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NOTES ON GRADING MONOIDS

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ABSTRACT. Throughout this paper, a semigroup S will denote a torsion free grading monoid, and it is a non-zero semigroup with 0. The operation is written additively. The aim of this paper is to study semigroup version of an integral domain ([1],[3],[4] and [5]).

1. Introduction

Let S be an additive commutative semigroup with identity(denoted by 0), that is a monoid. A monoid S is said to be *cancellative* if $x + y = x + z$ with x, y and $z \in S$ implies $y = z$ and S is said to be *torsion-free* if $nx = ny$ with $x, y \in S$ and $n \in N$ implies $x = y$ where N denotes the set of all positive integers. A cancellative monoid is called a *grading monoid* [10,p.112]. In this paper, a semigroup S will denote a torsion free grading monoid,and it is a non-zero semigroup with 0. The operation is written additively.

A nonempty subset B of a semigroup S is called an *additive system* if it satiesfies the following condition $b_1, b_2 \in B \Rightarrow b_1 + b_2 \in B$. For an additive system B, the *quotient semigroup* S_B is defined as follows: $\{s - b \mid s \in S, b \in B\}$. Especially, if $B = S$, then the quotient semigroup $S_S = \{s_1 - s_2 \mid s_1, s_2 \in S\}$ is called the *quotient group* of S, and is denoted by $q(s) = G$. T is called an *oversemigroup* of S if T is a subsemigroup of G containing S.

An *ideal* of S is a nonempty suset I of S such that $s + I = \{s + i | i \in I\} \subseteq I$ for each $s \in S$. For an ideal I, J of S, set $I^{-1} = \{x \in G | x + I \subseteq$

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$S\}$. Let J be an ideal of S. Set $rad(J) = \{s \in S | ns \in J$ for some $n \in Z_0\}$. I is called a *radical ideal* of S if $I = rad(I)$. For each $x \in S$, set $(x) = x + S$. An ideal of S is principal if $I = (x)$. An ideal P of S is *prime* if $x + y \in P$ implies $x \in P$ or $y \in P$ for $x, y \in S$. An element s of S is called a unit if $s + u = 0$ for some $u \in S$. Also set $M = \{m \in S |\ m$ is a non-unit element of $S\}$. Then M is the unique maximal ideal of S. A semigroup S is called *valuation semigroup* if either $\alpha \in S$ or $-\alpha \in S$ for each $\alpha \in G$. Throuhout this paper, we may refer to [6],[7],[8] and [9].

2. Results

A semigroup S is called *seminormal semigroup* if for each $x \in G$ such that there is a positive integer n with $mx \in S$ for all $m \geq n$ then $x \in S$, or equvalently, if $2\alpha, 3\alpha \in S$ for $\alpha \in G$ then $\alpha \in G$(cf.[3],[6],[8] and [9]).

THEOREM 2.1. *Let S be a seminormal semigroup with quotient group G and I be an ideal of S. Then $(rad(I) : rad(I))$ is a seminormal and $(rad(I) : rad(I)) = \{x \in G | nx \in (S : I)$ for all $n \geq 1\}$.*

Proof. Clearly $(rad(I) : rad(I))$ is a subsemigroup of $(S : I)$. We first show that $(rad(I) : rad(I)) = \{x \in G | nx \in (S : I)$ for all $n \geq 1\}$. Let $x \in G$ such that $mx \in (S : I)$ for all $m \geq 1$ and $a \in rad(I)$. Then $na \in I$ for some positive integer n. Hence, for all $m \geq n$, $m(x + a) \in S$. By seminormality of S we have $x + a \in S$. Also $(n+1)(x+a) = (n+1)x+na+a \in (S : I)+I+rad(I) \subseteq rad(I)$. Thus $x+a \in rad(I)$. Therefore $x \in (rad(I) : rad(I))$. The converse incusion is clear. Finally,we will show that $(rad(I) : rad(I))$ is seminormal. Let $nx \in (rad(I) : rad(I))$ for all $n \geq 1$. Since $nx \in (R : I)$ for all $n \geq 1$, we have $x \in (rad(I) : rad(I))$. Therefore $(rad(I) : rad(I))$ is seminormal semigroup.

COROLLARY 2.2. *If S is a seminormal semigroup and I is a radical ideal, then the semigroup $(I : I)$ is seminormal semigroup.*

THEOREM 2.3. *Let S be a seminormal semigroup with quotient group G and I be a prime ideal of S. Then P^{-1} is a subsemigroup of G if and only if $P^{-1} = (P : P)$.*

Proof. Since one direction is trivial, we assume P^{-1} is a subsemigroup of G and $P^{-1} \neq (P : P)$. Let $J = (S : P^{-1})$. We claim J = P. Since $P + P^{-1} \subseteq S$, we have $P \subseteq J$. Let $a \in J$, then $a + P^{-1} \subseteq S \subseteq (P : P)$ and so $(a + P^{-1}) + P = a + (P^{-1} + P) \subseteq P$. since P is prime and $P + P^{-1} \nsubseteq$, $a \in P$. Whence, $J = P$. This is contradictsthe fact that $P^{-1} \neq (P : P)$.

THEOREM 2.4. *Let S be a seminormal semigroup with quotient group G and I be an ideal of S for which I^{-1} is a semigroup. Then*

(1) $rad(I)^{-1} = (rad(I) : rad(I))$;

(2) $I^{-1} = (rad(I) : I) = (J : I)$ *for each prime* $I \subseteq J$.

Proof. (1) since $rad(I) \subseteq S$, $(rad(I) : rad(I)) \subseteq rad(I)^{-1}$. To prove $rad(I)^{-1} \subseteq (rad(I) : rad(I))$, let $x \in (rad(I))^{-1}$ and $a \in rad(I)$.. Then $na \in I$ for some positive integer n. Since $(rad(I))^{-1} \subseteq I^{-1}$ and I^{-1} is a semigroup, we have $2nx \in I^{-1}$. Hence $2nx + na \in I^{-1} + I \subseteq S$, whence $2n(x + a) = (2nx + na) + na \in S + I$. Since $x + a \in S$, this implies that $x + a \in rad(I)$. Therefore $rad(I)^{-1} = (rad(I)rad(I))$.

(3) It is enough to establish the inclusion $I^{-1} = (J : I)$. for each prime $I \subseteq J$ Let $x \in I^{-1}$. Since I^{-1} is a semigroup,we have $2x \in I^{-1}$, it follows that $2x + I \subseteq S$ and $2(x + I) = (2x + I) + I \subseteq I \subseteq J$. Since $x + I \subseteq S$, we have $x + I \subseteq J$. Thus $I + I^{-1} \subseteq J$, $I^{-1} \subseteq (J : I) \subseteq (S : I) = I^{-1}$, we have $I^{-1} = (J : I)$. Since this is true for each J, we have $I + I^{-1} \subseteq rad(I)$. Therefore $I^{-1} = (rad(I) : rad(I))$.

A prime ideal P of S is called *strongly prime* if $x, y \in G$ and $x + y \in P$ implies that $x \in P$ or $y \in P$. S is called *pseudo-valuation semigroup* if every prime ideal of S is strongly prime[3].

The following Lemma is useful restatement of definition of strongly prime ideal in semigroup S.

LEMMA 2.5. *Let P be a prime ideal of a semigroup with quotient group G. Then P is strongly prime ideal if and only if $-x + P \subseteq P$ for each $x \in G \backslash S$.*

Proof. Suppose that I is strongly prime. Letf $x \in G \backslash S$ and $p \in P$ Since $p = (p - x) + x \in P$ and P is strongly prime ideal, we have $(p - x) \in P$ or $x \in P$. Since $x \notin S$ we must have $p - x \in P$. Thus

$-x + P \subseteq P$. To prove opposite implication, assume $-x + P \subseteq P$ whenever $x \in G \backslash S$, and let $a + b \in P$. If $a, b \in S$ there is nothing to prove. Hence we may assume $a \notin S$ so that $-a + P \subseteq P$ and $b = -a + a + b \in P$. T

THEOREM 2.6. *Let P be an ideal in semigroup S with quotient group G. Then following statements are equivalent.*

(1) *P is strongly prime;*
(2) *$G \backslash P$ additive system;*
(3) *P is prime and is comparable to each (principal) fractional ideal of S.;*
(4) *$P : P$ is valuation semigroup with maximal ideal P;*
(5) *P is a prime ideal in some valuation oversemigroup of S.*

Proof. Clearly (1) and (2) are equivalent. (1) $\Rightarrow$ (3)Suppose that P is strongly prime ideal. Let $x \in G \backslash P$. Then $x + (-x + P) \subseteq P$. Since P is strongly prime, $-x + P \subseteq P$ and hence $P \subseteq x + P \subseteq x + S$. (3) $\Rightarrow$ (2) Let $x, y \in S$. Suppose that $x + y \in P$. Now $x \in S$ implies $P \subsetneqq X + S$, So $-x + P \subsetneqq S$. Then $y = -x + (x + y) \in -x + P \subseteq S$. Similarly, $x \in S$. But then we get contradiction that either $x \in P$ or $y \in P$ since P is prime. (1) $\Rightarrow$ (4). Suppose that $x \in G \backslash P$. From the proof of (1) $\Rightarrow$ (3), we see that $-x + P \subset P$, and hence $-x \in (P : P)$. From this it easily follows that $P : P$ is a valuation semigroup with P as its maximal ideal. Finally, the implications (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1) are obvious.

In the following Theorem we characterize pseudo-valuation semigroup with the maximal ideals

THEOREM 2.7. . *Let (S, M) be a semigroup. The following statement are equivalent:*

(1) *S is pseudo-valuation semigroup*
(2) *For each pair I, J of ideals of S, either $I \subseteq J$ or $M + J \subseteq M + I$.;*
(3) *For each pair I, J of ideals of S, either $I \subseteq J$ or $M + J \subseteq I$.;*
(4) *M is strongly prime.*

Proof. (1)$\Rightarrow$ (2) Assume $I \nsubseteq J$. Let $a \in I \backslash J$. For each $b \in J$ we have $a - b \notin S$, so that $-(a - b) + M \subseteq M$ and $M + b \subseteq M + a \subseteq M + I$. It follows that $M + J \subseteq M + I$.

(2) $\Rightarrow$ (3) straightforward.
(3) $\Rightarrow$ (4) Let $a, b \in S$ with $a - b \notin S$. By Lemma 2.5 ,this is enough to show that $-(a-b) + M \subsetneq M$. Since $a - b \notin S$ we have $(a) \nsubseteq (b)$ whence $M + b \subseteq (a)$ and $-(a-b) + M \subseteq S$. If $-(a-b) + M = S$ then $M = S + (a-b)$ and $a - b \notin S$, This is a contradiction. Hence $-(a-b) + M \subsetneq M$. Therefore M is stronly prime ideal
(4) $\Rightarrow$ (1) Let $x \in G$, $x \notin S$, and let P be a prime ideal. Again ,by Lemma 2.5, it is enough to show that $-x + P \subseteq P$. Let $p \in P$. Since $P \subseteq M$, we have $-x + p \in M$. Hence $-x + p - x \in M$, whence $2(-x+p) = (-x+p) + (-x+p) \in P$. Since P is prime and $-x+p \in S$, we therefore have $-x + p \in P$.

REFERENCES

[1] D.D.Anderson and David F. Anderson, Multiplicatively closed subsets of Fields, Houston Journal of Mathematics 13(1) (1987), 429-439.
[2] R.Gilmer, Multiplicative Ideal Theory, Marcel Dekker,, 1988.
[3] E.G. Hedstrom an E.G. Houston, pseudo-valuation domain, Pacific Journal of Mathematics 75(1) (1978), 137-147.
[4] Evan G. Houston, Salah-Eddine Kabbaj,Tomas G. Lucas and Abdeslam Mimouni, emph When is the dual of an ideal a ring, Journal of Algebra 225 (2000), 429-450.
[5] James A. Huckaba and Ira J. Papick, When the dual of an ideal is a ring, manuscripta math. 37 (1982), 67-85.
[6] Lee,D.S. and Park,C.H.,Some remarks on semigroups, Far East J.Math.Sci. ,11(3), (2003), 269-275.
[7] Matsuda R.and Kanemisu M.,Primary ideals and valuation ideals in semigroups, Southeast Asian Bulletin of Mathematics 26(2002), 433-437.
[8] Matsuda R.and Kanemisu M.,On seminormal semigroups, Arc.Math.,69 (1997),279-285.
[9] Matsuda R., On Seminormal semigroups and Pesudo-valuation Semigroups, Jp Jounal of Algebra, Number Theory and Applications 1(1) (2001, 25-39.
[10] D.G.Northcott, Lessons on Rings, Modules and Multiplicities, Cambridge Univ. Press,(1968).

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