On the Tail Series Laws of Large Numbers for Independent Random Elements in Banach Spaces

Banach 공간에서 독립인 확률요소들의 Tail 합에 대한 대수의 법칙에 대하여

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요약

본 연구에서는, Banach 공간의 값을 갖는 확률요소들의 합 $S_n = \sum_{i=1}^{n} V_i$이 수렴하는 경우에, Tail 합 $T_n = S - S_{n-1} = \sum_{i=n}^{\infty} V_i$에 대한 대수의 법칙을 고찰하여 S_n이 하나의 확률변수 S로 수렴하는 속도를 연구한다. 좀 더 구체적으로 말하자면, 확률변수들의 Tail 합과 확률요소들의 Tail 합에 대한 극한 성질의 유사성을 연구하여, Banach 공간에서 독립인 확률요소들의 Tail 합에 대한 약 대수의 법칙과 하나의 수렴법칙이 동등함을 기술하는 기존의 정리를 다른 대체적인 방법으로 증명한다.

■ 중심어 : | 수렴 속도 | Banach 공간의 확률요소들의 합 | Tail 합 | 강 대수의 법칙 | 약 대수의 법칙 |

Abstract

For the almost certainly convergent series $S_n = \sum_{i=1}^{n} V_i$ of independent random elements in Banach spaces, by investigating tail series laws of large numbers, the rate of convergence of the series S_n to a random variable S is studied in this paper. More specifically, by studying the duality between the limiting behavior of the tail series $T_n = S - S_{n-1} = \sum_{i=n}^{\infty} V_i$ of random variables and that of Banach space valued random elements, an alternative way of proving a result of the previous work, which establishes the equivalence between the tail series weak law of large numbers and a limit law, is provided in a Banach space setting.

■ keyword : | Rate of Convergence | Series of Random Elements in Banach Space | Tail Series | Strong Law of Large Numbers | Weak Law of Large Numbers |

I. INTRODUCTION

Let $\{V_n, n \geq 1\}$ be a sequence of independent random elements defined on a probability space $\{\Omega, \mathbb{F}, P\}$ and taking values in a real separable Banach space X with norm $\|\cdot\|$. As usual, their partial sums are denoted by $S_n = \sum_{i=1}^{n} V_i,\ n \geq 1.$

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Woyczyński [14][15], Jain [7], Giné et al. [6], and Etemadi [5] among others provided conditions under which the series S_n converges almost certainly (a.c.) (to a random element). When the series S_n converges a.c. to a random element S, then (set $S_0 = 0$)

$$T_n = S - S_{n-1} = \sum_{j=n}^{\infty} V_j,\ n \geqq 1$$

is a well-defined sequence of random elements (referred to as the *tail series*) with

$$T_n \to 0\ a.c.$$

The sequence of random elements $\{V_n, n \geq 1\}$ is said to obey the tail series *weak law of large numbers* (WLLN) (resp., *strong law of large numbers* (SLLN)) with respect to the norming constants $\{b_n, n \geq 1\}$ if the tail series $\{T_n, n \geq 1\}$ is well defined and for a given sequence of positive constants with $b_n = o(1)$,

$$\frac{T_n}{b_n} \xrightarrow{P} 0 \tag{1}$$

$$(resp., \frac{T_n}{b_n} \to 0\ a.c.). \tag{2}$$

In this paper, for independent summands of Banach space valued random elements, we shall be concerned with the rate in which S_n converges to S, or equivalently, in which the tail series T_n converges to 0.

Pioneering work on the limiting behavior of the tail series $\{T_n, n \geq 1\}$ was conducted by Chow and Teicher [1] wherein they obtained a tail series law of the iterated logarithm of independent random variables. The tail series problem in Banach space setting was investigated by Dianliang [3][4], Nam et al. [11] and Hu et al. [12]. Also, Rosalsky and Rosenblatt [13] provided conditions

$$\frac{\sup_{j \geq n} \| T_j \|}{b_n} \xrightarrow{P} 0 \tag{3}$$

to hold for a given sequence of positive constants $\{b_n, n \geq 1\}$. When $0 < b_n \downarrow$, Rosalsky and Rosenblatt [13] observed that the tail series SLLN (2) implies the limit law (3) and that (2) is indeed even equivalent to the apparently stronger limit law

$$\frac{\sup_{j \geq n} \| T_j \|}{b_n} \to 0 a.c. \tag{4}$$

Rather than taking the monotone decreasing sequence of positive constants, let us employ the sequence of positive constants $\{b_n, n \geq 1\}$ which is *quasi-monotone decreasing* in the sense that there exists a positive constant $C < \infty$ such that

$$b_j \leqq Cb_n \ whenever\ j \geqq n \geqq 1 \tag{5}$$

(Of course if $b_n \downarrow$, then (5) holds with $C = 1$). Then, for the quasi-monotone decreasing sequence of positive constants $\{b_n, n \geq 1\}$ it follows from

$$\frac{\sup_{j \geq n} \| T_j \|}{b_n} \leq C \sup_{j \geq n} \frac{\| T_j \|}{b_j}$$

that the tail series SLLN (2) implies the limit law (3) and that (2) is indeed equivalent to the apparently stronger limit law (4), thereby extending Rosalsky and Rosenblatt's [13] observation to the wider class of norming constants.

For an a.c. convergent series of random variables, Nam and Rosalsky [10] proved apropos of the tail series of independent summands that the tail series WLLN (1) and apparently stronger limit law (3) are indeed equivalent when $0 < b_n \downarrow$, and they provided an example showing that without the monotonicity condition on $\{b_n, n \geq 1\}$, the tail series WLLN (1) does not imply the limit law (3). This example reveals that (1) does not necessarily imply (3) without the quasi-monotonicity condition (5). It is important to note, in the random variable case, that the key inequality used in order to prove the Nam and Rosalsky [10] equivalence between (1) and (3) is the tail series analogue of the classical Lévy inequality for partial sums. As in the case of random variables, Nam et al' [11] equivalence (Theorem 2 below) between (1) and (3) in a Banach space setting can be reproved by employing the tail series analogue (Theorem 1 below) of Lévy inequality for random elements. As will become apparent, the formulation and proof of the ensuing Theorem 1 owe much to the work of Nam et al. [11].

II. MAINSTREAM

Not only is Theorem 1 a tail series analogue of the Lévy inequality for Banach space valued random elements, but it also an extension of tail series Lévy inequality from the random variable case to the case of Banach space setting, so it demonstrates the duality between the limiting behavior of the tail series of random variables and that of Banach space valued random elements and it may be of separate interest.

Theorem 1. *Let* $\{V_n, n \geq 1\}$ *be a sequence of independent and symmetric random elements in a real separable Banach space with* $\sum_{n=1}^{\infty} V_n$ *converging a.c. Then the tail series* $\{T_n = \sum_{j=n}^{\infty} V_j, n \geq 1\}$ *is a well-defined sequence of random elements satisfying*

$$P\left\{\sup_{j\geq n}\|T_j\| \geq \varepsilon\right\} \leq 2P\{\|T_n\| \geq \varepsilon\}, \varepsilon > 0.$$

In order to prove Theorem 1, Lemmas 1 and 2, which are the classical Lévy inequality(see, e.g., Laha and Rohatgi [8]) for random elements taking values in a Banach space and a modification of it, are needed.

Lemma 1 (Laha and Rohatgi [8]). *Let* $\{V_i, 1 \leqq i \leqq n\}$ *be a sequence of independent and symmetric random elements in a real separable Banach space. Then setting* $S_j = \sum_{i=1}^{j} V_i, 1 \leq j \leq n,$

$$P\{\max_{1\leq j\leq n}\|S_j\| \geq \varepsilon\} \leq 2P\{\|S_n\| \geq \varepsilon\}, \varepsilon > 0.$$

Lemma 2. *Let* $\{V_i, k \leqq i \leqq n\}$ *be a sequence of independent and symmetric random elements in a real separable Banach space. Then setting*

$$S_{j,n} = \sum_{i=j}^{n} V_i, k \leq j \leq n,$$

$$P\{\max_{k\leq j\leq n}\|S_{j,n}\| \geq \varepsilon\} \leq 2P\{\|S_{k,n}\| \geq \varepsilon\}, \varepsilon > 0.$$

Proof. Set $S_j^{(n)} = \sum_{i=1}^{j} V_{n+1-i}, 1 \leq j \leq n+1-k.$

Note at the outset that

$$\{S_{j,n}, j = k,\ldots,n\} = \{S_j^{(n)}, j = n+1-k,\ldots,1\}.$$

Then for $\epsilon > 0$,

$$P\{\max_{k\leq j\leq n}\|S_{j,n}\| \geq \varepsilon\} = P\{\max_{1\leq j\leq n+1-k}\|S_j^{(n)}\| \geq \varepsilon\}$$

$$\leq 2P\{\|S_{n+1-k}^{(n)}\| \geq \varepsilon\} \text{ (by Lemma 1)}$$

$$\leq 2P\{\|S_{k,n}\| \geq \varepsilon\} \quad \square$$

Proof of Theorem 1. Let $1 \leq n < N < M$. For $n \leq j \leq N$, set

$$S_{j,M} = \sum_{i=j}^{M} V_i.$$

Then

$$\|S_{j,M}\| - \|T_j\| \leq \|S_{j,M} - T_j\| \to 0 \, a.c. \text{ as } M \to \infty$$

implying

$$\|T_j\| = \lim_{M\to\infty}\|S_{j,M}\| a.c. \qquad (6)$$

Thus, the tail series $\{ T_n = \sum_{j=n}^{\infty} V_j, n \geq 1 \}$ is a well-defined sequence of random elements. Observe that for $\epsilon > 0$,

$$P\left\{\max_{n\le j\le N}\|T_j\| > \varepsilon\right\} = P\left\{\max_{n\le j\le N}\lim_{M\to\infty}\|S_{j,M}\| > \varepsilon\right\} \text{(by (6))}$$

$$= P\left\{\lim_{M\to\infty}\max_{n\le j\le N}\|S_{j,M}\| > \varepsilon\right\}$$

$$\le \liminf_{M\to\infty} P\left\{\max_{n\le j\le N}\|S_{j,M}\| > \varepsilon\right\}$$

(by Theorem 8.1.3 of Chow and Teicher [2])

$$\le \liminf_{M\to\infty} P\left\{\max_{n\le j\le M}\|S_{j,M}\| > \varepsilon\right\}$$

$$\le 2\liminf_{M\to\infty} P\left\{\|S_{n,M}\| \ge \varepsilon\right\} \text{ (by Lemma 2)}$$

$$\le 2\limsup_{M\to\infty} P\left\{\|S_{n,M}\| \ge \varepsilon\right\}$$

$$\le 2P\left\{\lim_{M\to\infty}\|S_{n,M}\| \ge \varepsilon\right\}$$

(by Theorem 8.1.3 of Chow and Teicher [2])

$$= 2P\{\|T_n\| \ge \varepsilon\}.$$

Letting $N\to\infty$ yields

$$P\left\{\sup_{j\ge n}\|T_j\| > \varepsilon\right\} \le 2P\{\|T_n\| > \varepsilon\}.$$

Now, replace ϵ by $\epsilon - \frac{1}{m}$ (for integer $m > \frac{1}{\epsilon}$) and then the lemma follows by letting $m\to\infty$. □

In Theorem 2 below, for a sequence of quasi-monotone decreasing constants, Nam et al. [11] extended Nam and Rosalsky's [10] result which pertained to the random variables case to the case of Banach space valued random elements, by virtue of the maximal inequality of Etemadi [5] in a Banach space setting.

Theorem 2 (Nam et al. [11]). *Let* $\{ V_n, n \geq 1 \}$ *be a sequence of independent random* elements in a real separable Banach space *with* $\sum_{n=1}^{\infty} V_n$ *converging a.c. and tail series* $T_n = \sum_{j=n}^{\infty} V_j, n \geq 1$, *and let* $\{b_n, n \geq 1\}$ *be a sequence of positive constants which is quasi-monotone decreasing in the sense that (5) holds. Then the tail series WLLN (1) and the limit law (3) are equivalent.*

Recalling that, in the random variable case, when $b_n \downarrow$, the key inequality used in order to prove the Nam and Rosalsky [10] equivalence between (1) and (3) is the classical Lévy inequality, Theorem 2 can be reproved by hiring Lévy inequality (Lemma 1) in a Banach space setting. As discussed in Nam et al. [11], the proof of the theorem introduces a symmetrization procedure.

Proof. Since (3) clearly implies (1), it need to be established that (1) implies (3). Observe at the outset that (see, e.g., Loéve [9])

$$\frac{\|T_n\|}{b_n} \xrightarrow{P} 0 \Rightarrow med\left(\frac{\|T_n\|}{b_n}\right) \to 0.$$

Then for arbitrary $\epsilon > 0$, there exist an integer N_ϵ such that

$$med\left(\frac{\|T_n\|}{b_n}\right) < \frac{\varepsilon}{2C} \text{ for all } n \geq N_\epsilon.$$

Thereby

$$\sup_{j\ge n} med\left(\frac{\|T_j\|}{b_j}\right) \le \frac{\varepsilon}{2C}.$$

Thus, for all $n \geq N_\epsilon$

$$P\left\{\frac{\sup_{j\geq n}\|T_j\|}{b_n}\geq\varepsilon\right\}\leq P\left\{\frac{\sup_{j\geq n}\|T_j\|}{b_n}\geq\frac{\varepsilon}{2}+C\sup_{j\geq n} med\left(\frac{\|T_j\|}{b_j}\right)\right\}$$

$$=P\left\{\frac{\sup_{j\geq n}\|T_j\|}{b_n}-C\sup_{j\geq n} med\left(\frac{\|T_j\|}{b_j}\right)\geq\frac{\varepsilon}{2}\right\}$$

$$\leq P\left\{\sup_{j\geq n}\left[\frac{\|T_j\|}{b_n}-C\frac{med\|T_j\|}{b_j}\right]\geq\frac{\varepsilon}{2}\right\}$$

$$\leq P\left\{\sup_{j\geq n}\left[\|T_j\|-med\|T_j\|\right]\geq\frac{\varepsilon b_n}{2}\right\}$$

(since $b_j \leqq Cb_n$)

$$\leq 2P\left\{\sup_{j\geq n}\|T_j^s\|\geq\frac{\varepsilon b_n}{2}\right\}$$

(by Symmetrization inequality of Loéve [9])

$$\leq 4P\left\{\|T_j^s\|\geq\frac{\varepsilon b_n}{2}\right\}$$ (by Theorem 1)

$$\leq 8P\left\{\frac{\|T_n\|}{b_n}\geq\frac{\varepsilon}{4}\right\}$$

(by Weak symmetrization inequality of Loéve [9])
$=o(1)$ (by (1)). □

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