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ON CONJUGATE POINTS OF THE GROUP $H(2,1)$

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Abstract. Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra which has an inner product $\langle , \rangle$ and has an orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ for its center $\mathfrak{z}$ and the orthogonal complement $\mathfrak{v}$ of $\mathfrak{z}$. Then Each element Z of $\mathfrak{z}$ defines a skew symmetric linear map $J_Z : \mathfrak{v} \longrightarrow \mathfrak{v}$ given by $\langle J_Z X, Y \rangle = \langle Z, [X,Y] \rangle$ for all $X, Y \in \mathfrak{v}$. Let γ be a unit speed geodesic in a 2-step nilpotent Lie group $H(2,1)$ with its Lie algebra $\mathfrak{n}(2,1)$ and let its initial velocity $\gamma'(0)$ be given by $\gamma'(0) = Z_0 + X_0 \in \mathfrak{z} \oplus \mathfrak{v} = \mathfrak{n}(2,1)$ with its center component Z_0 nonzero. Then we showed that $\gamma(0)$ is conjugate to $\gamma(\frac{2n\pi}{\theta})$, where n is a nonzero intger and $-\theta^2$ is a nonzero eigenvalue of $J_{Z_0}^2$, along γ if and only if either X_0 is an eigenvector of $J_{Z_0}^2$ or $\mathrm{ad}X_0 : \mathfrak{v} \longrightarrow \mathfrak{z}$ is not surjective.

1. Introduction

Let $\mathfrak{n}$ denote a finite dimensional Lie algebra over the real numbers. The Lie algebra $\mathfrak{n}$ is called 2-step nilpotent Lie algebra if $[X,[Y,Z]] = 0$ for any $X, Y, Z \in \mathfrak{n}$. A Lie group N is said to be 2-step nilpotent if its Lie algebra $\mathfrak{n}$ is 2-step nilpotent. Throughout, N will denote a simply connected, 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ having center $\mathfrak{z}$. We shall use $\langle , \rangle$ to denote either an inner product on $\mathfrak{n}$ or the induced left-invariant Riemannian metric tensor on N. Let $\mathfrak{v}$ denote the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$.

Each element Z of $\mathfrak{z}$ defines a skew symmetric linear map $J_Z : \mathfrak{v} \longrightarrow \mathfrak{v}$ given by $J_Z(X) = (\mathrm{ad}X)^*(Z)$ for all $X \in \mathfrak{v}$, where $(\mathrm{ad}X)^*$ is the

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adjoint of ad$X$ relative to the inner product $\langle\ ,\ \rangle$. Equivalently and more usefuly J_Z is defined by the equation

$$\langle J_Z(X), Y\rangle = \langle [X,Y], Z\rangle$$

for all $X, Y \in \mathfrak{v}$. One of simple examples of such groups is the 3-dimensional Heisenberg group H_3 with its Lie algebra $\mathfrak{n}_3$ with an orthornomal basis X, Y, Z. and with the only nonzero bracket operations $[X,Y] = -[Y,Z] = Z$. The group H_3 may be generalized in the different two ways: H-type groups and groups $H(p,q)$. A 2-step nilpotent Lie group N with its Lie algebra $\mathfrak{n}$ is called H-type if it satisfies

$$J_Z^2 = -\langle Z, Z\rangle I \text{ for all } Z \in \mathfrak{z}.$$

And The group $H(p,q)$ is defined as the simply connected 2-step nilpotent Lie group whose Lie algebra $n(p,q)$ of dimension $p+q+pq$ has an orthonoramal basis

$$\{X_1, X_2, \cdots, X_p\} \cup \{Y_1, Y_2, \cdots, Y_q\} \cup \{Z_{ij} | i = 1, \cdots, p,\ j = 1, \cdots, q\}.$$

In $\mathfrak{n}(p,q)$, the bracket operations are given by

$$[X_i, Y_j] = -[X_i, Y_j] = Z_{ij}$$

for all $i = 1, \cdots, p$ and $j = 1, \cdots, q$ and all other brackets are zero. The first general studies for 2-step nilpotent Lie groups were done by P.Eberlein [1][2] and some works about conjugate points in 2-step nilpotent Lie groups followed. Especially, in 1997, Walschap showed [11] that for a nonsingular 2-step nilpotent Lie group with one dimensional center, the cut locus and the conjugate locus coincide, and he made an explicit determination of all first conjugate points in such a group. Gornet and Mast showed [3] that the first cut point of the starting point $\gamma(0)$ along a unit speed geodesic γ with initial velocity $\gamma'(0) = X_0 + Z_0$ for $X_0 \in \mathfrak{v}$ and $Z_0 \in \mathfrak{z}$ in a simply connected 2-step nilpotent Lie group N does not occur before length $\frac{2\pi}{\theta(Z)}$, where $\theta(Z)$ is the biggest of the norms of the eigenvalues of the skew-symmetric map J_z. Jang and Park later gave explicit formulas for all conjugate points along geodesics in any 2-step nilpotent Lie groups with one dimensional center [5]. And J. Kim [8] calculated all conjugate points of H-type groups. These last two works are generalized in a pseudo-Riemmannian version by Jang, Parker and Park[6][7]. Lee[10] considered a class of 2-step nilpotent Lie groups $(N, \langle , \rangle)$ satisfying the following more general condition

$$J_Z^2 = \langle SZ, Z\rangle A \text{ for all } Z \in \mathfrak{z}, \tag{1.1}$$

where S is a positive definite symmetric operator on $\mathfrak{z}$ and A is a negative defnite symmetric operator on $\mathfrak{v}$ and calculated all conjugate points for these groups. Note that this class of 2-step nilpotent Lie groups contains all groups with one dimensional center and all H-type groups. A result of Gornet and Mast mentioned above tells that along a geodesic γ in a 2-step nilpotent Lie group N with initial velocity $\gamma'(0) = Z_0 + X_0 \in \mathfrak{z} \oplus \mathfrak{v}$, $Z_0 \neq 0$ there no conjugate point occurs before length $\frac{2\pi}{\theta(Z)}$, where $\theta(Z)$ is the biggest of the norms of the eigenvalues of the skew-symmetric map J_Z. But in most important cases including groups satisfying (1.1), we can see that such geodesics have its first conjugate point at length $\frac{2\pi}{\theta(z)}$, where $\theta(z)$ is the biggest of the norms of the eigenvalues of the skew-symmetric map J_Z. This leads to the question that "Does every geodesic in a 2-step nilpotent Lie group with initial velocty of nonzero center component have its first conjugate point at length $\frac{2\pi}{\theta(Z)}$, where $\theta(Z)$ is the biggest of the norms of the eigenvalues of the skew-symmetric map J_Z?" In this paper we investigated some conjugate points in the group $H(2,1)$ and found that in some geodesics with initial velocity of nonzero center component the first conjugate point does not occur at length $\frac{2\pi}{\theta(Z)}$, where $\theta(Z)$ is the biggest of the norms of the eigenvalues of the skew-symmetric map J_Z . To study conjugate points, we use the Jacobi operator.

DEFINITION 1.1. Along the geodesic γ, the *Jacobi operator* is given by

$$R_{\dot\gamma} \bullet = R(\bullet\,, \dot\gamma)\dot\gamma\,,$$

where R denotes the Riemmannian curvature tensor.

For the reader's convenience, we recall that a *Jacobi field* along γ is a vector field along γ which is a solution of the *Jacobi equation*

$$\nabla_{\dot\gamma}^2 Y(t) + R_{\dot\gamma} Y(t) = 0$$

along γ, where ∇ denotes the Riemannian connection. The point $\gamma(t_0)$ is *conjugate* to the point $\gamma(0)$ if and only if there exists a nontrivial

Jacobi field Y along γ such that $Y(0) = Y(t_0) = 0$. The multiplicity of $\gamma(t_0)$ is equal to the number of lineary independent of Jacobi fields $Y(t)$ with $Y(0) = Y(t_0) = 0$. We will identify an element of $\mathfrak{n}$ with a left invariant vector field on N since T_eN may be identified with $\mathfrak{n}$, where e denotes the identity element of N. Since N is endowed with a left invariant metric, we will only consider Jacobi fields and conjugate points along geodesics emanating from the identity element of N. For the reader's convenience, we provide the statement of Proposition 2.1 from [6].

PROPOSITION 1.2. *Let γ be a geodesic in a simply connected 2-step nilpotent group N with $\gamma(0) = e$ and $\dot{\gamma}(0) = Z_0 + X_0 \in \mathfrak{z} \oplus \mathfrak{v} = \mathfrak{n}$. A vector field $Y(t) = Z(t) + e^{tJ}U(t)$ along γ, where $Z(t) \in \mathfrak{z}$ and $U(t) \in \mathfrak{v}$ for each t, is a Jacobi field if and only if*

$$\dot{Z}(t) - [e^{tJ}U(t), e^{tJ}X_0] = \zeta\,, \tag{1.2}$$

$$e^{tJ}\ddot{U}(t) + e^{tJ}J\dot{U}(t) - J_\zeta e^{tJ}X_0 = 0\,, \tag{1.3}$$

where $J = J_{Z_0}$ and $\zeta \in \mathfrak{z}$ is a constant.

2. Conjugate points of $H(2,1)$

We may assume that the Lie algebra $\mathfrak{n}(2,1)$ is spanned by an orthonormal basis $\{X_1, X_2, X_3\} \cup \{Z_1, Z_2\}$ and the only nonzero brackets are

$$[X_1, X_3] = -[X_3, X_1] = Z_1,\ [X_2, X_3] = -[X_3, X_2] = Z_2.$$

Let $H(2,1)$ be a simply connected two step nilpotent Lie group with its Lie algebra $\mathfrak{n}(2,1)$ and let γ be a unit geodesic in $H(2,1)$ such that $\gamma(0) = e$, $\dot{\gamma}(0) = X_0 + Z_0 \in \mathfrak{v} \oplus \mathfrak{z} = \mathfrak{n}(2,1)$ and $Z_0 \neq 0$. Then without loss of generality we may work under the assumption $Z_0 = \theta Z_1$ for a nonzero constant θ and $X_0 = x_{01}X_1 + x_{02}X_2 + x_{03}X_3$ for constants x_{01}, x_{02}, x_{03}. Suppose that a vector field $Y(t) = Z(t) + e^{tJ}U(t)$ is a *Jacobi field* along γ, where $J = J_{Z_0}$. Then by solving the equations (1.2) and (1.3) under the condotion $U(0) = 0$, $Z(0) = 0$ and $\zeta =$

$c\theta Z_1+\theta' Z_2$, where c and θ' are constants,we have

$$\begin{aligned} U(t) \;=\; &\{a_1(\cos t\theta-1)+a_2\sin t\theta+ctx_{01}-\frac{x_{02}\theta'}{\theta}t\cos t\theta\}X_1 \\ &+\{\frac{\theta'}{\theta^2}x_{01}\sin t\theta+\frac{\theta'}{\theta^2}x_{03}(\cos t\theta-1)+\alpha t\}X_2 \\ &+\{-a_1\sin t\theta+a_2(\cos t\theta-1)+\frac{x_{02}\theta'}{\theta}t\sin t\theta+ctx_{03}\}X_3 \qquad (2.1)\end{aligned}$$

and

$$\begin{aligned} Z(t) \;=\; &\{\frac{1}{\theta}(a_1x_{03}-a_2x_{01}-\frac{1}{\theta^2}\theta' x_{02}x_{01})\sin t\theta \\ &+\frac{1}{\theta}(a_2x_{03}+a_1x_{01}+\frac{1}{\theta^2}\theta' x_{02}x_{03})(1-\cos\theta) \\ &+\frac{\theta' x_{02}}{\theta^2}t(x_{03}\sin t\theta-x_{01}\cos t\theta)+(a_2x_{01}-a_1x_{03})t+c\theta t\}Z_1 \\ &+\{\alpha x_{01}(-\frac{t}{\theta}\cos t\theta+\frac{1}{\theta^2}\sin t\theta)+\alpha x_{03}(\frac{t}{\theta}\sin t\theta+\frac{1}{\theta^2}(\cos t\theta-1)) \\ &+\frac{1}{\theta}(\frac{\theta' x_{01}x_{03}}{\theta^2}-a_1x_{02})(\cos t\theta-1)+\frac{1}{\theta}(a_2x_{02}-\frac{\theta' x_{03}^2}{\theta^2})\sin t\theta \\ &+\frac{\theta'}{2\theta^2}(x_{01}^2+x_{03}^2)t-a_2x_{02}t+\frac{\theta'}{4\theta^3}\sin 2t\theta(x_{03}^2-x_{01}^2) \\ &-\frac{\theta'}{2\theta^3}x_{01}x_{03}(\cos 2t\theta-1)+\theta' t\}Z_2, \qquad (2.2)\end{aligned}$$

where a_1, a_2, and α are arbitrary constants. Suppose that $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ and $t_0=\frac{2n\pi}{\theta}$. Then we may assume that $U(\frac{2n\pi}{\theta})=0$ and $Z(\frac{2n\pi}{\theta})=0$ in (2.1) and (2.2). These conditions imply that

$$cx_{01}-\frac{x_{02}\theta'}{\theta}=0, \qquad (2.3)$$

$$\alpha=cx_{03}=0,$$

$$\frac{\theta' x_{01}x_{02}}{\theta^2}+a_2x_{01}-a_1x_{03}+c\theta=0, \qquad (2.4)$$

and

$$\frac{\theta'}{2\theta^2}(x_{01}^2+x_{03}^2)-a_2x_{02}+\theta'=0. \qquad (2.5)$$

Since $cx_{03} = 0$, we can consider 2-cases: $x_{03} \neq 0$ or $x_{03} = 0$. Suppose that $x_{03} \neq 0$. Then we have $c = 0$, from which and (2.3) it follows that $x_{02}\theta' = 0$. Multiplying θ' at both sides of (2.5), we have

$$\frac{\theta'^2}{2\theta^2}(x_{01}^2 + x_{03}^2) - a_2 x_{02}\theta' + \theta'^2 = 0.$$

This and $x_{02}\theta' = 0$ imply

$$\theta'^2\{1 + \frac{x_{01}^2 + x_{03}^2}{2\theta^2}\} = 0$$

which implies that $\theta' = 0$. Thus,if $x_{02} \neq 0$, then from (2,3) we have $a_2 = 0$. This and (2.4) imply that $a_1 = 0$. So all constants in (2.1) and (2.2) are zero, which means that that $U(t) \equiv 0$ and $Z(t) \equiv 0$. In other words there is no nonzero *Jacobi field* $Y(t)$ along γ which satisfies $Y(0) = Y(\frac{2n\pi}{\theta})$, which contradicts to the assumption $\gamma(0)$ is conjugate $\gamma(\frac{2n\pi}{\theta})$ along γ. Therefore we can conclude that if $x_{02} \neq 0$ and $x_{03} \neq 0$, then $\gamma(0)$ does not have conjugate points at $t = \frac{2n\pi}{\theta}$. And if $x_{03} \neq 0$ and $x_{02} = 0$, then we have $c = \alpha = \theta' = 0$ and $a_1 = \frac{a_2 x_{01}}{x_{03}}$ from equations (2.3)-(2.5) and $\alpha = cx_{03} = 0$. This imply that if $x_{02} = 0$ and $x_{03} \neq 0$, then $\gamma(0)$ has a conjugate point of multiplicity 1 at every $t = \frac{2n\pi}{\theta}$. Now assume that $x_{03} = 0$. Then from (2.4) we get

$$c = -\frac{\theta' x_{01} x_{02}}{\theta^3} - a_2\frac{x_{01}}{\theta}.$$

Substituting this into (2.3), we find

$$-\frac{\theta' x_{01}^2 x_{02}}{\theta^3} - a_2\frac{x_{01}^2}{\theta} - \frac{x_{02}\theta'}{\theta} = 0$$

or

$$-\frac{\theta' x_{01}^2 x_{02}}{\theta^2} - a_2 x_{01}^2 - x_{02}\theta' = 0. \tag{2.6}$$

From (2.5) and $x_{01}^2 + x_{02}^2 + \theta^2 = 1$, we find

$$\theta' = \frac{2\theta^2}{1+\theta^2} a_2 x_{02}.$$

Substituting this into (2.6), we get

$$a_2\left\{\frac{2\theta^2 x_{02}^2}{1+\theta^2}(1 + x_{01}^2) + x_{01}^2\right\} = 0.$$

If $x_{01}^2 + x_{02}^2 \neq 0$, then we have $a_2 = 0$. From this $c = \theta' = 0$ follows. In this case a_1 in (2.1) and (2.2) is arbitrary. Suppose that $x_{01} = x_{02} = 0$, then we may assume a_1 and a_2 are arbitrary. Thus we have the following.

PROPOSITION 2.1. *Let γ be a geodesic in a group $H(2,1)$ with $\gamma(0) = e$ and $\dot\gamma(0) = Z_0 + X_0 \in \mathfrak{z} \oplus \mathfrak{v} = \mathfrak{n}$ and $Z_0 \neq 0$. Then for the nonzero eigen value θ of the map J_{Z_0} $\gamma(0)$ has its conjugate points at points $t = \frac{2n\pi}{\theta}$ along γ if and only if either X_0 is an eigenvector of $J^2 = J_{Z_0}^2$ or the map $adX_0 : \mathfrak{v} \longrightarrow \mathfrak{z}$ given by $adX_0(Y) = [X_0, Y]$ for $Y \in \mathfrak{v}$ is not surjective. The multiplicties of conjugate points are 1 (or 2) if X_0 is nozero (or zero).*

In this paper we do not calculate all conjugate points of the group $H(2,1)$. But we show that if x_0 is an eigenvector of $J_{Z_0}^2$, then $\gamma(0)$ has conjugate points at some values of t which are different from $\frac{2n\pi}{\theta}$. Actually we will calculate under more general condition. First of all we need to recall the definition of Heigenberglike groups.

DEFINITION 2.2. A two step nilpotent Lie group N with its Lie algebra n and a left invariant metric $\langle , \rangle$ is said to be Hesenberglike if it satisfies

$$[X, J_Z X] = cZ$$

where c is a constant for all $Z \in \mathfrak{z}$ and any eigenvector X of J_Z^2

PROPOSITION 2.3. *Let N be a simply connected two step nilpotent Heigeberlike group with its Lie algebra $\mathfrak{n}$ and a leftinvariant metric. And let γ be a unit spped geodesic in N with initial velocity $\gamma'(0) = Z_0 + X_0 \in \mathfrak{z} \oplus \mathfrak{v} = \mathfrak{n}$. Suppose that X_0 is an eigenvector of $J_{Z_0}^2$ with nonzero eigenvalue $-\theta^2$. Then $\gamma(0)$ has a conjugate point at every point t contained in the set $\{t | \langle X_0, X_0 \rangle \frac{\theta t}{2} \cot \frac{\theta t}{2} = 1\}$.*

Proof. For a number t_0 contained in $\{t | \langle X_0, X_0 \rangle \frac{\theta t}{2} \cot \frac{\theta t}{2} = 1\}$ let

$$U(t) = ctX_0 + (e^{-tJ} - I)(e^{-t_0 J} - I)^{-1}(ct_0 X_0)$$

and

$$Z(t) = \alpha(t) Z_0$$

for a constant c, where

$$\alpha(t) = ct + \frac{\langle JX_0, (-J)^{-1}(e^{tJ} - I)(e^{-t_0 J} - I)^{-1} ct_0 X_0 - t(e^{t_0 J} - I)^{-1} ct_0 X_0 \rangle}{\langle Z_0, Z_0 \rangle}$$

Note that $J = J_{Z_0}$ is invertible on the eingenspace correspoinding the eigenvalue θ. Then by a direct computation we can see that $U(t)$ and $Z(t)$ satisfy (1.2) and (1.3). Also we can confirm that $U(0) = U(t_0) = 0$ and $Z(0) = Z(t_0) = 0$. □

COROLLARY 2.4. *Let γ be a unit spped geodesic in $H(2,1)$ with initial velocity $\gamma'(0) = Z_0 + X_0 \in \mathfrak{z} \oplus \mathfrak{v} = \mathfrak{n}(2,1)$. Suppose that X_0 is an eigenvector of $J_{z_0}^2$ with nonzero eigenvalue $-\theta^2$. Then $\gamma(0)$ has a conjugate point at every point t contained in the set $\{t | \langle X_0, X_0 \rangle \frac{\theta t}{2} \cot \frac{\theta t}{2} = 1\}$.*

Proof. For nonzero vecotors $X \in \mathfrak{v}$ and $Z \in \mathfrak{z}$ in the Lie algebra $\mathfrak{n}(2,1)$ we can see easily that $[X, J_Z X]$ is a multiple of the vector Z when the vector X is an eigenvector of J_Z^2. So the conclusion follows from Proposition2.3. □

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