \).

Taking the supremum, gives

(2.10) \[ \sup_{0 \leq t \leq T} [||u(t)||^2_{L^2(\Omega)} + ||u(t)||^2_{H^1(\Omega)}] \]
\[ \leq C_2(||u_1||^2_{L^2(\Omega)} + ||u_0||^2_{H^1(\Omega)} + ||f||^2_{L^2(0,T,L^2(\Omega))}) \]

Using the state equation, we obtain

\[ \|u_{tt}\|^2_{L^2(0,T;H^{-1}(\Omega))} \leq C_3 \int_0^T (||f(t)||^2_{L^2(\Omega)} + ||u_{tt}\|^2_{H^1(\Omega)}) dt \]
\[ \leq C_4 T(||u_1||^2_{L^2(\Omega)} + ||u_0||^2_{H^1(\Omega)} + ||f||^2_{L^2(0,T,L^2(\Omega))}) \]

where \( C_3, C_4 \) are constants independent of \( a \).

We need to get higher order regularity of the weak solution of problem (1.1) in order to solve optimal problem. So, using Galerkin approximation method we obtain the higher order regularity of the weak solution.
Lemma 2.4. (Improved Regularity) (See, [2]). Assume that \( u \) is the weak solution of problem (1.1), if \( a = a(x) \in U_M, u_0 \in H^2(\Omega), u_1 \in H^1_0(\Omega), f \in L^2(\Omega_T), f_t \in L^2(\Omega_T) \), then \( u_t \in L^\infty(0,T; H^1(\Omega)), u_{tt} \in L^\infty(0,T; L^2(\Omega)), u_{ttt} \in L^2(0,T; H^{-1}(\Omega)) \) and we have the estimate:

\[
\text{ess sup}_{0 \leq t \leq T} \left( \| u(t) \|_{H^1(\Omega)} + \| u_t(t) \|_{H^1_0(\Omega)} + \| u_{tt}(t) \|_{L^2(\Omega)} \right) + \| u_{ttt}(t) \|_{L^2(\Omega_T)} + \| u_{ttt}(t) \|_{H^{-1}(\Omega)} \leq C(\| f \|_{H^1(\Omega_T; L^2(\Omega))} + \| u_0 \|_{H^1(\Omega)} + \| u_1 \|_{H^1_0(\Omega)})
\]

with \( C \) depending only on \( \Omega, T, \varepsilon_0, \) and \( M \).

Proof: Construct a sequence of approximations by selecting smooth functions \( \sigma_k = \sigma_k(x), (k = 1, 2, \ldots) \) such that

\[
\{ \sigma_k \}_{k=1}^\infty \text{ is a basis of } H^1_0(\Omega),
\]

\[
\{ \sigma_k \}_{k=1}^\infty \text{ is an orthonormal basis of } L^2(\Omega),
\]

and \( \{ \sigma_k \} \) are eigenfunctions for \(- \Delta\) on \( H^1_0(\Omega)\) corresponding to the eigenvalue \( \lambda_k \). For integer \( m \), write

\[
(2.12) \quad u_m(t) = \sum_{k=1}^m d_k^m(t) \sigma_k(x)
\]

where \( d_k^m(t) \) satisfy

\[
(2.13) \quad d_k^m(0) = \langle u_0(x), \sigma_k \rangle_{L^2}, d_k^m(0) = \langle u_1(x), \sigma_k \rangle_{L^2}, k = 1, 2, \ldots, m
\]

and

\[
(2.14) \quad \langle u_{mzt}(t), \sigma_k \rangle + \langle \nabla u_m(t), \nabla \sigma_k \rangle + \langle a(x) u_{mzt}(t), \sigma_k \rangle = \langle f(t), \sigma_k \rangle.
\]

Using the orthogonality of \( \{ \sigma_k \} \) in \( L^2(\Omega) \) and substituting the sum for \( u_m \) from (2.12) into (2.14), Eq. (2.14) becomes a system of ordinary differential equation (ODE)

\[
(2.15) \quad d_{ktt}^m(t) + \sum_{l=1}^m \langle \nabla \sigma_l, \nabla \sigma_k \rangle d_{l}^m(t) + \sum_{l=1}^m \langle a(x) \sigma_l, \sigma_k \rangle d_{l}^m(t) = \langle f(t), \sigma_k \rangle.
\]
0 ≤ t ≤ T, k = 1, 2, ..., m with initial conditions (2.13). By standard existence theorem for ODE, there exists a unique absolutely continuous solution satisfying (2.13) and (2.14).

Note that each $d^m_k(t)$ has more regularity since the coefficients are time independent and $f_t \in L^2(\Omega_T)$, and we will use $d^m_{k\alpha}(t) \in L^2(0, T)$.

**Step 1.** In the proof of Lemma 2.3, we have already derived the bounds

$$
\sup_{0 ≤ t ≤ T} \left( ||u_m(t)||_{H^1(\Omega)} + ||u_{mt}(t)||_{L^2(\Omega)} + ||u_{mst}(t)||_{L^2(0,T;H^{-1}(\Omega))} \right)
\leq C( ||f||_{L^2(0,T;L^2(\Omega))} + ||u_0||_{H^1(\Omega)} + ||u_1||_{L^2(\Omega)} )
$$

Passing to limits as $m = m_i \to \infty$, we deduce

$$
\sup_{0 ≤ t ≤ T} \left( ||u(t)||_{H^1(\Omega)} + ||u_t(t)||_{L^2(\Omega)} + ||u_{tt}(t)||_{L^2(0,T;H^{-1}(\Omega))} \right)
\leq C( ||f||_{L^2(0,T;L^2(\Omega))} + ||u_0||_{H^1(\Omega)} + ||u_1||_{L^2(\Omega)} )
$$

**Step 2.** Assume now the hypotheses of given initial conditions. Fix a positive integer $m$, and next differentiate the identity (2.14) with respect to $t$. Writing $\tilde{u}_m := u_{mt}$ we obtain

$$
\langle \tilde{u}_{mtt}(t), \sigma_k \rangle + \langle \nabla \tilde{u}_{mt}(t), \nabla \sigma_k \rangle + \langle a(x) \tilde{u}_{mt}(t), \sigma_k \rangle = \langle f_t(t), \sigma_k \rangle.
$$

Multiplying (2.18) by $d^m_{k\alpha}(t)$ and adding for $k = 1, 2, \ldots, m$, we discover

$$
\langle \tilde{u}_{mst}(t), \tilde{u}_{mt}(t) \rangle + \langle \nabla \tilde{u}_m(t), \nabla \tilde{u}_{mt}(t) \rangle + \langle a(x) \tilde{u}_{mt}(t), \tilde{u}_{mt}(t) \rangle = \langle f_t(t), \tilde{u}_{mt}(t) \rangle.
$$

Rewrite the above equality as

$$
\frac{1}{2} \frac{d}{dt} \left( ||\tilde{u}_{mt}(t)||_{L^2(\Omega)}^2 + ||\nabla \tilde{u}_m(t)||_{L^2(\Omega)}^2 \right) + \langle a(x) \tilde{u}_{mt}(t), \tilde{u}_{mt}(t) \rangle = \langle f_t(t), \tilde{u}_{mt}(t) \rangle.
$$

Using $a(x) ≥ \varepsilon_0 > 0$ and integrating to time $t$, we obtain
\[ (2.21) \int_{\Omega} (\tilde{u}_{mt}(t))^2 \, dx + \int_{\Omega} |\nabla \tilde{u}_{m}(t)|^2 \, dx + \varepsilon_0 \int_0^T \int_{\Omega} (\tilde{u}_{ms}(s))^2 \, dx \, ds \]
\[ \leq \frac{1}{\varepsilon_0} \int_0^T \int_{\Omega} (f_s(x,s))^2 \, dx \, ds + \int_{\Omega} (\tilde{u}_{mt}(0))^2 \, dx + \int_{\Omega} |\nabla \tilde{u}_{m}(0)|^2 \, dx. \]

Thus we have

\[ (2.22) \quad \sup_{t \leq T} \left( \int_{\Omega} (\tilde{u}_{mt}(t))^2 \, dx + \int_{\Omega} |\nabla \tilde{u}_{m}(t)|^2 \, dx \right) \]
\[ + \varepsilon_0 \int_0^T \int_{\Omega} (\tilde{u}_{ms}(s))^2 \, dx \, ds \]
\[ \leq C(\varepsilon_0) \int_0^T \int_{\Omega} (f_s(x,t))^2 \, dx \, dt + \int_{\Omega} (\tilde{u}_{mt}(0))^2 \, dx \]
\[ + \int_{\Omega} |\nabla \tilde{u}_{m}(0)|^2 \, dx. \]

Using the properties of \( \sigma_k \), we have

\[ (2.23) \quad \int_{\Omega} (\tilde{u}_{mt}(0))^2 \, dx + \int_{\Omega} |\nabla \tilde{u}_{m}(0)|^2 \, dx \]
\[ \leq C_2 (\|u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{H^1(\Omega)}). \]

Combining (2.22)-(2.23) and including \( u_{mt} = \tilde{u}_m \), we conclude that

\[ (2.24) \quad \sup_{0 \leq t \leq T} (\|u_{mt}(t)\|^2_{L^2(\Omega)} + \|\nabla u_{mt}(t)\|^2_{L^2(\Omega)}) + \|u_{mt}\|^2_{L^2(0,T;L^2(\Omega))} \]
\[ \leq C_3 (\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2_{H^1(\Omega)}). \]

Passing to limits as \( m = m_k \to \infty \), we deduce

\[ (2.25) \quad \sup_{0 \leq t \leq T} (\|u_{mt}(t)\|^2_{L^2(\Omega)} + \|\nabla u_{mt}(t)\|^2_{L^2(\Omega)}) + \|u_{mt}\|^2_{L^2(0,T;L^2(\Omega))} \]
\[ \leq C_3 (\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2_{H^1(\Omega)}). \]

**Step 3.** Now

\[ (2.26) \quad (-\Delta u_m, \sigma_k) = (f - u_{mtt} - au_{mt}, \sigma_k) \quad (k = 1, 2, \ldots, m). \]
Recall we are taking \( \{ \sigma_k \} \) to be the complete collection of eigenfunctions for \(-\Delta\) on \( H^1_0(\Omega)\). Multiplying (2.26) by \( \lambda_k d_k^n(t) \) and summing \( k = 1, 2, \ldots, m \), we deduce

\[
(2.27) \quad (-\Delta u_m, -\Delta u_m) = (f - u_{mnt} - au_{nt}, -\Delta u_m) \quad (k = 1, 2, \ldots, m).
\]

Using \( a(x) \leq M \), we obtain

\[
(2.28) \quad \|\Delta u_m\|_{L^2(\Omega)}^2 \leq C_4(\|f\|_{L^2(\Omega)}^2 + \|u_{nt}\|_{L^2(\Omega)}^2 + \|u_{nt}\|_{L^2(\Omega)}^2).
\]

Since \( \Delta u_m = 0 \) on \( \partial \Omega \) and \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) (see, [2]), we employ the inequality

\[
(2.29) \quad \|\Delta u\|_{L^2(\Omega)}^2 \leq \|\Delta u\|_{L^2(\Omega)}^2.
\]

Combining (2.28) and (2.29), we obtain

\[
(2.30) \quad \|u_m\|_{H^2(\Omega)}^2 \leq C_6(\|f\|_{L^2(\Omega)}^2 + \|u_{nt}\|_{L^2(\Omega)}^2 + \|u_{nt}\|_{L^2(\Omega)}^2).
\]

Thus we deduce from (2.16), (2.24) and (2.30) that

\[
(2.31) \quad \sup_{0 \leq t \leq T} (\|u_m(t)\|_{H^2(\Omega)}^2 + \|u_{nt}(t)\|_{H^2(\Omega)}^2 + \|u_{nt}(t)\|_{L^2(\Omega)}^2) \leq C_6(\|f\|_{H^1(0,T;L^2(\Omega))} + \|u_0\|_{H^2(\Omega)}^2 + \|u_1\|_{H^3(\Omega)}).
\]

Here we estimated \( \|u_m(0)\|_{H^2(\Omega)} \leq C|u_0|_{H^2(\Omega)} \).

Passing to limits as \( m \rightarrow m \rightarrow \infty \), we derived the same found for \( u \).

**Step 4.** It remains to show \( u_{nt} \in L^2(0,T; H^{-1}(\Omega)) \). To do so, take \( v \in H^1_0(\Omega) \), with \( \|v\|_{H^1(\Omega)} \leq 1 \), and write \( v = v_1 + v_2 \) where \( v \in \text{span}\{\sigma_k\}_{k=1}^m \) and \( (v_2, \sigma_k) = 0 \) \( (k = 1, 2, \ldots, m) \). Since the functions \( \{\sigma_k\}_{k=1}^m \) are orthogonal in \( H^1_0(\Omega) \), \( \|v_1\|_{H^1(\Omega)} \leq \|v\|_{H^1(\Omega)} \leq 1 \).

Utilizing (2.14), we deduce for a.e. \( 0 \leq t \leq T \) that

\[
(u_{nt}, v_1) + (-\Delta u_{nt}, v_1) + (au_{nt}, v_1) = (f_t, v_1).
\]

Then for a.e. \( 0 \leq t \leq T \)

\[
\langle u_{nt}, v \rangle = \langle u_{nt}, v \rangle = \langle u_{nt}, v_1 \rangle = \langle f_t, v_1 \rangle + \langle \Delta u_{nt}, v_1 \rangle - \langle au_{nt}, v_1 \rangle.
\]

and
Combining (2.30) and (2.30) we obtain

\[(u_{mtt}, v) \leq C_7(\|f\|_{L^2(\Omega)} + \|\Delta u_{mtt}\|_{L^2(\Omega)} + \|u_{mtt}\|_{L^2(\Omega)})\]

since \(\|v\|_{H^2(\Omega)} \leq 1\).

Combining (2.30) and (2.30) we obtain

\[(u_{mtt})_{H^{-1}(\Omega)} \leq C_7(\|f\|_{L^2(\Omega)} + \|\Delta u_{mtt}\|_{L^2(\Omega)} + \|u_{mtt}\|_{L^2(\Omega)})\]

\[\leq C_7(\|f\|_{H^1(\Omega); L^2(\Omega)} + \|u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}).
\]

And also \(u_{mtt}\) is bounded in \(L^2(0, T; H^{-1}(\Omega))\). Passing to limits as \(m = m_i \to \infty\), we deduce that \(u_{mtt} \in L^2(0, T; H^{-1}(\Omega))\). Combining the estimates above yields the desired result. \(\Box\)

Now we obtain the existence of an optimal control.

**Theorem 2.1.** There exists an optimal control \(a^* \in U_M\) which minimizes the objective functional \(J(a)\) for \(a \in U_M\).

**Proof.** Let \(\{a^n\} \subset U_M\) be a minimizing sequence such that

\[(2.34) \lim_{n \to \infty} J(a^n) = \inf_{a \in U_M} J(a)\]

Denote \(u^n = u(a^n)\). By Lemma 2.4 we have

\[\begin{align*}
es \sup_{0 \leq t \leq T} (||u(t)||_{H^2(\Omega)} + ||u_t(t)||_{H^2(\Omega)} + ||u_{tt}(t)||_{L^2(\Omega)}) \\
+ ||u_{ttt}(t)||_{L^2(0, T; H^{-1}(\Omega))} \leq C(\|f\|_{H^1(\Omega); L^2(\Omega)} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^2(\Omega)})
\end{align*}\]

On a subsequence, by weak compactness there exist \(u^*\) in \(C([0, T]; H^2(\Omega))\) such that

\[(2.35) \begin{align*}
&u^n \rightharpoonup u^* \text{ weakly* in } L^\infty(0, T; H^2(\Omega)) , \\
&u^n_t \rightharpoonup u^*_t \text{ weakly* in } L^\infty(0, T; H^2_0(\Omega)), \\
&\alpha^n \to \alpha^* \text{ weakly in } L^2(\omega) \text{ and weakly in } L^2(\Omega), \\
&u^n_{tt} \rightharpoonup u^*_{tt} \text{ weakly in } L^2(0, T; L^2(\Omega)).
\end{align*}\]

Since \(H^2(\Omega), H^2_0(\Omega) \hookrightarrow L^2(\Omega)\) are compact embedding, we have \(u^n \rightharpoonup u^*\) strongly in \(L^\infty(0, T; L^2(\Omega))\) and . By the definition of weak solution,
we have
\[ < u^n_t, \phi > = -\int_{\Omega} [\nabla u^n \cdot \nabla \phi + a^n u^n \phi - f \phi] dx \]
for any \( \phi \in H_0^1(\Omega) \) and a.e. \( 0 \leq t \leq T \). Since
\[ u_t^n \to u^*_t \text{ strongly in } L^2(\Omega_T), \]
\[ a^n \to a^* \text{ weakly in } L^2(\omega) \text{ and } L^2(\Omega), \]
then
\[ u_t^n a^n \to u_t^* a^* \text{ weakly in } L^2(\omega \times [0, T]) \text{ and } L^2(\Omega \times [0, T]). \]
Passing to the limit as \( n \to \infty \) in the weak formulation of \( u^n \), we obtain
\[ < u^*_t, \phi > = -\int_{\Omega} [\nabla u^* \cdot \nabla \phi + a^* u^*_t \phi - f \phi] dx. \]
Thus \( u^* = u(a^*) \) is the solution of state Eq. (1.1) with control \( a^* \). Since
\[ J(a^*) = \frac{1}{2} \int_{\omega \times (0, T]} (u(a^*) - z_d)^2 dx dt + \frac{\beta}{2} \int_{\omega} (a^*)^2(x) dx, \]
using lower-semicontinuity of \( L^2 \) norm with respect to weak convergence, we have
\[ J(a^*) \leq \lim_{n \to \infty} \frac{1}{2} \int_{\omega \times [0, T]} (u(a^n) - z_d)^2 dx dt + \lim_{n \to \infty} \frac{\beta}{2} \int_{\omega} (a^n)^2(x) dx \]
\[ \leq \liminf_{n \to \infty} J(a^n) = \inf_{a \in \mathcal{U}} J(a). \]
Finally, we conclude that \( a^* \) is an optimal control. \( \square \)

**Remark** (i) This problem is not related to the structure shape of domain \( \Omega \); (ii) the dimension of domain \( \Omega \subset \mathbb{R}^n \) are \( n = 2, 3, 4 \) and this problem is more generalized space than Boris and Lenhart([4]); (iii) optimal solution \( a^* \) has nonlinear in \( \omega \) and linear property in \( \Omega - \omega \) and also only be include important information data in regional area \( \omega \).
3. Characterization of the Optimal Control

We now derive the optimality system by differentiating the objective functional $J(a)$ with respect to the control $a$. Since $u = u(a)$ is involved in $J(a)$, we must first prove the appropriate differentiability of the mapping
\[ a \rightarrow (u(a), u_t(a)) = \hat{u}(a). \]

**Lemma 3.1.** The mapping
\[ a \in U_M \rightarrow \hat{u}(a) = (u(a), u_t(a)) \in L^2(0,T; \tilde{\mathcal{H}}), \tilde{\mathcal{H}} = H^2(\Omega) \times H^1_0(\Omega) \]
is differentiable in the following sense:
\[ \frac{\hat{u}(a + \varepsilon l) - \hat{u}(a)}{\varepsilon} \rightarrow \hat{\psi} \]
weakly in $L^2(0,T; \tilde{\mathcal{H}})$ as $\varepsilon \to 0$, for any $a$ satisfying $a + \varepsilon l \in U_M$ for $\varepsilon$ small and $l \in U_M$. Moreover $\hat{\psi} = (\psi, \psi_t)$ in $L^2(0,T; \tilde{\mathcal{H}})$ is a weak solution of the following problem:

(3.1) \[ \begin{align*}
\psi_{tt} &= \Delta \psi - a \psi_t - lw_t, \quad \text{in } \Omega_T, \\
\psi &= 0, \quad \text{on } \partial \Omega \times [0,T], \\
\psi(0) &= \psi_t(0) = 0, \quad \text{in } x \in \Omega
\end{align*} \]

where $u = u(a)$.

**Proof.** Denote $u^\varepsilon = u(a + \varepsilon l)$ and $u = u(a)$, then $(u^\varepsilon - u)/\varepsilon$ is a weak solution of

\[ \begin{align*}
\frac{(u^\varepsilon - u)}{\varepsilon}_{tt} &= \Delta \frac{(u^\varepsilon - u)}{\varepsilon} - a\frac{(u^\varepsilon - u)}{\varepsilon}_t - lw^\varepsilon_t \quad \text{in } \Omega_T, \\
\frac{(u^\varepsilon - u)}{\varepsilon}_t &= 0 \quad \text{in } \partial \Omega \times [0,T], \\
\frac{(u^\varepsilon - u)}{\varepsilon} &= 0 \quad \text{in } t = 0, x \in \Omega, \\
\frac{(u^\varepsilon - u)}{\varepsilon}_t &= 0 \quad \text{in } t = 0, x \in \Omega.
\end{align*} \]

Using Lemma 2.4, we get
An optimal control for the wave equation

\begin{align*}
\text{ess sup}_{0 \leq t \leq T} & \left[ \| \frac{u^\varepsilon(t) - u(t)}{\varepsilon} \|_{H^1(Q)} + \| \frac{u^\varepsilon(t) - u(t)}{\varepsilon} u(t) \|_{H^1_0(Q)} \\
+ \| \frac{u^\varepsilon(t) - u(t)}{\varepsilon} u(t) \|_{L^2(Q)} \right] \\
& \leq C \| u_0 \|_{L^2(\Omega)} + CT \| u_0 \|_{L^2(\Omega)} \sup_{0 \leq t \leq T} \| u^\varepsilon(t) \|_{L^2(\Omega)} \\
& \leq C_T \| u_0 \|_{L^\infty} \sup_{0 \leq t \leq T} \| u^\varepsilon(t) \|_{H^1_0(\Omega)} \\
& \leq C_T \| u_0 \|_{L^\infty} \sup_{0 \leq t \leq T} \| u^\varepsilon(t) \|_{H^1_0(\Omega)} \\
& \leq C_T \| u_0 \|_{L^\infty} \sup_{0 \leq t \leq T} \| u^\varepsilon(t) \|_{H^1_0(\Omega)} \\
& \leq C_T \| u_0 \|_{L^\infty}
\end{align*}

where \( C_T \) depend on the \( L^\infty \) bound on \( l \), but it is independent of \( \varepsilon \).

Hence on a subsequence, by weak compactness, we have

\begin{align*}
\frac{u^\varepsilon(t) - u(t)}{\varepsilon} & \to \psi_t \text{ weakly}^* \text{ in } L^\infty(0, T; H^2(\Omega)), \\
\frac{u^\varepsilon(t) - u(t)}{\varepsilon} u(t) & \to \psi u \text{ weakly}^* \text{ in } L^\infty(0, T; H^1_0(\Omega)) \text{ and strongly in } L^2(\Omega),
\end{align*}

and

\begin{align*}
\frac{u^\varepsilon(t) - u(t)}{\varepsilon} u(t) & \to \psi u_t \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)).
\end{align*}

By the definition of weak solution, we have

\begin{align*}
< \frac{u^\varepsilon(t) - u(t)}{\varepsilon}, \phi > & + \int_\Omega \nabla \left( \frac{u^\varepsilon(t) - u(t)}{\varepsilon} \right) \cdot \nabla \phi \, dx + \int_\Omega a(x) \frac{u^\varepsilon(t) - u(t)}{\varepsilon} \phi \, dx \\
& = - \int_\Omega l u^\varepsilon \phi \, dx
\end{align*}

for any \( \phi \in H^1_0(\Omega) \), and a.e. \( 0 \leq t \leq T \). Letting \( \varepsilon \to 0 \) and using \( u^\varepsilon \to u \) strongly in \( L^2(\Omega) \), we obtain that \( \psi \) is the weak solution of problem (3.1). \( \square \)

Now we are ready to derive the necessary conditions that characterize an optimal control.

Theorem 3.1. Given an optimal control \( a \) and corresponding \( \tilde{u} = \tilde{u}(a) = (u, u_t) \), there exists a weak solution \( \tilde{p} = (p, p_t) \) in \( \mathcal{H} \) to the
adjoint problem:

\[
\begin{align*}
    p_t &= \Delta p + ap_t + \chi_\omega (u - z_d), \quad \text{in } \Omega \times [0,T), \\
    p &= 0 \quad \partial \Omega \times [0,T] \\
    p(T) &= p_t(T) = 0 \quad x \in \Omega.
\end{align*}
\]

Furthermore, \( a \) satisfies

\[
(3.3) \quad a = \begin{cases} 
    \max \left( \varepsilon_0, \min \left( \frac{I^\omega \rho(t) \, dt}{\beta}, M \right) \right), & \text{in } \omega \\
    0, & \text{in } \Omega - \omega 
\end{cases}
\]

**Proof.** Let \( a \in U^*_M \) be an optimal control and \( \tilde{u} = \tilde{u}(a) \) be the corresponding optimal solution. Let \( a + \varepsilon l \in U^*_M \) for \( \varepsilon > 0 \) and \( \tilde{u}^\varepsilon = \tilde{u}(a + \varepsilon l) \) be the corresponding weak solution of Eq. (1.1). We compute the directional derivative of the objective functional \( J(a) \) with respect to \( a \) in the direction of \( l \). Since \( J \) is supposed to attain its minimum for \( a \), we have

\[
0 \leq \lim_{\varepsilon \to 0^+} \frac{J(a + \varepsilon l) - J(a)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \left( \int_0^T \int_\omega \left[ (u^\varepsilon - z_d)^2 - (u - z_d)^2 \right] dx \, dt + \frac{\beta}{2\varepsilon} \int_\omega [(a + \varepsilon l)^2 - a^2] dx \right)
\]

\[
= \lim_{\varepsilon \to 0^+} \left( \int_0^T \int_\omega \frac{(u^\varepsilon - u)(u^\varepsilon + u - 2z_d)}{2\varepsilon} dx \, dt + \frac{\beta}{2} \int_\omega 2al + (\varepsilon l)^2 dx \right)
\]

\[
= \int_\omega 3.4 \int_\omega \psi(u - z_d) dx \, dt + \int_\omega \beta al \, dx,
\]

where \( \psi \) is defined in Lemma 3.1. Taking \( u - z_d \) as source like \( f \) and using the similar argument in Lemma 2.1, we obtain the existence and uniqueness of solution of problem (3.2). Assume that \( p \) is the solution of problem (3.2) i.e., \( p \) satisfies

\[
(3.5) \quad < p_n(t), \phi > + \int_\Omega \nabla p(t) \cdot \nabla \phi \, dx - \int_\Omega ap_t(t) \phi \, dx
\]

\[
= \int_\Omega \chi_\omega (u - z_d) \phi \, dx
\]
for any $\phi \in H^1_0(\Omega)$, and a.e. $0 \leq t \leq T$. Then

$$0 \leq \int_0^T \int_\omega \psi(u - z_d) dx \, dt + \int_\Omega \beta a \chi_\omega dx$$

by integration by parts twice in time and using $\psi(0) = \psi_T(0) = 0$ and $p(T) = P_t(T) = 0$. Then from Eq. (3.1) for $\psi$, we obtain

$$0 \leq \int_\Omega l(x) \left( \int_0^T -p(x, t) u_t(x, t) dt + \beta a(x) \chi_\omega(x) \right) dx.$$  

Note that $l = l(x)$ is an arbitrary function with $a + \varepsilon l \in U_M$ for all small $\varepsilon$. By a standard control argument involving the sign of the variation $l$ depending on the size of $a$, we obtain the desired characterization of $a$, namely,

$$\begin{cases} 
\max \left( \varepsilon_0, \min \left( \frac{L^T p_{\Omega d t}}{\beta}, M \right) \right), & \text{in } \omega \\
0, & \text{in } \Omega - \omega 
\end{cases}$$  

(3.6)

Thus proof is complete. \qed

REFERENCES


Department of Mathematics  
University of Ulsan  
Ulsan 680-749  
Korea  
*E-mail*: yonghann@hotmail.com