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NIL SUBSETS IN BCH-ALGEBRAS

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ABSTRACT. Using the notion of nilpotent elements, the concept of nil subsets is introduced, and related properties are investigated. We show that a nil subset on a subalgebra (resp. (closed) ideal) is a subalgebra (resp. (closed) ideal). We also prove that in a nil algebra every ideal is a subalgebra.

1. Introduction

In 1966, Y. Imai and K. Iséki [8] and K. Iséki [9] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 1983, Q. P. Hu and X. Li [5, 6] introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. They have studied some properties of these algebras. Certain other properties have been studied by B. Ahmad [1], M. A. Chaudhry [2], W. A. Dudek and J. Thomys [4]. In 1992, W. Huang [7] introduced a nil ideals in BCI-algebras. The present authors [13, 14] studied some properties of this concepts. But nil ideals in BCH-algebras have not been studied yet. In this paper, we introduce the concept of nil subsets by using nilpotent elements, and investigate some related properties. We show that a nil subset on a subalgebra (resp. (closed) ideal) is a subalgebra (resp. (closed) ideal). We also prove that in a nil algebra every ideal is a subalgebra.

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2. Preliminaries

A *BCH-algebra* is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(1) $x * x = 0$,
(2) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(3) $(x * y) * z = (x * z) * y$

for all x, y, z in X.

In any BCH-algebra X, the following hold.

(4) $(x * (x * y)) * y = 0$,
(5) $x * 0 = 0$ implies $x = 0$,
(6) $0 * (x * y) = (0 * x) * (0 * y)$,
(7) $x * 0 = x$.

In what follows, the letter X denotes a BCH-algebra unless otherwise specified.

A non-empty subset S of X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset A of X is called an *ideal* of X if $0 \in A$ and if $x * y, y \in A$ imply that $x \in A$. Note that an ideal of a BCH-algebra may not be a subalgebra. An ideal A of X is said to be *closed* if $0 * x \in A$ for all $x \in A$.

For any elements x, y in X, let us write $x * y^n$ for $(\cdots((x * y) * y) * \cdots) * y$ where y occurs n times.

3. Main Results

DEFINITION 3.1. An element x in X is said to be *nilpotent* if $0 * x^n = 0$ for some positive integer n. An ideal A of X is called a *nil ideal* of X if every element of A is nilpotent. Inparticular, if every element in X is nilpotent, then X is called a *nil algebra*.

LEMMA 3.2. *For any x in X and any positive integer n, we have*

$$0 * (0 * x)^n = 0 * (0 * x^n).$$

Proof. The lemma is trivial for $n = 1$. Now let us assume that the lemma is true for a positive integer n. Then

$$\begin{aligned} 0*(0*x^{n+1}) &= 0*((0*x^n)*x) \\ &= (0*(0*x^n))*(0*x) \\ &= (0*(0*x)^n)*(0*x) \\ &= 0*(0*x)^{n+1}, \end{aligned}$$

ending the proof. □

LEMMA 3.3. *For any x, y in X and any positive integer n, we have*

$$0*(x*y)^n = (0*x^n)*(0*y^n).$$

Proof. By (6), the lemma holds for $n = 1$. Now let us assume that the lemma is true for positive integer n. By using (3), (6) and Lemma 3.2, we have

$$\begin{aligned} 0*(x*y)^{n+1} &= (0*(x*y)^n)*(x*y) \\ &= ((0*x^n)*(0*y^n))*(x*y) \\ &= ((0*(x*y))*x^n)*(0*y^n)) \\ &= ((0*(0*y^n))*x^{n+1})*(0*y) \\ &= ((0*(0*y)^n)*(0*y))*x^{n+1} \\ &= (0*((0*y)^{n+1}))*x^{n+1} \\ &= (0*(0*y^{n+1}))*x^{n+1} \\ &= (0*x^{n+1})*(0*y^{n+1}). \end{aligned}$$

This completes the proof. □

Let S be any non-empty subset of X. For any positive integer k, we define a *k-nil subset* on S as follows:

$$N_k(S) := \{x \in S \mid 0*x^k = 0\}.$$

THEOREM 3.4. *If S is a subalgebra of X, then so is the k-nil subset $N_k(S)$ on S for every positive integer k.*

Proof. Let $x, y \in N_k(S)$. Then $x, y \in S$, $0*x^k = 0$ and $0*y^k = 0$. Hence, by Lemma 3.3, we have that

$$0*(x*y)^k = (0*x^k)*(0*y^k) = 0*0 = 0$$

and $x * y \in S$ because S is a subalgebra. Therefore $x * y \in N_k(S)$, which proves that $N_k(S)$ is a subalgebra of X. □

COROLLARY 3.5. *The k-nil subset $N_k(X)$ on X is a subalgebra of X for every positive integer k.*

PROPOSITION 3.6. *Let S be a subalgebra of X and let k be a positive integer. If $x \in N_k(S)$, then $0 * x \in N_k(S)$.*

Proof. If $x \in N_k(S)$, then $x \in S$ and $0 * x^k = 0$. It follows from Lemma 3.2 that

$$0 * (0 * x)^k = 0 * (0 * x^k) = 0 * 0 = 0$$

and $0 * x \in S$ because S is a subalgebra. Hence $0 * x \in N_k(S)$. □

The following example shows that the converse of Proposition 3.6 may not be true.

EXAMPLE 3.7. Let $X = \{0, 1, 2, 3\}$ be a set with Cayley table as follows:

$*$	0	1	2	3
0	0	0	3	3
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

Then $(X; *, 0)$ is a BCH-algebra. We note that $S = \{0, 3\}$ is a subalgebra of X. Since $0 * (0 * 2)^2 = 0 * 3^2 = (0 * 3) * 3 = 3 * 3 = 0$, we get $0 * 2 \in N_2(S)$ but $2 \notin N_2(S)$ because $2 \notin S$.

THEOREM 3.8. *Let A be an ideal of X. Then the k-nil subset $N_k(A)$ on A is an ideal of X for any positive integer k.*

Proof. It is clear that $0 \in N_k(A)$. Let $x*y \in N_k(A)$ and $y \in N_k(A)$. Then $x * y, y \in A$, $0 * (x * y)^k = 0$ and $0 * y^k = 0$. Since A is an ideal, we have $x \in A$ and

$$0 * x^k = (0 * x^k) * 0 = (0 * x^k) * (0 * y^k) = 0 * (x * y)^k = 0.$$

Hence $x \in N_k(A)$, and therefore $N_k(A)$ is an ideal of X. □

COROLLARY 3.9. *The k-nil subset $N_k(X)$ on X is an ideal of X for any positive integer k.*

Note that, in a BCH-algebra, every closed ideal is a subalgebra([2]). Following Proposition 3.6 and Theorem 3.8, we have

THEOREM 3.10. *If A is a closed ideal of X, then the k-nil subset $N_k(A)$ on A is a closed ideal of X for every positive integer k.*

THEOREM 3.11. *Let S be a subset of X and let k and r be positive integers. If $k|r$, then $N_k(S) \subset N_r(S)$.*

Proof. If $k|r$, then $r = kq$ for some positive integer q. Let $x \in N_k(S)$. Then

$$0 * x^r = 0 * x^{kq} = (\cdots((0 * \underbrace{x^k) * x^k) * \cdots) * x^k}_{q \text{ times}} = 0.$$

This means that $x \in N_r(S)$, so that $N_k(S) \subset N_r(S)$. □

COROLLARY 3.12. *For any positive integers k and r such that $k|r$, we have $N_k(X) \subset N_r(X)$.*

PROPOSITION 3.13. *Let S be a subalgebra of X. If x and y are nilpotent elements in S, then $x * y$ is also a nilpotent element in S.*

Proof. Suppose that x and y are nilpotent elements in S. Then there exist positive integers m and n such that $0*x^m = 0$ and $0*y^n = 0$, respectively. Let $k = lcm\{m, n\}$. Then $mt = k = ns$ where s and t are positive integers such that $(s,t) = 1$. It follows from Lemma 3.3 that

$$0 * (x * y)^k = (0 * x^k) * (0 * y^k) = (0 * x^{mt}) * (0 * y^{ns}) = 0 * 0 = 0.$$

Clearly $x * y \in S$. Thus $x * y$ is a nilpotent element in S. □

COROLLARY 3.14. *If x and y are nilpotent elements of X, then so is $x * y$.*

THEOREM 3.15. *Every X contains a maximal nil ideal which is also a subalgebra of X.*

Proof. Let $N(X) := \{x \in X \mid x \text{ is a nilpotent element}\}$. Clearly $0 \in N(X)$. Assume that $x * y \in N(X)$ and $y \in N(X)$. Then there exist positive integers k and r such that $0 * (x * y)^r = 0$ and $0 * y^k = 0$. It follows from Theorem 3.11 that $0 * (x * y)^{kr} = 0$ and $0 * y^{kr} = 0$, that is, $x * y \in N_{kr}(X)$ and $y \in N_{kr}(X)$. By Corollary 3.9, we get $x \in N_{kr}(X) \subseteq N(X)$. Therefore $N(X)$ is an ideal of X. Now we show that $N(X)$ is a subalgebra of X. Let $x, y \in N(X)$. By using Corollary 3.12, we can assume that $x, y \in N_k(X)$ for some positive integers k. It follows from (1), (3) and Proposition 3.6 that $(x*y)*x = 0*y \in N_k(X)$. Since $N_k(X)$ is an ideal of X, we conclude that $x * y \in N_k(X)$ and hence $N_k(X)$ is a subalgebra of X. □

Finally we give a condition for an ideal to be a subalgebra.

THEOREM 3.16. *In a nil algebra, every ideal is a subalgebra.*

Proof. Let A be an ideal of a nil algebra X and let $x, y \in A$. Then there exists a positive integer n such that $0 * y^n = 0$, i.e.,

$$(\cdots((0 * y) * y) * \cdots) * y = 0 \text{ where } y \text{ occurs } n \text{ times.}$$

Since A is an ideal, it follows that $0 * y \in A$ so that $(x * y) * x = (x * x) * y = 0 * y \in A$. Hence $x * y \in A$, ending the proof. □

REFERENCES

[1] B. Ahmad, *On classification of BCH-algebras,* Math. Japonica **35(5)** (1990), 801–804.
[2] M. A. Chaudhry, *On BCH-algebras,* Math. Japonica **36(4)** (1991), 665–676.
[3] M. A. Chaudhry and H. Fakhar-ud-din, *Ideals and filters in BCH-algebras,* Math. Japonica **44(1)** (1996), 101–112.
[4] W. A. Dudek and J. Thomys, *On decomposition of BCH-algebras,* Math. Japonica **35** (1990), 1131–1138.
[5] Q. P. Hu and X. Li, *On BCH-algebras,* Math. Seminar Notes**11** (1983), 313–320.
[6] Q. P. Hu and X. Li, *On proper BCH-algebras,* Math. Japonica **30** (1985), 659–661.
[7] W. Huang, *Nil-radical in BCI-algebras,* Math. Japonica **37(2)** (1992), 363–366.
[8] Y. Imai and K. Iséki, *On axiom systems of propositional calculi XIV,* Proc. Japan Academy **42** (1966), 19–22.
[9] K. Iséki, *An algebra related with a propositional calculus,* Proc. Japan Academy **42** (1966), 26–29.
[10] K. Iséki, *On BCI-algebras,* Math. Seminar Notes **8** (1980), 125–130.
[11] K. Iséki and S. Tanaka, *Ideal theory of BCK-algebras,* Math. Japonica **21** (1976), 351–366.

[12] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japonica **23** (1978), 1–26.
[13] Y. B. Jun, J. Meng and E. H. Roh, *On nil ideals in BCI-algebras*, Math. Japonica **38(5)** (1993), 1051–1056.
[14] Y. B. Jun and E. H. Roh, *Nil ideals in BCI-algebras*, Math. Japonica **41(2)** (1995), 297–302.

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