ESTIMATION OF THE NUMBER OF ROOTS ON THE COMPLEMENT

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Abstract. Let $f : (X, A) \rightarrow (Y, B)$ be a map of pairs of compact polyhedra. A surplus Nielsen root number $SN(f; X \setminus A, c)$ is defined which is lower bound for the number of roots on $X \setminus A$ for all maps in the homotopy class of $f$. It is shown that for many pairs this lower bound is the best possible one, as $SN(f; X \setminus A, c)$ can be realized without by-passing condition.

1. INTRODUCTION

Zhao considered the minimum number $MF[f; X \setminus A]$ of fixed points on the complement $X \setminus A$ and defined the Nielsen number on the complementary space of a given map $f : (X, A) \rightarrow (X, A)$, $N(f; X \setminus A)$ which is a lower bound for $MF[f; X \setminus A]$ and has the same basic properties as $N(f; X, A)([8])$. Zhao[9] introduced a new concept “surplus Nielsen number”, $SN(f; X \setminus A)$, which is a lower bound for the number of fixed points on $X \setminus A$ for all maps in the homotopy class of $f$. And he showed that for many pairs this lower bound is the best possible one, as $SN(f; X \setminus A)$ can be realized without the by-passing condition.

This paper is a analogy of Zhao[9]. To determine the minimal number $MR[f; X \setminus A, c]$ of roots at $c \in B$ on $X \setminus A$ for all maps in the homotopy class of a given map $f : (X, A) \rightarrow (Y, B)$, the Nielsen root number on the complementary space $N(f; X \setminus A, c)$ is introduced in Yang[7], which is a lower bound for $MR[f; X \setminus A, c]$.

It is the purpose of this paper to introduce a better lower bound for $MR[f; X \setminus A, c]$, which can be realized without the hypothesis that $A$ can be by-passed. The method used here follows that of Zhao[9]. After some preparation in section 2, the
surplus Nielsen root number of \( f \) on \( X \setminus A \), \( SN(f; X \setminus A, c) \), is defined (Definition 3.1), \( SN(f; X \setminus A, c) \geq N(f; X \setminus A, c) \). In section 4, we shall prove that \( SN(f; X \setminus A, c) = MR[f; X \setminus A, c] \) if \( X \) and every component of \( X \setminus A \) is a manifold with dimension different from 2.

2. Root Classes on the Subspace

Let \( f : X \to Y \) be a map of compact polyhedron, and let \( U \) be a subset of \( X \) which has finitely many arcwise connected components. A root class of \( f|_U : U \to Y \) is said to be a root class of \( f \) on \( U \).

**Definition 2.1.** Two roots \( x_0 \) and \( x_1 \) of \( f : X \to Y \) on \( U \) are said to belong to the same root class of \( f \) on \( U \) if there exists a path \( \alpha \) in \( U \) from \( x_0 \) to \( x_1 \) such that \( e_c \simeq f \cdot \alpha \rel{0,1} \).

It is obvious that every root class of \( f \) on \( U \) belongs to a root class of \( f \). The root classes of \( f \) on \( U \) have the same basic properties as original root classes. We repeat some basic properties of root classes, which can be found in [4].

**Proposition 2.2.** The root set \( \Gamma(f|_U) \) of \( f \) on \( U \) splits into a disjoint union of root classes on \( U \).

**Proposition 2.3.** Every root class of \( f \) on \( U \) is an open subset of \( \Gamma(f|_U) \).

**Definition 2.4.** Let \( H : f_0 \simeq f_1 : X \to Y \) be a homotopy. For \( x_0 \in \Gamma(f_0|_U) \) and \( x_1 \in \Gamma(f_1|_U) \), we say that \( x_0 \) and \( x_1 \) are \( H \)-related on \( U \) if there exists a path \( \beta \) in \( U \) from \( x_0 \) to \( x_1 \) such that \( H(\beta(t), t) \simeq e_c \rel{0,1} \).

**Proposition 2.5.** Let \( H : f_0 \simeq f_1 : X \to Y \) be a homotopy. Let \( x_0 \) belong to a root class \( R_0 \) of \( f_0 \) on \( U \) and let \( x_1 \) belong to a root class \( R_1 \) of \( f_1 \) on \( U \). Let \( x_0 \) and \( x_1 \) be \( H \)-related on \( U \). Then \( x_0' \) and \( x_1' \) are \( H \)-related on \( U \) for any \( x_0' \in R_0 \) and \( x_1' \in R_1 \).

**Definition 2.6.** For a root class \( R_0 \) of \( f_0 \) on \( U \) and a root class \( R_1 \) of \( f_1 \) on \( U \), we say that \( R_0 \) and \( R_1 \) are \( H \)-related on \( U \) if there exist \( x_0 \in R_0 \) and \( x_1 \in R_1 \) such that \( x_0 \) and \( x_1 \) are \( H \)-related on \( U \).

By Proposition 2.5, this definition is independent of the choice of \( x_0 \) and \( x_1 \).

For a subset \( D \subset X \times I \), the subset \( D_t = \{ x \in X \mid (x, t) \in D \} \) of \( X \) will be called the \( t \)-slice of \( D \).
Proposition 2.7. Let $H : f_0 \simeq f_1 : X \to Y$ be a homotopy, and let $\mathbb{R}_0$ and $\mathbb{R}_1$ be root classes of $f_0$ and $f_1$ on $U$ respectively. Then $\mathbb{R}_0$ and $\mathbb{R}_1$ are $H$-related on $U$ if and only if they are respectively the 0- and 1-slices of a single root class of $H$ on $U \times I$.

3. Surplus Nielsen Root Number $SN(f; X \setminus A, c)$

Let $(X, A)$ be a pair of compact polyhedra, then $(X \setminus A)$ consists of finitely many components and every component of $X \setminus A$ is arcwise connected and semilocally 1-connected. Let us consider the map $f : (X, A) \to (Y, B)$, and a homotopy of the form $H : (X \times I, A \times I) \to (Y, B)$.

Definition 3.1. A root class $\mathbb{R}$ of $f$ on $X \setminus A$ is said to be a nonsurplus root class of $f$ on $X \setminus A$ if there is a point $x_0 \in \mathbb{R}$ and there is a path $\alpha : I, 0, I \setminus \{1\}, 1 \to X, x_0, X \setminus A$. $A$ such that

$$f \cdot \alpha \simeq e_c : I, 0, 1 \to Y, c, B.$$

A root class of $f$ on $X \setminus A$ which is not a nonsurplus root class of $f$ on $X \setminus A$ is said to be a surplus root class of $f$ on $X \setminus A$.

By Definition 2.1 and Definition 3.1, we have

Corollary 3.2. A root $x_0$ of $f$ on $X \setminus A$ belongs to a nonsurplus root class if and only if there exists a path from $x_0$ to $A$ satisfying the conditions of Definition 3.1.

Theorem 3.3. The number of surplus root classes of $f$ on $X \setminus A$ is finite, each of them is a compact subset of $X$.

Proof. Let $\mathbb{R}$ be a surplus root class of $f$ on $X \setminus A$, we shall prove $\mathbb{R}$ is compact.

Suppose $x_0 \in X - \mathbb{R}$, it suffices to find a neighborhood $V$ of $x_0$ in $X$ such that $V \cap \mathbb{R} = \emptyset$.

(i) If $x_0 \notin \Gamma(f; c)$, we can take $V = X - \Gamma(f; c)$.

(ii) If $x_0 \in \Gamma(f; X \setminus A, c)$, then $x_0$ belongs to a root class $\mathbb{R}'$ of $f$ on $X - A$. By Proposition 2.2 and 2.3, $\mathbb{R}' \cap \mathbb{R} = \emptyset$ and there is a neighborhood $V$ of $x_0$ in $X \setminus A$ such that $V \cap \Gamma(f; X \setminus A, c) \subset \mathbb{R}'$. Since $X \setminus A$ is an open subset of $X$, $V$ is also a neighborhood of $x_0$ in $X$ and $V \cap \mathbb{R} \subset \mathbb{R}' \cap \mathbb{R} = \emptyset$. 

(iii) If $x_0 \in A \cap \Gamma(f, c)$. Let $C$ be the component of $X \setminus A$ containing $R$. Assume that $x_0 \in \text{cl}(C)$, otherwise we can take $V = X - \text{cl}(C)$. Pick a neighborhood $W$ of $c$ such that every loop in $W$ is trivial in $Y$. There is an arclwise connected neighborhood $V$ of $x_0$ such that $V \subset f^{-1}(W)$. Suppose $x_1 \in V \cap \Gamma(f |_C)$, take a path $\alpha$ in $V$ from $x_1$ to $x_0$ with $\alpha(I - 1) \subset C$, then $f \cdot \alpha$ in $W$, hence

$$f \cdot \alpha \simeq e_c : I, 0, 1 \to Y, c, B$$

Thus, $x_1$ is in a nonsurplus root class of $f$ on $X \setminus A$, and this implies that $V \cap R = \emptyset$.

From the proof above, we also get that the union of all the surplus root classes of $f$ on $X \setminus A$ is a compact set. As in [3, p. 7, Corollary 1.13], we get that the number of surplus root classes of $f$ on $X \setminus A$ is finite. □

From this theorem, we can define the index of a surplus root class of $f$ on $X \setminus A$ in the same way as in [4], which is a homomorphism from $H^*_*(X) \to H^*_*(Y, Y - \{c\})$.

**Definition 3.4.** A surplus root class $R$ of $f$ on $X \setminus A$ is essential if $\text{ind}(f, R) \neq 0$; inessential if $\text{ind}(f, R) = 0$. The number of essential surplus root classes of $f$ on $X \setminus A$ is called the *surplus Nielsen root number* of $f$ on $X \setminus A$, denoted $SN(f; X \setminus A, c)$.

**Lemma 3.5.** Let $H : f \simeq g : (X, A) \to (Y, B)$ be a homotopy, let $R_0$ and $R_1$ be root classes of $f$ and $g$ on $X \setminus A$ respectively, and let $R_0$ and $R_1$ be $H$-related on $X \setminus A$. Then the following conditions are equivalent:

(i) $R_0$ is a surplus root class of $f$ on $X \setminus A$,

(ii) $R_1$ is a surplus root class of $g$ on $X \setminus A$,

(iii) $R_0$ and $R_1$ are respectively the 0- and 1-slices of a single surplus root class of $H$ on $(X \times I) - (A \times I)$.

**Proof.** Since $R_0$ and $R_1$ are $H$-related on $X \setminus A$, we can assume, by Proposition 2.7, that $R_0$ and $R_1$ are respectively the 0- and 1-slices of a single root class $R$ on $(X \times I) - (A \times I)$.

If $R_0$ is a nonsurplus root class of $f$ on $X \setminus A$, then there is a path

$$\alpha : I, 0, 1 \to X, x_0, X \setminus A, A$$

such that

$$e_c \simeq f \cdot \alpha : I, 0, 1 \to Y, c, B,$$
where \( x_0 \in \mathbb{R}_0 \). Define a map \( i_0 : X \to X \times I \) and \( j_0 : Y \to Y \times I \) by \( i_0(x) = (x, 0) \) and \( j_0(y) = (y, 0) \) respectively. Then we get a path

\[
i_0 \cdot \alpha : I, \ 0, I - \{1\}, 1 \to X \times I, (x_0, 0), (X \times I) - (A \times I), A \times I
\]

with

\[
\mathbb{H} \cdot (i_0 \cdot \alpha) = j_0 \cdot (f \cdot \alpha) \simeq j_0 \cdot e_c : I, \ 0, 1 \to Y \times I, (c, 0), B \times I.
\]

Note that \((x_0, 0) \in \mathbb{R}\), it follows that \(\mathbb{R}\) is a nonsurplus root class of \(H\) on \((X \times I) - (A \times I)\).

Furthermore, if \(\mathbb{R}\) is a nonsurplus root class on \((X \times I) - (A \times I)\), then there is a path

\[
\beta : I, \ 0, I - \{1\}, 1 \to X \times I, (x', s), (X \times I) - (A \times I), A \times I
\]

with \((x', s) \in \mathbb{R}\) such that

\[
e_{c(s,s)} \simeq \mathbb{H} \cdot \beta : I, \ 0, 1 \to Y \times I, (c, s), A \times I.
\]

Thus, we \(x' \in \Gamma(f_s)\), where \(f_s\) is the \(s\)-slice of \(H\), i.e. \(f_s(x) = H(x, s)\). For a \(x_1 \in \mathbb{R}_1\), \(x_1\) and \(x'\) are respectively in the 1- and \(s\)-slices of \(\mathbb{R}\), and then they are \(H'\)-related on \(X - A\), where \(H'(x, t) = H(x, 1 - t + s \cdot t)\) is a homotopy from \(g\) to \(f_s\). By Definition 2.4, there is a path \(\gamma\) in \(X \setminus A\) from \(x_1\) to \(x'\) such that

\[
H'(\gamma(t), t) \simeq \gamma(t)rel\{0, 1\},
\]

i.e.,

\[
H(\gamma(t), 1 - t + s \cdot t) \simeq \gamma(t)rel\{0, 1\}.
\]

Define maps \(p : X \times I \to X\) and \(q : Y \times I \to Y\) by \(p(x, t) = x, q(y, t) = y\) respectively, then the product of \(\gamma\) and \(p \cdot \beta\)

\[
\gamma(p \cdot \beta) : I, \ 0, I - \{1\}, 1 \to X, x_1, X \setminus A, A
\]

is a path from \(x_1\) to \(A\), and

\[
g \cdot \gamma(t) = H(\gamma(t), 1) \simeq H(\gamma(t), 1 - t + s \cdot t) \simeq \gamma(t),
\]

\[
g \cdot (p \cdot \beta) = H(p \cdot \beta, 1) \simeq q \cdot (\mathbb{H} \cdot \beta) \simeq q \cdot e_{c(s,s)}.
\]

Moreover, we get

\[
e_{c(s,s)} \simeq g(\gamma(p \cdot \beta)) : I, \ 0, 1 \to Y, c, B.
\]

Thus, \(\mathbb{R}_1\) is a nonsurplus root class of \(g\) on \(X \setminus A\).

The converse is the same. 

By this lemma we have
Theorem 3.6 (Homotopy invariance). If two maps \( f \simeq g : (X, A) \to (Y, B) \) are homotopic, then \( SN(f; X \setminus A, c) = SN(g; X \setminus A, c) \).

The next theorem follows directly from Theorem 3.3, Theorem 3.6 and the properties of the root index.

Theorem 3.7. \( SN(f; X \setminus A, c) \) is a nonnegative integer. Any map which is homotopic to \( f : (X, A) \to (Y, B) \) has at least \( SN(f; X \setminus A, c) \) roots on \( X \setminus A \). Thus, \( SN(f; X \setminus A, c) \leq MR[f; X \setminus A, c] \).

Theorem 3.8. Let \( f : (X, A) \to (Y, B) \) be a map of pairs of compact polyhedra, then \( SN(f; X \setminus A, c) \geq N(f; X \setminus A, c) \). If \( A \) can be by-passed in \( X \), then \( SN(f; X \setminus A, c) = N(f; X \setminus A, c) \).

Proof. By Definition 3.1 and [8, Theorem 2.3], a root class of \( f \) on \( X \setminus A \) which is contained in a weakly noncommon root class is a surplus one, all of them lie in an open subset \( X \setminus A \) of \( X \). By additivity of the root index, an essential weakly noncommon root class contains at least one essential surplus root class of \( f \) on \( X \setminus A \). Thus, \( SN(f; X \setminus A, c) \geq N(f; X \setminus A, c) \).

If \( A \) can be by-passed in \( X \), then a root class of \( f \) will contain at most one root class of \( f \) on \( X \setminus A \). By [8, Lemma 3.5], every surplus root class of \( f \) on \( X \setminus A \) is contained in a weakly noncommon root class, therefore \( SN(f; X \setminus A, c) = N(f; X \setminus A, c) \). \( \square \)

Following example shows that our new lower bound \( SN(f; X \setminus A, c) \) can be greater strictly than \( N(f; X \setminus A, c) \).

Example 3.9. Let \( X = S^1 = \{ e^{\theta i} \} \), and let \( A = \{ e^0, e^{\pi i} \} \). A map \( f : (X, A) \to (X, A) \) is given by \( f(e^{\theta i}) = (e^{-2\theta \pi i}) \). The point \( c = e^{\pi i} \).

As a map on \( X \), \( f \) is homotopic to a map \( g : X \to X \) given by \( g(e^{\theta i}) = e^0 \). Hence, \( f \) has no essential root class. It follows that \( N(f; X \setminus A, c) = 0 \). But, two roots \( \{ e^{\frac{\pi}{2} i}, e^{\frac{3\pi}{2} i} \} \) at \( c \) lie in different components of \( X - A \) and has non-zero indices. Thus, \( SN(f; X \setminus A, c) = 2 \).

4. Minimum Theorem for \( SN(f; X \setminus A, c) \)

Lemma 4.1 Let \( (X, A) \) and \( (Y, B) \) be pairs of compact polyhedra, where every component of \( X \setminus A \) is a PL manifold with dimension greater than 2. Let \( x_0 \) be
an isolated root of a root finite map \( f : (X, A) \rightarrow (Y, B) \) on \( X \setminus A \). Suppose \( \alpha : I, 0, I - \{1\}, 1 \rightarrow X, x_0, X \setminus A, A \) is a path from \( x_0 \) to \( A \) with \( \Gamma(f) \cap \alpha(I) = \{x_0\} \) and
\[
e_e \simeq f \cdot \alpha : I, 0, 1 \rightarrow Y, c, B.
\]

Then \( f \) is homotopic to a map \( f' : (X, A) \rightarrow (Y, B) \) with
\[
\Gamma(f') = (\Gamma(f) - \{x_0\}) \cap \{\alpha(1)\}.
\]

**Proof.** By a perturbation, we can assume that \( \alpha \) is a PL arc which containing no roots except for starting point. Let \( H : (I \times I, \{1\} \times I) \rightarrow (Y, B) \) be the homotopy from \( e_e \) to \( f \cdot \alpha \), i.e. \( H(t, 0) = c \) and \( H(t, 1) = f \cdot \alpha(t) \). Since \((Y, B)\) is a simplicial pair, we may assume that \( c \) is a vertex of \( Y \) and \{\( H(1, s) \)\}_{0 \leq s \leq 1} \) is a PL arc in \( B \) from \( c \) to \( f(x_0) \). Choose a conic neighborhood \( N(f(x_0), \varepsilon) = \{y \in Y | d(y, f(x_0)) < \varepsilon\} \) of \( f(x_0) \) such that \( N \cap H(\{1\}, I) \) is a line segment. We define a homotopy \( G : (X \times I, A \times I) \rightarrow (Y, B) \) by
\[
G(x, t) = \begin{cases} 
  f(x) & \text{if } f(x) \notin N(c, \varepsilon t) \\
  \left(\frac{2}{\varepsilon t}d(f(x), f(x_0)) - 1\right)f(x) \\
  + (2 - \frac{2}{\varepsilon t}d(f(x), f(x_0)))f(x_0) & \text{if } 0 < \frac{\varepsilon t}{2} < d(f(x), f(x_0)) \leq \varepsilon t \\
  H(1, 1 - t + \frac{2}{\varepsilon t}d(f(x), f(x_0))) & \text{if } 0 \leq d(f(x), f(x_0)) \leq \frac{\varepsilon t}{2}
\end{cases}
\]
(cf. [4]). We define \( g : (X, A) \rightarrow (Y, B) \) by \( g(x) = G(x, 1) \), then \( g \) is homotopic to \( f \) with \( \Gamma(g) = \Gamma(f) \cup \{\alpha(1)\} \) and \( e_e \simeq g \cdot \alpha \).

By using the method in [6] and [2], we can combine the root \( x_0 \) to \( \alpha(1) \).

**Theorem 4.2.** Let \((X, A)\) and \((Y, B)\) be pairs of compact polyhedra such that

1. \( X \) and \( Y \) are PL manifolds with same dimension,
2. every component of \( X \setminus A \) is a PL manifold with dimension greater than 2.

Then every map \( f : (X, A) \rightarrow (Y, B) \) is homotopic to a map \( g : (X, A) \rightarrow (Y, B) \) with \( SN(f; X \setminus A, c) \) roots on \( X \setminus A \).

**Proof.** By transversality and homotopy extension, we can assume that \( f \) is root-finite and that all roots of \( f \) on \( X \setminus A \) lie in maximal simplexes. We can unite roots belonging to the same root class of \( f \) on \( X \setminus A \) as in [6]. Suppose \( x_0 \) lies in a nonsurplus root class, then there is a path
\[
\alpha : I, 0, I - \{1\}, 1 \rightarrow X, x_0, X \setminus A, A
\]
such that

\[ e_c \simeq f \cdot \alpha : I, 0, 1 \to Y, c, B. \]

By Lemma 4.1, we shall move the root \( x_0 \) to \( \alpha(1) \in A \). As \( X \) has no local cut point, we can take the paths with different terminal points which are not roots. Finally, delete root classes on \( X \setminus A \) which consist of a single root of index zero by the usual method (\cite[p.123, Theorem 4]{[1]}). Then we get a map \( g : (X, A) \to (Y, B) \) with \( SN(f; X \setminus A, c) \) roots on \( X - A \).

\[ \square \]

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