GENERALIZED ANTI FUZZY SUBGROUPS

YOUNG BAE JUN AND SEOK ZUN SONG

ABSTRACT. Using the notion of anti fuzzy points and its besideness to and non-quasi-coincidence with a fuzzy set, new concepts of an anti fuzzy subgroup are introduced and their inter-relations are investigated.

1. Introduction

The concept of fuzzy sets was first initiated by Zadeh [3]. Since then it has become a vigorous area of research in engineering, medical science, social science, physics, statistics, graph theory, etc. Biswas [2] gave the idea of anti fuzzy subgroups, and obtained some results. In this paper, we introduce new concepts of an anti fuzzy subgroup by using the notion of anti fuzzy points and its besideness to and non-quasi-coincidence with a fuzzy set, and investigate their inter-relations

2. Anti fuzzy subgroups

Given a fuzzy set $\Phi$ in a set $G$ and for every $\alpha \in [0, 1]$, the subsets $C(\Phi; \alpha) := \{x \in G \mid \Phi(x) \leq \alpha\}$ and $O(\Phi; \alpha) := \{x \in G \mid \Phi(x) < \alpha\}$ are called the closed $\alpha$-cut and the open $\alpha$-cut of $\Phi$, respectively.

PROPOSITION 2.1. Let $\Phi$ and $\Psi$ be fuzzy sets in a set $G$. Then

(i) $C(\Phi; 1) = G$.
(ii) $(\forall \alpha, \beta \in [0, 1]) (\alpha \leq \beta \Rightarrow C(\Phi; \alpha) \subseteq C(\Phi; \beta))$.

Received July 10, 2006.
2000 Mathematics Subject Classification: 06D35, 08A72.
Key words and phrases: Anti fuzzy point, besideness to, non-quasi-coincidence, $(\Omega, \cup)$-anti fuzzy subgroup.
(iii) \( \Phi \subseteq \Psi \Rightarrow (\forall \alpha \in [0, 1]) C(\Psi; \alpha) \subseteq C(\Phi; \alpha) \).
(iv) \( (\forall \alpha, \beta \in [0, 1]) (C(\Phi; \alpha) = \cap_{\alpha \leq \beta} O(\Phi; \beta)) \).
(v) \( (\forall \alpha \in [0, 1]) (C(\Phi \cup \Psi; \alpha) = C(\Phi; \alpha) \cap C(\Psi; \alpha)) \).
(vi) \( (\forall \alpha \in [0, 1]) (C(\Phi \cap \Psi; \alpha) = C(\Phi; \alpha) \cup C(\Psi; \alpha)) \).

Proof. Straightforward. \( \square \)

In what follows, let \( G \) denote a group with \( e \) as the identity element unless otherwise specified.

**Definition 2.2.** [2] A fuzzy set \( \Phi \) in \( G \) is called an anti fuzzy subgroup of \( G \) if it satisfies the following assertions:

1. \( (\forall x, y \in G) (\Phi(xy) \leq \max\{\Phi(x), \Phi(y)\}) \)
2. \( (\forall x \in G) (\Phi(x) = \Phi(x^{-1})) \)

If \( \Phi \) is an anti fuzzy subgroup of \( G \), then \( \Phi(e) \leq \Phi(x) \) for all \( x \in G \) (see [2]).

**Proposition 2.3.** [2] Let \( \Phi \) be a fuzzy set in \( G \). Then \( \Phi \) is an anti fuzzy subgroup of \( G \) if and only if each closed \( \alpha \)-cut \( C(\Phi; \alpha) \) of \( \Phi \) is a subgroup of \( G \).

The \( C(\Phi; \alpha) \)'s are called level subgroups, which can be empty and form a chain in the subgroup lattice \( (S, \subseteq) \) of \( G \). By Hausdorff's axiom (or Zorn's lemma) there is a maximal chain \( C \) in \( S \).

The order of the set of membership values of a fuzzy set \( \Phi \) is called the order of \( \Phi \) and is denoted by \( |\Phi| \).

**Theorem 2.4.** Let \( G \) be with \( |G| \geq 2 \). Let \( \Phi \) be a fuzzy set in \( G \) defined by

\[
\Phi(x) := \begin{cases} 
\alpha_1 & \text{if } x = e, \\
\alpha_2 & \text{otherwise}, 
\end{cases}
\]

for any \( x \in G \) and \( 0 \leq \alpha_1 < \alpha_2 \leq 1 \). Then \( \Phi \) is an anti fuzzy subgroup of \( G \) of order 2.

Proof. Let \( x, y \in G \). Note that \( x \neq e \) implies \( x^{-1} \neq e \), and \( x = e \) implies \( x^{-1} = e \). Hence \( \Phi(x) = \Phi(x^{-1}) \). Assume that \( y \neq x^{-1} \). If \( x \neq e \) or \( y \neq e \), then \( xy \neq e \). Thus \( \Phi(xy) = \alpha_2 = \max\{\Phi(x), \Phi(y)\} \). If \( x = e \) and
y = e, then \( xy = e \) and so \( \Phi(xy) = \alpha_1 = \max\{\Phi(x), \Phi(y)\} \). Now suppose \( y = x^{-1} \). Then \( xy = e \) and hence \( \Phi(xy) = \alpha_1 \leq \max\{\Phi(x), \Phi(y)\} \).

Therefore \( \Phi \) is an anti fuzzy subgroup of \( G \) of order 2.

**Theorem 2.5.** Let \( G \) be a finite cyclic group of prime order. Then an anti fuzzy subgroup \( \Phi \) of \( G \) with \( |\Phi| = 2 \) must be of the form (3).

*Proof.* If \( |G| = 2 \), say \( G = \{e, x\} \), then

\[
\Phi(e) = \Phi(xx) \leq \max\{\Phi(x), \Phi(x)\} = \Phi(x)
\]

which contradicts (1).

Case 1. \( \Phi(e) = \alpha_2 \) and \( \Phi(x) = \alpha_1 \) for some \( x \in G \) where \( 0 \leq \alpha_1 < \alpha_2 \leq 1 \). Since \( \Phi \) is an anti fuzzy subgroup of \( G \), it follows from (2) that \( \alpha_1 = \Phi(x) = \Phi(x^{-1}) \) so that

\[
\Phi(x^{-1}) = \Phi(e) = \alpha_2 > \alpha_1 = \max\{\Phi(x), \Phi(x^{-1})\}
\]

It follows that \( \Phi(xz) = \Phi(e) \) so that \( xz \in H \). Hence \( H \) is a subgroup of \( G \) and it is proper since \( y \notin H \). This contradicts our assumption, and the proof is complete.

By a *proper anti fuzzy subgroup* we mean an anti fuzzy subgroup which is neither of order 1 nor of order 2 with the form (3).

**Theorem 2.6.** A group \( G \) with no proper subgroups cannot have proper anti fuzzy subgroups.

*Proof.* Suppose that \( \Phi \) is an anti fuzzy subgroup of \( G \) which has the form given in either Case 1 or Case 2 in the proof of Theorem 2.5. In Case 1, the closed \( \alpha \)-cut \( C(\Phi; \alpha_1) \) is a subgroup of \( G \) by Proposition 2.3, but \( e \notin C(\Phi; \alpha_1) \) since \( \alpha_1 < \alpha_2 \). This is a contradiction. In Case 2, the closed \( \alpha \)-cut \( C(\Phi; \alpha_1) \) is a subgroup of \( G \) by Proposition
2.3, and it is proper by having at least two elements $x$ and $e$. This
contradicts our assumption. Now let $\Phi$ be an anti fuzzy subgroup
of $G$ with $|\Phi| \geq 3$, and of course with $|G| \geq 3$. Then there
must exist three different elements, say $x, y, z$, with different membership
values, say $\Phi(x) = \alpha_1 < \Phi(y) = \alpha_2 < \Phi(z) = \alpha_3$. The closed $\alpha$-
cuts $C(\Phi; \alpha_1)$, $C(\Phi; \alpha_2)$, and $C(\Phi; \alpha_3)$ are different subgroups of $G$
and $C(\Phi; \alpha_1) \subset C(\Phi; \alpha_2) \subset C(\Phi; \alpha_3)$. At least one of them
is proper subgroup of $G$. This completes the proof.

**Theorem 2.7.** Let $\Phi$ be an anti fuzzy subgroup of $G$. Then a
maximal subgroup chain $C = \{H_i \mid i \in \Lambda\}$ of $G$, where $\Lambda$ is any index
set, fulfills:

(i) $\forall y \in H_{i+1} \setminus H_i \ (\exists \alpha_i \in [0, 1]) \ (\Phi(y) = \alpha_i)$

(ii) $\forall x \in H_i \setminus H_{i-1} \ (\forall y \in H_{i+1} \setminus H_i) \ (\Phi(x) \leq \Phi(y))$.

**Proof.** Consider a maximal subgroup chain $C$ of $G$ containing all
different closed $\alpha$-cuts $C(\Phi; \alpha_i)$ of $\Phi$. Such chain always exists since
every chain can be extended to a maximal one.

(i) If $|H_{i+1} \setminus H_i| = 1$ for some $i$, then the statement is trivial. In
the case $|H_{i+1} \setminus H_i| \geq 2$, we suppose $\Phi(y_1) \neq \Phi(y_2)$, say $\Phi(y_1) = \alpha_1 < \Phi(y_2) = \alpha_2$ for some $y_1, y_2 \in H_{i+1} \setminus H_i$. Take the closed $\alpha$-
cuts $C(\Phi; \alpha_1)$ and $C(\Phi; \alpha_2)$. By Proposition 2.3, they form different
nonempty level subgroups of $G$ with $H_i \subset C(\Phi; \alpha_1) \subset C(\Phi; \alpha_2) \subset H_{i+1}$, which contradicts the maximality of the given chain.

(ii) By (i), $\Phi(y) = \alpha_i$ for all $y \in H_{i+1} \setminus H_i$, and $\Phi(x) = \alpha_{i-1}$ for all
$x \in H_i \setminus H_{i-1}$ with $0 \leq \alpha_i \leq \alpha_{i-1} \leq 1$. Suppose $\alpha_i < \alpha_{i-1}$. For the
corresponding level subgroups we get

$$H_i = C(\Phi; \alpha_i) \subset C(\Phi; \alpha_{i-1}) = H_{i-1},$$

which is a contradiction.

3. Redefined anti fuzzy subgroups

**Definition 3.1.** A fuzzy set $\Phi$ in $G$ of the form

$$\Phi(y) = \begin{cases} \alpha \in [0, 1) & \text{if } y = x, \\ 1 & \text{otherwise,} \end{cases}$$
Generalized anti fuzzy subgroups

is called an anti fuzzy point with support $x$ and value $\alpha$ and is denoted by $x^\alpha$.

**Definition 3.2.** Let $\Phi$ be a fuzzy set in a group $G$. An anti fuzzy point $x^\alpha$ is said to beside to $\Phi$, written $x^\alpha \prec \Phi$, if $\Phi(x) \leq \alpha$. An anti fuzzy point $x^\alpha$ is said to non-quasi-coincident with $\Phi$, written $x^\alpha \not\equiv \Phi$, if $\Phi(x) + \alpha < 1$. If $x^\alpha \prec \Phi$ and (resp. or) $x^\alpha \equiv \Phi$, then we write $x^\alpha \leq \bigvee^\alpha \Phi$ (resp. $x^\alpha \equiv \bigvee^\alpha \Phi$).

**Lemma 3.3.** For any fuzzy set $\Phi$ in $G$, the condition (1) is equivalent to the condition:

$$(\forall x, y \in G) (\forall \alpha, \beta \in [0, 1]) (x^\alpha \prec \Phi, y^\beta \prec \Phi \Rightarrow (xy)^{\max\{\alpha, \beta\}} \prec \Phi).$$

**Proof.** Assume that (1) is valid and let $x, y \in G$ and $\alpha, \beta \in [0, 1)$ be such that $x^\alpha \prec \Phi$ and $y^\beta \prec \Phi$. Then $\Phi(x) \leq \alpha$ and $\Phi(y) \leq \beta$. It follows from (1) that

$$\Phi(xy) \leq \max\{\Phi(x), \Phi(y)\} \leq \max\{\alpha, \beta\}$$

so that $(xy)^{\max\{\alpha, \beta\}} \prec \Phi$. Now suppose that (4) is true. Note that $x^{\Phi(x)} \prec \Phi$ and $y^{\Phi(y)} \prec \Phi$ for all $x, y \in G$. Using (4), we have $(xy)^{\max\{\Phi(x), \Phi(y)\}} \prec \Phi$ and so $\Phi(xy) \leq \max\{\Phi(x), \Phi(y)\}$. This completes the proof. \qed

In what follows $\Omega$ and $\hat{\mathcal{O}}$ denote any one of $\prec, \not\equiv, \prec \equiv \bigvee^\alpha \equiv \Phi$, unless otherwise specified.

**Definition 3.4.** A fuzzy set $\Phi$ in $G$ is called an $(\Omega, \hat{\mathcal{O}})$-anti fuzzy subgroup of $G$, where $\Omega \neq \prec \equiv \bigvee^\alpha \equiv \Phi$, if it satisfies:

(b1) $(\forall x, y \in G) (\forall \alpha, \beta \in [0, 1)) (x^\alpha \Omega \Phi, y^\beta \Omega \Phi \Rightarrow (xy)^{\max\{\alpha, \beta\}} \Omega \Phi)$

(b2) $(\forall x \in G) \Phi(x) = \Phi(x^{-1}).$

In Definition 3.4, the case $\Omega = \prec \equiv \bigvee^\alpha \equiv \Phi$ is omitted because there exists a fuzzy set $\Phi$ in $G$ such that $\{x^\alpha \mid x^\alpha \equiv \bigvee^\alpha \equiv \Phi\}$ is empty. In fact, if $\Phi(x) \geq 0.5$ for all $x \in G$, then $\Phi$ is such a fuzzy set.
**Example 3.5.** (I) Consider the Klein's 4-group \( G = \{ e, a, b, c \} \) with the following multiplication table.

\[
\begin{array}{cccc}
  e & a & b & c \\
  e & e & a & b \\
  a & a & c & b \\
  b & b & c & e \\
  c & c & b & a \\
\end{array}
\]

Let \( \Phi : G \to [0,1] \) be defined by \( \Phi(e) = 0.4, \Phi(a) = 0.3 \) and \( \Phi(b) = \Phi(c) = 0.6 \). Then \( \Phi \) is a \((\prec, \prec \vee \rangle)-\text{anti fuzzy subgroup of } G \). We note that \( \Phi \) is not an \((\Omega, \Upsilon)-\text{anti fuzzy subgroup of } G \) for every \((\Omega, \Upsilon) \in \{(\prec, \prec), (\prec \vee \rangle, (\prec \vee \rangle, \prec \vee \rangle)\} \) since

(i) \( a^{0.3} \prec \Phi \) and \( a^{0.33} \prec \Phi \), but \( (aa)^{\max\{0.3,0.33\}} = a^{0.33} \prec \Phi \),

(ii) \( a^{0.55} \prec \Phi \) and \( b^{0.35} \prec \Phi \), but \( (ab)^{\max\{0.55,0.35\}} = b^{0.55} \prec \Phi \),

(iii) \( a^{0.55} \prec \Phi \) and \( c^{0.33} \prec \Phi \), but \( (ac)^{\max\{0.55,0.33\}} = b^{0.55} \prec \Phi \).

(II) Let \( G = \{ e, a, b \} \) be the group defined by the multiplication table

\[
\begin{array}{ccc}
  e & a & b \\
  e & e & a \\
  a & a & b \\
\end{array}
\]

Let \( \Phi : G \to [0,1] \) be defined by \( \Phi(e) = 0.3, \Phi(a) = 0.2 \) and \( \Phi(b) = 0.1 \). Then \( \Phi \) satisfies the following implication:

(5)
\[
(\forall x, y \in G)(\forall \alpha, \beta \in [0,1])(x^\alpha \prec \Phi, y^\beta \prec \Phi \Rightarrow (xy)^{\max\{\alpha,\beta\}} \prec \Phi). \]

But since \( \Phi(a^{-1}) \neq \Phi(a) \), \( \Phi \) is not a \((\prec, \prec \vee \rangle)-\text{anti fuzzy subgroup of } G \).

**Theorem 3.6.** If \( \Phi \) is a non-constant \((\Omega, \Upsilon)-\text{anti fuzzy subgroup of } G \), then

(i) \( \Phi(e) \neq 1 \),

(ii) \( G^* := \{ x \in G \mid \Phi(x) \neq 1 \} \) is a subgroup of \( G \).

**Proof.** (i) Assume that \( \Phi(e) = 1 \). Since \( \Phi \) is non-constant, there exists \( x \in G \) such that \( \Phi(x) = \alpha < 1 \). If \( \Omega \in \{\prec, \prec \vee \rangle\} \), then \( x^\alpha \Omega \Phi \)
and $(x^{-1})^\Phi$. But

$$\Phi(xx^{-1}) = \Phi(e) = 1 > \alpha = \max\{\alpha, \alpha\}$$

and

$$\Phi(xx^{-1}) + \max\{\alpha, \alpha\} = \Phi(e) + \alpha = 1 + \alpha > 1.$$

Thus $(xx^{-1})^\Phi$, a contradiction. Note that $x^\Phi < \Phi$ and $(x^{-1})^\Phi < \Phi$. Since

$$\Phi(xx^{-1}) = \Phi(e) = 1 > \max\{0,0\} \quad \text{and} \quad \Phi(xx^{-1}) + \max\{0,0\} = 1,$$

we have $(xx^{-1})^\Phi$, a contradiction. Therefore $\Phi(e) \neq 1$.

(ii) Let $x, y \in G^*$. Then $\Phi(x) \neq 1$ and $\Phi(y) \neq 1$. Assume that $\Phi(xy) = 1$. If $\Omega \in \{<,\leq\}$, then $x^\Phi\Phi \subset \Phi$ and $y^\Phi\Phi \subset \Phi$. Since $\Phi(xy) \notin \max\{\Phi(x), \Phi(y)\}$ and

$$\Phi(xy) + \max\{\Phi(x), \Phi(y)\} = 1 + \max\{\Phi(x), \Phi(y)\} \neq 1,$$

we have $(xy)^\Phi$, a contradiction. Note that $x^\Phi < \Phi$ and $y^\Phi < \Phi$, but

$$\Phi(xy) + \max\{\Phi(x), \Phi(y)\} = 1 + \max\{\Phi(x), \Phi(y)\} \neq 1$$

and $\Phi(xy) = 1 \notin \max\{\Phi(x), \Phi(y)\}$. Hence $(xy)^\Phi$, a contradiction. Consequently $\Phi(xy) \neq 1$ and thus $xy \in G^*$. Obviously $x \in G^*$ implies $x^{-1} \in G^*$. Therefore $G^*$ is a subgroup of $G$.

**Theorem 3.7.** Every non-constant $\langle x, \rangle$-anti fuzzy subgroup $\Phi$ of $G$ is the characteristic anti function of $G^*$, that is,

$$\Phi(x) = \begin{cases} 0 & \text{if } x \in G^*, \\ 1 & \text{otherwise.} \end{cases}$$

**Proof.** Assume that there exists $x \in G^*$ such that $\Phi(x) \neq 0$. Let $\alpha \in [0,1)$ be such that $\alpha > \max\{\Phi(e), \Phi(x), 1 - \Phi(x)\}$. Then $x^\alpha < \Phi$ and $e^\alpha < \Phi$, but

$$\Phi(xe) + \max\{\alpha, \alpha\} = \Phi(x) + \alpha < 1,$$

that is, $(xe)^\Phi$. This is a contradiction. Hence $\Phi(x) = 0$ for all $x \in G^*$, and therefore $\Phi$ is the characteristic anti function of $G^*$.

**Corollary 3.8.** Every non-constant $\langle x, \rangle$-anti fuzzy subgroup $\Phi$ of $G$ is the characteristic anti function of $G^*$.
Theorem 3.9. Every non-constant $(\Gamma, \Gamma')$-anti fuzzy subgroup $\Phi$ of $G$ is constant on $G^*$.

Proof. Assume that there exists $b \in G^*$ such that $\alpha = \Phi(b) \neq \Phi(e) = \beta$. Suppose $\alpha > \beta$ and choose $\alpha_1, \alpha_2 \in [0, 1)$ such that $\alpha_2 < 1 - \alpha < \alpha_1 < 1 - \beta$. Then $e^{\alpha_1} \in \Phi$ and $b^{\alpha_2} \in \Phi$, but $(eb)^{\max\{\alpha_1, \alpha_2\}} = b^{\alpha_1}e^{\alpha_2}$. This is a contradiction. If $\alpha < \beta$, then $b^{1-\beta} \in \Phi$ and $(b^{-1})^{1-\beta} \in \Phi$. On the other hand, $(bb^{-1})^{\max(1-\beta, 1-\beta)} = e^{1-\beta}e^{1-\beta}$. This is impossible. Hence $\Phi(b) = \Phi(e)$ for all $b \in G^*$, and therefore $\Phi$ is a constant on $G^*$.

Theorem 3.10. Let $H$ be a subgroup of $G$ and let $\Phi : G \to [0, 1]$ be such that

(i) $(\forall x \in G \setminus H) (\Phi(x) = 1)$,
(ii) $(\forall x \in H) (\Phi(x) = \Phi(x^{-1}) \leq 0.5)$.

Then $\Phi$ is a $(<, < \triangledown \Gamma, \Gamma')$-anti fuzzy subgroup of $G$.

Proof. Let $x, y \in G$ and $\alpha, \beta \in [0, 1)$ be such that $x^\alpha \Phi$ and $y^\beta \Phi$. Then $\Phi(x) \leq \alpha$ and $\Phi(y) \leq \beta$. Moreover, $x, y \in H$ by (i), and hence $xy \in H$. If $\max\{\alpha, \beta\} < 0.5$, then

$$\Phi(xy) + \max\{\alpha, \beta\} < 0.5 + 0.5 = 1$$

and so $(xy)^{\max\{\alpha, \beta\}} \Phi$. If $\max\{\alpha, \beta\} \geq 0.5$, then $\Phi(xy) \leq 0.5 \leq \max\{\alpha, \beta\}$ and thus $(xy)^{\max\{\alpha, \beta\}} \Phi$. Consequently, we have $(xy)^{\max\{\alpha, \beta\}} \Phi$. This completes the proof.

Theorem 3.11. Let $\Phi$ be a $(\Gamma, < \triangledown \Gamma, \Gamma')$-anti fuzzy subgroup of $G$ such that $\Phi$ is non-constant on $G^*$. Then $\Phi(x) \leq 0.5$ for all $x \in G^*$.

Proof. We first prove that there exists $x \in G$ such that $\Phi(x) \leq 0.5$. If possible, let $\Phi(x) > 0.5$ for all $x \in G$. Since $\Phi$ is non-constant on $G^*$, there is $x \in G^*$ such that $\alpha = \Phi(x) \neq \Phi(e) = \beta$. Assume that $\alpha < \beta$ and choose $\delta < 0.5$ such that $\alpha + \delta < 1 < \beta + \delta$. Then $x^\delta \in \Phi$ and $(x^{-1})^\delta \in \Phi$, but $(xx^{-1})^{\max\{\delta, 0\}} = e^{\delta} \Phi$. This is a contradiction. If $\alpha > \beta$, then we can choose $\delta < 0.5$ such that $\beta + \delta < 1 < \alpha + \delta$. Then $e^\delta \Phi$ and $e^\delta \in \Phi$, but $(ex)^{\max\{\delta, 0\}} = e^{\delta} \Phi$, which is a contradiction. Therefore $\Phi(x) \leq 0.5$ for some $x \in G$. Now we prove $\Phi(x) \leq 0.5$. If not, then $\gamma = \Phi(x) > 0.5$. Since there exists $x \in G$ such that $\alpha = \Phi(x) \leq 0.5$, we have $\alpha < \gamma$. Choose $\beta < \gamma$ such that
\[ \alpha + \beta < 1 < \gamma + \beta \]. Then \( x^\beta \in \Phi \) and \((x^{-1})^\beta \in \Phi \), but \((xx^{-1})^{\max\{\beta, \gamma\}} = e^\beta \leq \sqrt[3]{\Phi} \). This is impossible, and therefore \( \Phi(e) \leq 0.5 \). Finally suppose that \( \alpha = \Phi(x) > 0.5 \) for some \( x \in G^* \). Let us choose \( \alpha_1 > 0 \) such that
\[ \alpha > 0.5 + \alpha_1. \]
Then \( x^\beta \in \Phi \) and \( e^{0.5-\alpha_1} \in \Phi \), but
\[ (e.x)^{\max\{0, 0.5-\alpha_1\}} = x^{0.5-\alpha_1} \leq \sqrt[3]{\Phi}. \]
This is a contradiction. Hence \( \Phi(x) \leq 0.5 \) for all \( x \in G^* \).

**Proposition 3.12.** Let \( \Phi \) be a \((<, \leq \sqrt[3]{\cdot})\)-anti fuzzy subgroup of \( G \). Then
(i) \( \Phi \) satisfies the following inequality
\[ (\forall x, y \in G) \ (\Phi(xy) \leq \max\{\Phi(x), \Phi(y), 0.5\}). \]
(ii) If there exists \( x \in G \) such that \( \Phi(x) \leq 0.5 \), then \( \Phi(e) \leq 0.5 \).

**Proof.** (i) Let \( x, y \in G \). If \( \max\{\Phi(x), \Phi(y)\} > 0.5 \), then \( \Phi(xy) \leq \max\{\Phi(x), \Phi(y)\} \). For, assume that \( \Phi(xy) > \max\{\Phi(x), \Phi(y)\} \) and choose \( \alpha \in [0, 1] \) such that \( \max\{\Phi(x), \Phi(y)\} < \alpha < \Phi(xy) \). Then \( x^\alpha \in \Phi \) and \( y^\alpha \in \Phi \), but
\[ (xy)^{\max\{\alpha, \alpha\}} = (xy)^\alpha \leq \sqrt[3]{\Phi}, \]
a contradiction. Hence \( \Phi(xy) \leq \max\{\Phi(x), \Phi(y)\} \) whenever \( \max\{\Phi(x), \Phi(y)\} > 0.5 \). Now suppose that \( \max\{\Phi(x), \Phi(y)\} \leq 0.5 \). Then \( x^{0.5} \in \Phi \) and \( y^{0.5} \in \Phi \) which imply that
\[ (xy)^{\max\{0.5, 0.5\}} = (xy)^{0.5} \leq \sqrt[3]{\Phi}, \]
that is, \( (xy)^{0.5} \in \Phi \) or \( (xy)^{0.5} \in \Phi \). If \( \Phi(xy) > 0.5 \), then \( \Phi(xy) + 0.5 > 0.5 + 0.5 = 1 \) and so \( (xy)^{0.5} \in \Phi \). This is a contradiction. Hence \( \Phi(xy) \leq 0.5 \). Consequently, \( \Phi(xy) \leq \max\{\Phi(x), \Phi(y), 0.5\} \) for all \( x, y \in G \).

(ii) Let \( x \in G \) be such that \( \Phi(x) \leq 0.5 \). Then
\[ \Phi(e) = \Phi(xx^{-1}) \leq \max\{\Phi(x), \Phi(x^{-1}), 0.5\} = 0.5. \]
This completes the proof.

**Proposition 3.13.** Let \( G = \langle a \rangle \) be a cyclic group of finite order and let \( \Phi \) be a \((<, \leq \sqrt[3]{\cdot})\)-anti fuzzy subgroup of \( G \) such that \( \Phi(a) \leq 0.5 \). Then \( \Phi(x) \leq 0.5 \) for all \( x \in G \).
Proof. Let \( x \in G \). Since \( G = \langle a \rangle \), there exists \( m \in \mathbb{N} \) such that \( x = a^m \). Thus \( \Phi(a^2) \leq \max\{\Phi(a), 0.5\} = 0.5, \Phi(a^3) \leq \max\{\Phi(a^2), \Phi(a), 0.5\} = 0.5, \) and so on. It follows that \( \Phi(x) = \Phi(a^m) \leq \max\{\Phi(a^{m-1}), \Phi(a), 0.5\} = 0.5 \). This completes the proof.

**Theorem 3.14.** Every \((<\forall r, <\forall r)\)-anti fuzzy subgroup of \( G \) is a \((<, <\forall r)\)-anti fuzzy subgroup of \( G \).

**Proof.** Let \( \Phi \) be a \((<\forall r, <\forall r)\)-anti fuzzy subgroup of \( G \). For any \( x, y \in G \), let \( \alpha_1, \alpha_2 \in [0, 1) \) be such that \( x^{\alpha_1} < \Phi \) and \( y^{\alpha_2} < \Phi \). Then \( x^{\alpha_1} < \forall r \Phi \) and \( y^{\alpha_2} < \forall r \Phi \), which imply that \( (xy)^{\max\{\alpha_1, \alpha_2\}} < \forall r \Phi \). Hence \( \Phi \) is a \((<, <\forall r)\)-anti fuzzy subgroup of \( G \).

**Theorem 3.15.** Every \((<, <)\)-anti fuzzy subgroup of \( G \) is a \((<, <\forall r)\)-anti fuzzy subgroup of \( G \).

**Proof.** Straightforward.

Example 3.5 shows that the converses of Theorems 3.14 and 3.15 need not be true.

4. Acknowledgement

The authors are highly grateful to referees for their valuable comments and suggestions helpful in improving this paper.

**REFERENCES**


Y. B. Jun
Department of Mathematics Education (and RINS)
Gyeongsang National University
Chinju 660-701, Korea
E-mail: skywine@gmail.com
S. Z. Song
Department of Mathematics
Cheju National University
Cheju 690-756, Korea
E-mail: szsong@cheju.ac.kr