INTTEGRAL GEOMETRY
ON PRODUCT OF SPHERES II

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1. Introduction and Result

Let $G$ be a Lie group and $H$ a closed subgroup of $G$. We assume that $G$ has a left invariant Riemannian metric that is also right invariant under elements of $H$. Then $G/H$ is a homogeneous space with an invariant Riemannian metric. Consider now two submanifolds $M$ and $N$ of $G/H$, one fixed and the other moving under the action of $g \in G$. We always assume that $M$ and $N$ are in generic positions. This means that the dimension of the intersection $M \cap gN$ is nonnegative for almost all $g \in G$. Let $\text{vol}(M \cap gN)$ be an integral invariant of the submanifold $M \cap gN$. One of the basic problems in integral geometry is to find explicit formulas for integral of $\text{vol}(M \cap gN)$ over $G$ with respect to the invariant measure $d\mu_G(g)$ on $G$ in terms of known integral invariants of $M$ and $N$. Especially R. Howard [1] obtained a generalized Poincaré formula for Riemannian homogeneous spaces as follows:

Let $M$ and $N$ be submanifolds of $G/H$ with $\dim M + \dim N = \dim(G/H)$. Assume that $G$ is unimodular. Then

\[(1.1) \quad \int_G \| (M \cap gN) \| d\mu_G(g) = \int_{M \times N} \sigma_H(T^\perp_x M, T^\perp_y N) \, d\mu_{M \times N}(x,y),\]

where $\|X\|$ denotes the number of elements in a set $X$ and $\sigma_H(T^\perp_x M, T^\perp_y N)$ is defined by (2.1) in Section 2.
The formula (1.1) holds under the general situation. However, it is difficult to give an explicit description through the concrete computation of $\sigma_H(T_x^1 M, T_y^1 N)$, and only a little is known about it. In this paper, we attempt to explicitly describe this formula for two dimensional submanifolds in the product of unit sphere $S^2$. More precisely,

**Theorem 1.1.** Let $M$ and $N$ be submanifolds of $S^2 \times S^2$ of dimension 2. Assume that for almost all $g \in G$, $M$ and $gN$ intersect transversely. For any point $x \in M$ and $y \in N$, $\xi_x$ and $\eta_y$ denote the unit vector of $T_x M$ and $T_y N$, respectively. Then we have

$$\int_{SO(3) \times SO(3)} \| (M \cap gN) \| \mu_{SO(3) \times SO(3)}(g) = \int_{M \times N} \sigma(\xi, \eta) \mu_{M \times N}(x, y).$$

Here $\sigma(\xi, \eta)$ was introduced by the Gauss hypergeometric function in the Section 3.

2. Preliminaries

Here we shall review the generalized Poincaré formula on Riemannian homogeneous spaces given by R. Howard [1] and recall the Gauss hypergeometric function and the elliptic integrals.

Let $E$ be a finite dimensional real vector space with an inner product, and let $V$ and $W$ be two vector subspaces of $E$ with orthonormal bases $v_1, \ldots, v_p$ and $w_1, \ldots, w_q$ respectively. The angle between subspaces $V$ and $W$ is defined by

$$\sigma(V, W) = \| v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q \|,$$

where

$$\| x_1 \wedge \cdots \wedge x_k \|^2 = |\det \langle x_i, x_j \rangle|.$$  

This definition is independent of the choice of orthonormal bases. It is obvious that if $p + q = \dim E$ then

$$\sigma(V, W) = \sigma(V^\perp, W^\perp).$$

Let $G$ be a Lie group and $H$ a closed subgroup of $G$. We assume that $G$ has a left invariant Riemannian metric that is also invariant under the right actions of elements of $H$. This metric induces a $G$-invariant Riemannian metric on $G/H$. We denote by $o$ the origin of
$G/H$. If $x, y \in G/H$ and $V$ is a vector subspace of $T_x(G/H)$ and $W$ is a vector subspace of $T_y(G/H)$ then define $\sigma_H(V, W)$ by

$$\sigma_H(V, W) = \int_H \sigma((dg_x)_o^{-1}V, dh_o^{-1}(dg_y)_o^{-1}W) d\mu_H(h)$$

where $g_x$ and $g_y$ are elements of $G$ such that $g_xo = x$ and $g_yo = y$. This definition is independent of the choice of $g_x$ and $g_y$ in $G$ such that $g_xo = x$ and $g_yo = y$.

We list here the basic properties of the Gauss hypergeometric function that are needed in this paper only. For further details see [4].

The Gauss hypergeometric series, convergent for $|z| < 1$, is given by the power series

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \cdot \frac{\Gamma(b+n)}{\Gamma(b)} \cdot \frac{\Gamma(c+n)}{\Gamma(c+n)} \cdot \frac{z^n}{n!}$$

where $\Gamma$ is the gamma function. By analytic continuation $F(a, b, c; z)$ can be extended to define a function analytic and single-valued in the complex $z$ plane cut along the positive real axis from 1 to $\infty$. We remark that above series reduces to a polynomial of degree $n$ in $x$ when $a$ or $b$ is equal to $-n, (n = 0, 1, 2, \cdots)$. The series (2.2) is not defined when $c$ is equal to $-m, (m = 0, 1, 2, \cdots)$, provided $a$ or $b$ is not a negative integer $n$ with $n < m$. The hypergeometric equation

$$z(1-z)\frac{d^2u}{dz^2} + (c - (a + b + 1)z)\frac{du}{dz} - abu = 0$$

has the solution $u = F(a, b, c; z)$.

The six functions $F(a \pm 1, b, c; z)$, $F(a, b \pm 1, c; z)$ and $F(a, b, c \pm 1; z)$ are called contiguous to $F(a, b, c; z)$. Relations between $F(a, b, c; z)$ and any two contiguous functions have been given by Gauss. By repeated application of these relations the function $F(a + m, b + n, c + l; z)$ with integer $m, n, l$ can be expressed as a linear combination of $F(a, b, c; z)$ and one of its contiguous functions with coefficients which are rational functions of $a, b, c, z$. For examples,

$$azF(a + 1, b + 1, c + 1; z) = c [F(a, b + 1, c; z) - F(a, b, c; z)]$$

$$(c - 1)F(a, b, c - 1; z) = (c - a - 1)F(a, b, c; z) + aF(a + 1, b, c; z).$$
Among the special cases are
\begin{equation}
(1 - z)^t = \int F(-t, b, b; z),
\end{equation}
\begin{equation}
\arcsin z = z F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2\right).
\end{equation}
Furthermore C. F. Gauss evaluated, for \( \Re(c - a - b) > 0 \),
\begin{equation}
F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.
\end{equation}
In this paper, we may consider only when \( z \) is a real number.
We now recall that the incomplete elliptic integrals of the first and second kind are defined by, for \( 0 < k < 1 \),
\begin{equation}
F(\psi, k) = \int_0^\psi \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \, d\theta, \quad E(\psi, k) = \int_0^\psi \sqrt{1 - k^2 \sin^2 \theta} \, d\theta,
\end{equation}
respectively. If \( \psi = \pi/2 \) then the integrals are called the complete elliptic integral of the first and second kind, and are denoted by \( K(k) \) and \( E(k) \) or simply \( K \) and \( E \) respectively.

3. Proof of the Theorem 1.1

Let \( S^2 \) be the standard sphere of dimension 2. Throughout this section, to simplify notation, we will regard \( G \) and \( H \) as \( SO(3) \times SO(3) \) and \( SO(2) \times SO(2) \). The special orthogonal group \( SO(3) \) acts transitively on \( S^2 \). The isotropy subgroup of \( SO(3) \) at a point in \( S^2 \) is \( SO(2) \). Thus \( S^2 \times S^2 \) can be realized as a homogeneous space \( G/H \). Let \( \mathfrak{so}(3) \times \mathfrak{so}(3) \) be the Lie algebra of \( G \). Define an inner product on \( \mathfrak{so}(3) \times \mathfrak{so}(3) \) by
\begin{equation}
(X, Y) = -\frac{1}{2} \text{Trace}(XY), \quad (X, Y \in \mathfrak{so}(3) \times \mathfrak{so}(3)).
\end{equation}
We extend this inner product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{so}(3) \times \mathfrak{so}(3) \) to the left invariant Riemannian metric on \( G \). Then we obtain a bi-invariant Riemannian metric on \( G \). This bi-invariant Riemannian metric on \( G \) induces a \( G \)-invariant Riemannian metric on \( G/H \).
Let $M$ and $N$ be submanifolds of $S^2 \times S^2$ of dimension 2. By the formula (1.1), we have

\[ \int_G \mathbb{H}(M \cap gN) \, d\mu_G(g) = \int_{M \times N} \sigma_H(T_x M, T_y N) \, d\mu_{M \times M}(x, y). \]

For any point $x = (x_1, x_2) \in M$,

\[ T_x M = T_{(x_1, x_2)} M \subset T_{x_1} S^2 \oplus T_{x_2} S^2. \]

Thus $u_x$ can be realized as an unit vector of $T_x M$ just as follows:

\[ u_x = (u_1, u_2) \in T_{x_1} S^2 \oplus T_{x_2} S^2 = \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^4. \]

We here can transport $u_x$ to $(\cos \theta_1, 0, \sin \theta_1, 0)$, since the action of $H$ preserves the length of vectors. Thus we can take

\[ (\cos \theta_1, 0, \sin \theta_1, 0), (-\sin \theta_1 \cos \theta_2, \sin \theta_2 \cos \theta_1, \cos \theta_1 \cos \theta_2, \sin \theta_2 \sin \theta_3) \]

as an orthonormal basis of $T_x M$. Similarly we have

\[ (\cos \tau_1, 0, \sin \pi_1, 0), (-\sin \tau_1 \cos \tau_2, \sin \tau_2 \cos \tau_1, \cos \tau_1 \cos \tau_2, \sin \tau_2 \sin \tau_3) \]

as an orthonormal basis of $T_y N$.

In this choice of orthonormal bases, we can easily take one. But it is too much variables to calculate the $\sigma_H(\cdot, \cdot)$.

Now let $Gr_2^+(\mathbb{R}^4)$ be an oriented Grassmann manifold as a submanifold of $\wedge_2 \mathbb{R}^4$. We take an orientation on $\mathbb{R}^4$ such that $e_1, e_2, e_3, e_4$ is a positive basis of $\mathbb{R}^4$ and the inner product on $\wedge_2 \mathbb{R}^4$ induced by that on $\mathbb{R}^4$. Let $*$ be the Hodge star operator on $\wedge_2 \mathbb{R}^4$. Put

\[ \wedge_2^+ = \left\{ \xi \in \wedge_2 \mathbb{R}^4 \mid * \xi = \xi \right\}, \quad \wedge_2^- = \left\{ \xi \in \wedge_2 \mathbb{R}^4 \mid * \xi = -\xi \right\}. \]

Then we have an orthogonal direct sum decomposition

\[ \wedge_2 \mathbb{R}^4 = \wedge_2^+ \oplus \wedge_2^-. \]
We define orthonormal bases $A_1$ and $B_1$ of $\Lambda^2_+$ and $\Lambda^2_-$ by

$$
A_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4), \quad B_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4),
$$

$$
A_2 = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 - e_2 \wedge e_4), \quad B_2 = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 + e_2 \wedge e_4),
$$

$$
A_3 = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 + e_2 \wedge e_3), \quad B_3 = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 - e_2 \wedge e_3).
$$

Then we obtain

$$
\Lambda^2_+ = \text{Span}\{A_1, A_2, A_3\}, \quad \Lambda^2_- = \text{Span}\{B_1, B_2, B_3\}.
$$

By a simple calculation, we have

$$
Gr^o_2(\mathbb{R}^4) = S^2 \left( \frac{1}{\sqrt{2}} \right) \times S^2 \left( \frac{1}{\sqrt{2}} \right).
$$

Hence we can easily take orthonormal bases $\xi$ and $\eta$ as follows:

$$
\xi = \frac{1}{\sqrt{2}}(\cos \theta_1 A_1 + \sin \theta_1 A_2) + \frac{1}{\sqrt{2}}(\cos \theta_2 B_1 + \sin \theta_2 B_2),
$$

$$
\eta = \frac{1}{\sqrt{2}}(\cos \tau_1 A_1 + \sin \tau_1 A_2) + \frac{1}{\sqrt{2}}(\cos \tau_2 B_1 + \sin \tau_2 B_2),
$$

where $0 \leq \theta_1, \theta_2, \tau_1, \tau_2 \leq \pi$. We can simply write

$$
\sigma_H(T_x M, T_y N) = \sigma_H(\xi, \eta),
$$

since $\sigma_H(T_x M, T_y N)$ is dependent only on $\xi$ and $\eta$, that is, $\theta$ and $\tau$.

Now we work on the following integral

$$
\sigma_H(\xi, \eta) = \int_H |\xi \wedge k\eta| d\mu_H(h).
$$

We have set, to simplify notation,

$$
\cos \theta_i \cos \tau_i = c_{ii}, \quad \sin \theta_i \sin \tau_i = s_{ii}, \quad (i = 1, 2).
$$

Then we immediately obtain

$$
|\xi \wedge k\eta| = \frac{1}{2} |c_{11} + s_{11} \cos (\alpha + \beta) - c_{22} - s_{22} \cos (\alpha - \beta)|,
$$
since
\[
\eta = \frac{1}{\sqrt{2}} (\cos \tau_1 A_1 + \sin \tau_1 \cos (\alpha + \beta) A_2 + \sin \tau_1 \sin (\alpha + \beta) B_3)
\]
\[
+ \frac{1}{\sqrt{2}} (\cos \tau_2 B_1 + \sin \tau_2 \cos (\alpha - \beta) B_2 + \sin \tau_2 \sin (\alpha - \beta) B_3)
\]
for
\[
h = \begin{bmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 \\
\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & \cos \beta & -\sin \beta \\
0 & 0 & \sin \beta & \cos \beta
\end{bmatrix} \in SO(2) \times SO(2).
\]

Hence we have to evaluate the following integral.
\[
\sigma_H(\xi, \eta) = \int_0^{2\pi} \int_0^{2\pi} \frac{1}{2} \left| c_{11} + s_{11} \cos (\alpha + \beta) \
-c_{22} - s_{22} \cos (\alpha - \beta) \right| d\alpha d\beta,
\]

namely,
\[
\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \left| c_{11} - c_{22} + (s_{11} - s_{22}) \cos \alpha \cos \beta \
- (s_{11} + s_{22}) \sin \alpha \sin \beta \right| d\alpha d\beta.
\]

Since \( 0 \leq \theta_1, \theta_2, \tau_1, \tau_2 \leq \pi \), we have \( 0 \leq s_{11}, s_{22} \leq 1 \) and \(-1 \leq c_{11}, c_{22} \leq 1 \). And put \( a = s_{11} + s_{22}, b = s_{11} - s_{22}, c = c_{11} - c_{22} \) then
\( 0 \leq a \leq 2, \quad -1 \leq b \leq 1, \quad -2 \leq c \leq 2, \quad |a| \geq |b|. \)

Having set up these notations, we can now give lemma that is needed to calculate our result.

**Lemma 3.1.** Let \( S^1(r) \) be a circle with radius \( r \). If \( |a| \leq 1 \) then
\[
\int_{S^1(r)} |ra + x_1| \, d\mu_{S^1(r)}(x) = 4r^2 \left( a \arcsin a + \sqrt{1 - a^2} \right).
\]

We can easily show this lemma and omit its proof.
At first, we shall prove the case where \( c = 0 \).
In this case, we will assume that \( a = 0 \). Then we have \( b = 0 \) since \( s_{11} = s_{22} = 0 \). Therefore we have
\[
\sigma_H(\xi, \eta) = 0.
\]

We suppose that \( a > 0 \). Then we have
\[
\sigma_H(\xi, \eta) = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \left| \sqrt{b^2 \cos^2 \alpha + a^2 \sin^2 \alpha \cos \beta} \right| d\beta d\alpha
\]
\[
= 2 \int_0^{2\pi} \sqrt{b^2 \cos^2 \alpha + a^2 \sin^2 \alpha} d\alpha
\]
\[
= 8a \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \alpha} \, d\alpha \quad \text{\{put } k := \sqrt{1 - (b/a)^2} \text{\}\}}
\]
\[
= 8a \text{E}(k).
\]

Now we shall prove the case where \( c \neq 0 \).

**Case I** The case where \( 0 < |b| < a < |c| \).

In this case, we shall compute the following:
\[
\sigma_H(\xi, \eta) = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \left| c + b \cos \alpha \cos \beta - a \sin \alpha \sin \beta \right| d\beta d\alpha
\]
\[
= \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \left| c - \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha} \sin(\beta + \phi) \right| d\beta d\alpha.
\]

Here if \( a \neq |b| \) then we have \( \sin^2 \alpha \leq \frac{c^2 - b^2}{a^2 - b^2} \), since \( c^2 \geq a^2 \). Hence we have \( |c| \geq \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha} \), for all \( \alpha \in [0, 2\pi] \). Therefore we obtain
\[
\int_0^{2\pi} \left| c - \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha} \sin(\beta + \phi) \right| d\beta = \int_0^{2\pi} |c| d\beta = 2\pi |c|.
\]

If \( a = |b| \) then we have
\[
\int_0^{2\pi} |c - a \sin(\beta + \phi)| \, d\beta = 2\pi |c|.
\]

In this case, from (3.2) and (3.3), we immediately obtain
\[
\sigma_H(\xi, \eta) = \frac{1}{2} \int_0^{2\pi} 2\pi |c| \, d\alpha = 2\pi^2 |c|.
\]
Case II. The case where $0 < |c| \leq |b| \leq a$.

If $a = |b|$ then, by Lemma 3.1, we have

$$
\sigma_H(\xi, \eta) = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} |c - a \sin(\beta + \phi)| d\beta d\alpha
$$

$$
= \frac{1}{2} \int_0^{2\pi} \left( 4c \arcsin \left( \frac{c}{a} \right) + 4\sqrt{a^2 - c^2} \right) d\alpha
$$

(3.4)

$$
= 4\pi c \arcsin \left( \frac{c}{a} \right) + 4\pi \sqrt{a^2 - c^2}.
$$

Here if $a \neq |b|$ then we have $\sin^2 \alpha \geq \frac{c^2 - b^2}{a^2 - b^2}$, since $c^2 < a^2$. Hence we have $|c| \leq \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}$, for all $\alpha \in [0, 2\pi]$. Therefore, by Lemma 3.1, we get

$$
(3.5) \int_0^{2\pi} \left| c - \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha} \sin(\beta + \phi) \right| d\beta
$$

$$
= 4c \arcsin \left( \frac{c}{\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}} \right) + 4\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha - c^2}.
$$

Let us integrate on $[0, 2\pi]$ both term of (3.5). The integral of the second part of the right-hand side of (3.5) gives

$$
\int_0^{2\pi} \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha - c^2} d\alpha
$$

$$
= \int_0^{2\pi} \sqrt{(a^2 - c^2) - (a^2 - b^2) \cos^2 \alpha} d\alpha
$$

(3.6)

$$
= 4\sqrt{a^2 - c^2} E \left( \sqrt{\frac{a^2 - b^2}{a^2 - c^2}} \right).
$$

Now we compute the first part of the right-hand side of (3.5). To do this, we prepare the following lemma and formulas (3.7) and (3.8).

**Lemma 3.2.** For integer $m$, we have

$$
\int \sin^{2m} x \, dx = - \cos x \, F \left( \frac{1}{2}, \frac{1}{2} - m, \frac{3}{2}; \cos^2 x \right).
$$
We can easily show the above lemma, using the binomial theorem, 
the details are left to the reader.
From Lemma 3.2, it is obvious that

\[
(3.7) \quad \int_0^{\pi/2} \sin^{2m} x \, dx = \frac{(2m - 1)!!}{(2m)!!} \cdot \frac{\pi}{2}.
\]

where

\[
m!! = \begin{cases} 
m(m - 2) \cdots 4 \cdot 2, & m : \text{even}; \\
m(m - 2) \cdots 3 \cdot 1, & m : \text{odd}.
\end{cases}
\]

And, by a simple calculation and the binomial theorem, we obtain the 
following equality:

\[
(3.8) \quad \left( \frac{1}{1 + k^2 \sin^2 x} \right)^{2n+1} = \sum_{m=0}^{\infty} \frac{(2n + 2m - 1)!!}{(2m)!!(2n - 1)!!} (-k^2)^m \sin^{2m} x.
\]

From the Taylor expansion of \( \arcsin f(x) \) and (3.8), (3.7), we have

\[
\int_0^{2\pi} \arcsin \left( \frac{c}{\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}} \right) \, d\alpha \\
= \sum_{n=0}^{\infty} \frac{(2n - 1)!!}{(2n)!!} \frac{1}{2n + 1} \int_0^{2\pi} \theta^{2n+1} \left( \frac{1}{\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}} \right)^{2n+1} \, d\alpha \\
= \sum_{n=0}^{\infty} \frac{(2n - 1)!!}{(2n)!!} \frac{1}{2n + 1} \left| b \right|^{2n+1} \int_0^{2\pi} \left( \frac{1}{1 + k^2 \sin^2 \alpha} \right)^{2n+1} \, d\alpha \\
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2n - 1)!!(2n + 2m - 1)!!}{(2n + 1)!!(2m)!!(2n - 1)!!} \left( \frac{c}{\left| b \right|} \right)^{2n+1} (-k^2)^m \int_0^{2\pi} \sin^{2m} \alpha \, d\alpha \\
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2n - 1)!!(2n + 2m - 1)!!}{(2n + 1)!!(2m)!!(2n - 1)!!} \left( \frac{c}{\left| b \right|} \right)^{2n+1} (-k^2)^m \frac{2\sqrt{\pi} \Gamma \left( m + \frac{1}{2} \right) \Gamma \left( n + m + \frac{1}{2} \right) (-k^2)^m}{m! \Gamma \left( n + \frac{1}{2} \right) n!} \frac{2\pi}{m!} \\
= 2\pi \sum_{n=0}^{\infty} \frac{(2n - 1)!!}{(2n)!!} \frac{1}{2n + 1} \left( \frac{c}{\left| b \right|} \right)^{2n+1} F \left( \frac{1}{2}, n + \frac{1}{2}; 1; -k^2 \right)
\]
where the step going from the second to third line used putting $k^2 = (a^2 - b^2)/b^2$, and the fifth to sixth line used

$$(2n + 2m - 1)!! = \frac{2^{n+m}}{\sqrt{\pi}} \Gamma\left(n + m + \frac{1}{2}\right).$$

Summarizing, we obtain

$$\sigma_H(\xi, \eta) = 4\pi c \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1} \left(\frac{c}{|b|}\right)^{2n+1} \times F\left(\frac{1}{2}, n + \frac{1}{2}, 1; \frac{b^2 - a^2}{b^2}\right) + 8\sqrt{a^2 - c^2}E\left(\frac{a^2 - b^2}{\sqrt{a^2 - c^2}}\right).$$

**Remark 3.3.** It is trivial that the case where $a = |b|$ in just above equality goes to (3.4).

**Case III.** The case where $0 \leq |b| \leq |c| \leq a$.

In particular, if $0 < |b| = |c| = a$ then we have

$$\sigma_H(\xi, \eta) = \frac{1}{2} \int_{0}^{2\pi} \int_{0}^{2\pi} |c - a \sin(\beta + \phi)| \, d\beta \, d\alpha.$$

Since $|c/a| = 1$, we obtain

$$\sigma_H(\xi, \eta) = \frac{1}{2} \int_{0}^{2\pi} 2\pi |c| \, d\alpha = 2\pi^2 |c| = 2\pi^2 a.$$

It is sufficient to calculate the following:

$$\sigma_H(\xi, \eta) = \frac{1}{2} \int_{0}^{2\pi} \int_{0}^{2\pi} \left|c - \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha \sin(\beta + \phi)}\right| \, d\beta \, d\alpha,$$

where the case is $|b| \leq |c| < a$ or $|b| < |c| \leq a$.

In these cases, we immediately know that $0 \leq \frac{c^2 - b^2}{a^2 - b^2} \leq 1$. The inequality

$$|c| \geq \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}$$
is satisfied whenever \(0 \leq \alpha \leq \theta, \pi - \theta \leq \alpha \leq \pi + \theta, \) \(2\pi - \theta \leq \alpha \leq 2\pi,\)
where
\[
\theta = \arcsin \sqrt{c^2 - b^2 \over a^2 - b^2}.
\]
Then we obtain
\[
\int_0^{2\pi} \left| c - \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha \sin(\beta + \phi)} \right| \, d\beta = 2\pi |c|.
\]
Therefore we have
\[
\int_0^{2\pi} \left| c - \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha \sin(\beta + \phi)} \right| \, d\beta = 2\pi |c| \theta
\]
On the other hand, the inequality
\[
|c| \leq \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}
\]
holds for \(\theta \leq \alpha \leq \pi - \theta, \pi + \theta \leq \alpha \leq 2\pi - \theta.\) Then, by Lemma 3.1, we have
\[
\int_0^{2\pi} \left| c - \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha \sin(\beta + \phi)} \right| \, d\beta
\]
\[
= 4c \arcsin \left( \frac{c}{\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}} \right) + 4\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha - c^2}.
\]
We first integrate the second part of right-hand side of (3.11) on \(|\theta, \pi - \theta|\). Then we have
\[
\int_{\theta}^{\pi - \theta} \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha - c^2} \, d\alpha
\]
\[
= \int_{\theta}^{\pi - \theta} \sqrt{(a^2 - b^2) \sin^2 \alpha - (c^2 - b^2)} \, d\alpha.
\]
Here we put \((a^2 - b^2) \sin^2 \alpha - (c^2 - b^2) = (a^2 - c^2) \sin^2 \psi.\) Then, using the coordinate transformation, above integral is as follows:
\[
2(a^2 - c^2) \int_0^{\pi/2} \frac{1 - \cos^2 \psi}{\sqrt{(a^2 - c^2) \sin^2 \psi + (c^2 - b^2)}} \, d\psi.
\]
Integral geometry on product of spheres II

From

\[ \int_0^{\pi/2} \frac{d\psi}{\sqrt{(a^2 - c^2) \sin^2 \psi + (c^2 - b^2)}} = \frac{1}{\sqrt{a^2 - b^2}} \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - \sin^2 \theta \cos^2 \psi}} = \frac{1}{\sqrt{a^2 - b^2}} K(\sin \theta), \]

and, using putting \( \cos \psi = t, \)

\[ \int_0^{\pi/2} \frac{\cos^2 \psi \, d\psi}{\sqrt{(a^2 - c^2) \sin^2 \psi + (c^2 - b^2)}} = \int_0^1 \frac{t^2 \, dt}{\sqrt{(a^2 - b^2) - (a^2 - c^2) t^2 \sqrt{1 - t^2}}} = \frac{1}{\sqrt{a^2 - b^2}} \int_0^1 \frac{t^2}{\sqrt{1 - \sin^2 \theta t^2} \sqrt{1 - t^2}} \, dt = -\frac{\sqrt{a^2 - b^2}}{a^2 - c^2} \left\{ E(\sin \theta) - K(\sin \theta) \right\}, \]

we know that (3.12) becomes the following:

(3.13) \[ 2\sqrt{a^2 - b^2} \left\{ E(\sin \theta) - \frac{c^2 - b^2}{a^2 - b^2} K(\sin \theta) \right\}. \]

Next we compute the first part of right-hand side of (3.11) on \([\theta, \pi - \theta].\)

Since

(3.14) \[ \int_{\theta}^{\pi - \theta} \sin^{2m} \alpha \, d\alpha = 2 \cos \theta F \left( \frac{1}{2}, \frac{1}{2} - m, \frac{3}{2}; \cos^2 \theta \right), \]
we have

\[
\int_0^{\pi-\theta} \arcsin \left( \frac{c}{\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}} \right) \, d\alpha \\
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(2n-1)!!(2n+2m-1)!!}{(2n)!!(2n+1)(2m)!!(2n-1)!!} \left( \frac{c}{|b|} \right)^{2n+1} (-k^2)^m \\
\times \int_0^{\pi-\theta} \sin^{2m} \alpha \, d\alpha \\
= 2 \cos \theta \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1} \left( \frac{c}{|b|} \right)^{2n+1} \\
\times \sum_{m=0}^{\infty} \frac{(2n+2m-1)!!}{(2m)!!(2n-1)!!} (-k)^m F \left( \frac{1}{2}, \frac{1}{2} - m, \frac{3}{2}; \cos^2 \theta \right).
\]

Summarizing, we obtain

\[
\sigma_H(\xi, \eta) = 4\pi |c| \theta + 4\sqrt{a^2 - b^2} \left\{ E(\sin \theta) - \frac{c^2 - b^2}{a^2 - b^2} K(\sin \theta) \right\} + 8c \cos \theta \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1} \left( \frac{c}{|b|} \right)^{2n+1} \\
\times \sum_{m=0}^{\infty} \frac{(2n+2m-1)!!}{(2m)!!(2n-1)!!} (-k)^m F \left( \frac{1}{2}, \frac{1}{2} - m, \frac{3}{2}; \cos^2 \theta \right).
\]

These equalities bring the proof to a conclusion.

Last of all we here give our result in the following table:
<table>
<thead>
<tr>
<th>$a$, $b$, $c$</th>
<th>$\sigma_H(\xi, \eta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq</td>
<td>b</td>
</tr>
<tr>
<td>$0 \leq</td>
<td>c</td>
</tr>
<tr>
<td>$0 &lt;</td>
<td>b</td>
</tr>
<tr>
<td>$0 \leq</td>
<td>b</td>
</tr>
</tbody>
</table>

where $\theta := \arcsin \sqrt{\frac{c^2-b^2}{a^2-b^2}}$.

REFERENCES


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