Limiting Behavior of Tail Series of Independent Random Variable

독립인 확률변수들의 Tail 합의 극한 성질에 대하여

남윤우*, 장윤식**
Department of Computer Science and Statistics Air Force Academy*, SK Telecom**
Eunwoo Nam(ewnam@afa.ac.kr)*, Yoon-Sik Jang(yjang@sktelcom.com)**

해설

본 연구에서는, 서로 독립인 확률변수들의 합 $S_n$이 수렴하는 경우에, 확률변수들의 Tail 합
$T_n = S_n - S_{n-1} = \sum_{i=n}^{\infty} X_i$의 극한 성질을 연구함으로써, $S_n$이 하나의 확률변수 $S$로 수렴하는 속도를 연구한다.

Abstract

For the almost certain convergent series $S_n$ of independent random variables, by investigating the limiting behavior of the tail series, $T_n = S_n - S_{n-1} = \sum_{i=n}^{\infty} X_i$, the rate of convergence of the series $S_n$ to a random variable $S$ is studied in this paper. More specifically, the equivalence between the tail series weak law of large numbers and a limit law is established for a quasi-monotone decreasing sequence, thereby extending a result of previous work to the wider class of the norming constants.

Keyword : Rate of Convergence | Series of Random Variables | Tail Series | Almost Certain Convergence | Convergence in Probability

I. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables defined on a probability space $\Omega, \mathcal{F}, P$ and, as usual, their partial sums are denoted by $S_n = \sum_{i=1}^{n} X_i, n \geq 1$. If the series $S_n$ converges almost certainly (a.c.) to a random variable $S$, then (set $S_0 = 0$)
is a well-defined sequence of random variables (referred to as the tail series) with

$$T_n = S - S_{n-1} = \sum_{i=n}^{\infty} X_i, \quad n \geq 1$$

(1)

In this paper, we shall be concerned with the rate in which $S_n$ converges to $S$, or equivalently, in which the tail series $T_n$ converges to 0. The sequence of random variables $\{X_n, n \geq 1\}$ is said to obey the tail series weak law of large numbers (WLLN) (resp., strong law of large numbers (SLLN)) with respect to the norming constants $\{b_n, n \geq 1\}$ if the tail series $\{T_n, n \geq 1\}$ is well defined and for a given sequence of positive constants with $b_n = o(1)$,

$$\frac{T_n}{b_n} \overset{p}{\rightarrow} 0$$

(2)\hspace{1cm} (resp., $\frac{T_n}{b_n} \rightarrow 0\text{a.c.}$).

(3)

Pioneering work on the limiting behavior of the tail series $\{T_n, n \geq 1\}$ was conducted by Chow and Teicher[1] wherein they obtained a tail series law of the iterated logarithm (LIL). After numerous other investigations on the tail series LIL problem had been made, the tail series SLLN problem was studied by Klesov[3][4], Mikosch[5], Nam and Rosalsky[7] and Nam[6].

Recalling that (1) is equivalent to

$$\sup_{j \geq n} |T_j| \overset{p}{\rightarrow} 0,$$

Nam and Rosalsky[8] provided various sets of conditions in order for the limit law

$$\frac{\sup_{j \geq n} |T_j|}{b_n} \overset{p}{\rightarrow} 0$$

(4)

to hold for a given sequence of positive constants $\{b_n, n \geq 1\}$ and then this result was generalized by Sung and Volodin[11]. When $0 < b_n \downarrow$, Nam and Rosalsky[8] observed that the tail series SLLN (3) implies the limit law (4) and that (3) is indeed even equivalent to the apparently stronger limit law

$$\sup_{j \geq n} \frac{|T_j|}{b_n} \rightarrow 0\text{a.c.}$$

(5)

They also provided an example wherein the limit law (4) holds with $0 < b_n \downarrow$, but SLLN (3) fails. In their follow-up article, Nam and Rosalsky[9] proved apropos of the sequence of random variables that the tail series WLLN (2) and the limit law (4) are indeed equivalent when $0 < b_n \downarrow$, thereby establishing the validity of a conjecture posed by Nam and Rosalsky[8]. Moreover, Nam and Rosalsky[9] provided an example showing that without the monotonicity condition on $\{b_n, n \geq 1\}$ the tail series SLLN (3) (as well as tail series WLLN (2)) does not imply either of the limit laws (4) or (5).

Rather than taking the monotone decreasing sequence of positive constants, let us employ the sequence of positive constants $\{b_n, n \geq 1\}$ which is quasi-monotone decreasing in the sense that there exists a positive constant $C < \infty$ such that
\[ b_j \leq C b_n \text{ whenever } j \geq n \geq 1 \quad (6) \]

(Of course if \( b_n \downarrow \), then (6) holds with \( C = 1 \).

Then, for the quasi-monotone decreasing sequence of positive constants \( \{b_n, n \geq 1\} \) it follows from

\[
\sup_{j \geq n} \left| T_j \right| \leq C \sup_{j \geq n} b_j
\]

that the tail series SLLN (3) implies the limit law (4) and that (3) is indeed equivalent to the apparently stronger limit law (5), thereby extending the observation in Nam and Rosalsky[8].

The main purpose of the current work is to extend (in Theorem 1 below) the Nam and Rosalsky[9] equivalence between the tail series WLLN (2) and the limit law (4) from the monotone decreasing constant case to the case of the quasi-monotone decreasing constants. As will become apparent, the formulation and proof of the ensuing Theorem 1 owe much to the work of Nam and Rosalsky[9].

II. PRELIMINARY LEMMAS

Some lemmas are needed in order to establish the main result. Lemma 1 provides the tail series analogue of the classical Levy inequality of partial sums of independent random variables and is due to Nam and Rosalsky[9].

Lemma 1 (Nam and Rosalsky[9]). Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables with \( \sum_{n} X_n \) converging a.e. Then

the tail series \( \{T_n = \sum_{i=n}^{\infty} X_i, n \geq 1\} \) is a well-defined sequence of random variables, and for every \( \epsilon > 0 \) the inequalities

\[
P\left( \sup_{j \geq n} \left| T_j + \text{med}(S_{n,j-1}) \right| \geq \epsilon \right) \leq 2P\{T_n \geq \epsilon\}
\]

and

\[
P\left( \sup_{j \geq n} \left| T_j + \text{med}(S_{n,j-1}) \right| \geq \epsilon \right) \leq 2P\{T_n \geq \epsilon\}
\]

hold where (set \( S_{n,n-1} = 0 \))

\[
S_{n,j} = \sum_{i=n}^{j} X_i, j \geq n \geq 1.
\]

Lemmas 2 and 3 extend lemmas of Nam and Rosalsky[9] to the case of the quasi-monotone decreasing constants.

Lemma 2. Let \( \{Y_n, n \geq 1\} \) be a sequence of random variables and let \( \{b_n, n \geq 1\} \) be a sequence of positive constants which is quasi-monotone decreasing in the sense that (6) holds. If

\[
\frac{Y_n}{b_n} \overset{p}{\to} 0, \quad (7)
\]

then we obtain

\[
\sup_{j \geq n} \left| \text{med}(Y_j - Y_n) \right| = o(b_n).
\]

Proof. For arbitrary \( \epsilon > 0 \), (7) ensures that there exists an integer \( N_\epsilon \) such that
\[ P \left( \left| Y_n \right| \geq \frac{\varepsilon}{2b_n} \right) < \frac{1}{4} \quad \text{and} \quad \left| Y_n \right| \geq \frac{\varepsilon}{2C} \right) < \frac{1}{4} \]

for all \( n \geq N \). Then for all \( j \geq n \geq N \),

\[ P \left( \frac{|Y_j - Y_n|}{b_n} \geq \varepsilon \right) \leq P \left( \frac{|Y_j|}{b_n} \geq \varepsilon \right) + P \left( \frac{|Y_n|}{b_n} \geq \varepsilon \right) \leq P \left( \frac{|Y_j|}{b_j} \geq \frac{\varepsilon}{2} \right) + P \left( \frac{|Y_n|}{b_n} \geq \frac{\varepsilon}{2} \right) \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \]

This implies (see, e.g., Chow and Teicher[2], p. 72) that

\[ \frac{\text{med}(Y_j - Y_n)}{b_n} \leq \varepsilon \quad \text{for all} \quad j \geq n \geq N. \]

Thus

\[ \sup_{j \geq n} \frac{\text{med}(Y_j - Y_n)}{b_n} \leq \varepsilon \quad \text{for all} \quad n \geq N, \]

proving the lemma. \( \square \)

In Lemma 2, replacing \( \{ Y_n, n \geq 1 \} \) by a tail series \( \{ T_n, n \geq 1 \} \) yields the following lemma as an immediate corollary.

Lemma 3. Let \( \{ X_n, n \geq 1 \} \) be a sequence of independent random variables with \( \sum_{n=1}^{\infty} X_n \) converging a.c. and tail series \( T_n = \sum_{i=1}^{\infty} X_i, n \geq 1 \), and let \( \{ b_n, n \geq 1 \} \) be a sequence of positive constants which is quasi-monotone decreasing in the sense that (6) holds. If the tail series WLLN

\[ \frac{T_n}{b_n} \overset{p}{\to} 0 \]

holds, then

\[ \sup_{j \geq n} \text{med}(S_{n,j-1}) = o(b_n) \]

where (set \( S_{n,1} = 0 \))

\[ S_{n,j} = \sum_{i=1}^{j} X_i, j \geq n \geq 1. \]

III. THE MAIN RESULT

Combining Lemmas 1 and 3 accounted for yields Theorem 1, which establishes the equivalence between the tail series WLLN and a limit law for a sequence of quasi-monotone decreasing constants.

Theorem 1. Let \( \{ X_n, n \geq 1 \} \) be a sequence of independent random variables with \( \sum_{n=1}^{\infty} X_n \) converging a.c. and tail series \( T_n = \sum_{i=1}^{\infty} X_i, n \geq 1 \), and let \( \{ b_n, n \geq 1 \} \) be a sequence of positive constants which is quasi-monotone decreasing in the sense that (6) holds. Then the tail series WLLN

\[ \frac{T_n}{b_n} \overset{p}{\to} 0 \] (8)
and the limit law

\[ \sup_{j \geq n} \left| T_j \right| \frac{r}{b_n} \to 0 \]  

(9)

are equivalent.

Proof. Since (9) clearly implies (8), it is enough to show that (8) implies (9). For arbitrary \( \epsilon > 0 \), it follows from (8) and Lemma 1 (with \( \epsilon \) replaced by \( \epsilon b_n \)) that

\[ \sup_{j \geq n} \left| T_j + \text{med}(S_{n,j-1}) \right| \frac{r}{b_n} \to 0. \]  

(10)

Then

\[ \frac{\sup_{j \geq n} \left| T_j \right|}{b_n} \leq \frac{\sup_{j \geq n} \left| T_j + \text{med}(S_{n,j-1}) \right|}{b_n} + \frac{\sup_{j \geq n} \text{med}(S_{n,j-1})}{b_n} \to 0 \]

(by (10) and Lemma 3). □

Theorem 1, which is proved by employing the tail series analogue (Lemma 1) of the classical Levy inequality of independent random variables, is a complete counterpart in the random variable case of the Nam et al.[10] equivalence between (8) and (9) in a Banach space setting. An immediate application of above Theorem 1 to a theorem of Nam and Rosalsky [9] yields the following corollary, which extends a corollary of them to the quasi-monotone decreasing constant case, but the corollary is still a special case of a limit theorem of Nam and Rosalsky[8].

Corollary 1. Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables with \( E(X_n) = 0 \), \( n \geq 1 \) and let \( \{b_n, n \geq 1\} \) be a sequence of positive constants which is quasi-monotone decreasing in the sense that (6) holds. Assume that

\[ \sum_{j=1}^{\infty} E\left[|X_j|^p\right] = o(b_n^p) \text{ for some } 1 \leq p \leq 2 \]

Then the tail series \( \{T_n = \sum_{i=1}^{\infty} X_i, n \geq 1\} \) is a well-defined sequence of random variables obeying the limit law (9).

References


남 윤우(Eunwoo Nam) 정회원

- 1979년 3월 : 공군사관학교 국방관리학과(이학사)
- 1982년 2월 : 서울대대학교 자연과학수학과(이학사)
- 1985년 2월 : 서울대학교 자연과학계산통계학과(통계학석사)
- 1985년 3월~1988년 8월 : 공군사관학교 수학과 교수
- 1992년 12월 : University of Florida, Dept. of Statistics(통계학박사)
- 1992년 3월~현재 : 공군사관학교 전산통계학과 교수

<관심분야> : 확률론(Theory of Probability), 수렴이론(Theory of Convergence)

장윤식(Yoon-Sik Jang) 정회원

- 1998년 2월 : 광운대학교 전자공학과 대학원(공학석사)
- 현재 : SK텔레콤 상무이사

<관심분야> : 이동통신, 모바일콘텐츠