A New Approach for the Analysis Solution of Dynamic Systems Containing Fractional Derivative

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Fractional derivative models, which are used to describe the viscoelastic behavior of material, have received considerable attention. Thus it is necessary to put forward the analysis solutions of dynamic systems containing a fractional derivative. Although previously reported such kind of fractional calculus-based constitutive models, it only handles the particularity of rational number in part, has great limitation by reason of only handling with particular rational number field. Simultaneously, the former study has great unreliability by reason of using the complementary error function which can't ensure uniform real number. In this paper, a new approach is proposed for an analytical scheme for dynamic system of a spring-mass-damper system of single-degree of freedom under general forcing conditions, whose damping is described by a fractional derivative of the order $0<\alpha<1$ which can be both irrational number and rational number. The new approach combines the fractional Green's function and Laplace transform of fractional derivative. Analytical examples of dynamic system under general forcing conditions obtained by means of this approach verify the feasibility very well with much higher reliability and universality.

Key Words : Fractional Derivative, Analysis Solution, Viscoelastic Damping, Dynamic System

1. Introduction

In the present study, the distribution of fractional derivative models, which are used to describe the viscoelastic behavior of materials, has received considerable attention (Agrawal, 2001 ; Bagley and Torvik, 1983 ; Elshehawey et al., 2001 ; Enelund and Josefson, 1997 ; Enelund et al., 1999 ; Ingman and Suzdalnitsky, 2001). The use of fractional calculus-based constitutive models is motivated in large range by the fact that fractional derivative models describe the frequency dependence of the structural damping characteristics quite remarkably (Agrawal, 2001 ; Enelund et al., 1999), and fewer parameters are required to represent the material viscoelastic behavior, as compared to those required when using traditional Kelvin and Maxwell-based models (Bagley and Torvik, 1983 ; Enelund et al., 1999).

Several authors have applied such types of fractional calculus-based constitutive models to model the dynamic behavior of the dynamic sys-

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tem with viscoelastic damping. For example, fractional operators that have been applied to the analysis of damped vibrations of viscoelastic single-mass systems is reported (Rossikhin and Shitikova, 1997). Similarly a time-domain finite-element analysis of fractionally damped viscoelastic structures is also reported (Enelund and Josefson, 1997). In addition, the motion of an N-degree-of-freedom system is analyzed (Ingman and Suzdalnitsky, 2001). However, for the dynamic systems based on fractional calculus, it is difficult to obtain the analysis solutions due to the existence of the fractional derivative terms in motion equations. Regarding the analysis solution of Eq. (1), Elshehawey obtained an exact solution for $\alpha=1/2$ (Elshehawey et al., 2001). Agrawal analyzed the fractional Green's function, for $0\leq a/b<2$, a, b are integers by using the method reported by Miller (Miller, 1993). However, it seems quite difficult to obtain the analysis solutions when is an irrational number. Unfortunately, the analysis solutions appear contradictory (Elshehawey et al., 2001) when using the complementary error function Erfc $(-\lambda_i\sqrt{t})$, where $(-\lambda_i\sqrt{t})$ must be real. Since the roots of a fourth order algebra equation, λ_i, $i=1, 2, 3, 4$ are not easy to be real numbers simultaneously in most cases. Once the variable $(-\lambda_i\sqrt{t})$ is not real, the function Erfc $-(\lambda_i\sqrt{t})$, seems to have no meaning. Therefore, in fact the analysis solutions are unreliable (Elshehawey et al., 2001).

In the present study, a scheme for analysis the solution of a single-degree-of-freedom spring-mass-damper system whose damping is presented by a fractional derivative of the order $0<\alpha<1$ which can be both irrational numbers and rational is presented and discussed. The more extensive application range ensures the universality of this approach. According to the Laplace transform of fractional derivative, the fractional Green's function and the Duhamel-integral-type closed-form expression for the response of the system is obtained initially. Inverse Laplace transform and Taylor expansion theory are then used to obtain the expressions of fractional Green's function and its derivatives. The above methods, which are applied to the new approach, ensure the reliability of analysis solution. Thus, the responses under general forcing condition are analyzed, and the analytical expressions are obtained. Cases are also presented in order to verify the present technique and display the behavior of the responses of the fractional damping system.

2. Fractional Dynamic Model and General Solution

Consider the viscoelastic behavior of a single-degree-of-freedom oscillator, which includes a discrete mass and a viscoelastic spring, and is governed by fractional calculus law. In combination with Newton's second law, the equation of motion can be given as (Enelund et al., 1999; Ingman and Suzdalnitsky, 2001),

$$mD^2y(t)+cD^\alpha y(t)+ky(t)=f(t) \qquad y(0)=y_0,\ \dot{y}(0)=y_1 \tag{1}$$

where m, c, k, represent the mass, damping, and stiffness coefficients, respectively, $f(t)$ is the externally applied force. Accordingly, the viscous-damping term in these equations is replaced by $cD^2y(t)$, which is proportional to the α-order derivative of the displacement. As for the fractional α-order derivative of the function $y(t)$, using the extended Riemann-Liouville definition (Enelund et al., 1999; Oldham and Spanier, 1974; Samko, 1993; Xu and Tan, 2001), the equation can be written as

$$D^\beta y(t)=\frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_0^t(t-\tau)^{n-\beta-1}y(\tau)d\tau \qquad 0<n-\beta\leq 1 \tag{2}$$

where n is an integer. In the time domain, the presence of integro-differential operators make the computation of the structural equations more complicated than that of using the ordinary operators. Fortunately, by using the Laplace transform in the frequency domain, the model becomes much easier to handle (Enelund et al., 1999).

Hence, the Laplace transform is defined as

$$\bar{y}(s)=L[y(t)]=\int_0^{+\infty}y(t)e^{-st}dt \tag{3}$$

However, for the ordinary fractional order derivative, the equations are generalized as (Oldham, and Spanier, 1974),

$$\left[D^{\beta}y(t)\right]=s^{\beta}\tilde{y}(s)-\sum_{k=1}^{n-1}s^{k}D^{\beta-1-k}y(0) \quad (4)$$

where n is an integer such that $n-1<\beta\leq n$ When $0<\beta<1$, Eq. (4) can be rewritten as

$$L[D^{\beta}y(t)]=s^{\beta}\tilde{y}(s)-D^{\beta-1}y(0)$$

Compared with Eq. (1), the term $D^{\beta-1}y(0)$ is in fact a fractional integration and vanishes for any reasonable $y(t)$, which satisfies the motion equation of Eq. (1), then by substituting Eq. (1) into above formula, Eq. (5) is obtained as below,

$$L[D^{\beta}y(t)]=s^{\beta}\tilde{y}(s) \quad (5)$$

Applying the Laplace transform Eqs. (4) and (5) to Eq. (1) alternatively, leads to

$$(ms^{2}+cs^{\alpha}+k)\tilde{y}(s)=\tilde{f}(s)+smy_{0}+my_{1} \quad (6)$$

i.e.

$$\tilde{y}(s)=\tilde{G}(s)\tilde{f}(s)+\tilde{G}(s)smy_{0}+\tilde{G}(s)my_{1} \quad (7)$$

where

$$\tilde{G}(s)=1/(ms^{2}+cs^{\alpha}+k)$$

By using the inverse Laplace transform of Eq. (7), the following generalized equation is presented as

$$\begin{aligned}y(t)=&my_{0}\dot{G}(t)+my_{0}G(0)\delta(t)\\&+my_{1}G(t)+\int_{0}^{t}G(t-\tau)f(\tau)d\tau\end{aligned} \quad (8)$$

where Green's function $G(t)$ is the inverse Laplace transform of $\tilde{G}(s)$, and $\delta(t)$ is δ-function. The differentiative coefficient of $\delta(t)$ and $\int_{0}^{t}G(t-\tau)f(t)d\tau$ in Eq. (8) are given in Appendix (Eqs. (a1)). Differentiating Eq. (8) with respect to time and using the properties of the Green's function, Eq. (9) can be expressed as

$$\begin{aligned}\dot{y}(t)=&my_{0}\ddot{G}(t)+my_{0}G(0)\dot{\delta}(t)+my_{1}\dot{G}(t)\\&+f(t)G(0)+\int_{0}^{t}\dot{G}(t-\tau)f(\tau)d\tau\end{aligned} \quad (9)$$

3. Expression for the Fractional Green's Function

In order to verify the responses of Eqs. (8) and (9), the fractional Green's function $G(t)$ and its derivatives $\dot{G}$, $\ddot{G}$ must be determined. Here, for any $0<\alpha<1$ and $m\neq0$, c, k, the expression of the solution can be represented in analytical form.

The Laplace transform of fractional Green's function $G(t)$ is given by

$$\tilde{G}(s)=\frac{1}{ms^{2}+cs^{\alpha}+k'} \quad (10)$$

If $k=0$, it gives

$$\begin{aligned}\tilde{G}(s)&=\frac{1}{ms^{2}+cs^{\alpha}}\\&=\frac{1}{m}\frac{s^{(2-\alpha)-2}}{s^{2-\alpha}+c/m}\end{aligned} \quad (11)$$

Taking the inverse Laplace transform of Eq. (11), the expression is given by

$$G(t)=\frac{1}{m}tE_{2-\alpha,2}\left(-\frac{c}{m}t^{2-\alpha}\right) \quad (12)$$

where, the inverse Laplace transforms are given in terms of the generalized Mittag-Leffler function (Miller, 1993 ; Oldham and Spanier, 1974), which is defined by the power series. The power series in Eqs. (12) and (13) are given in Appendix (Eqs. (a2)).

The below inverse transform is defined as,

$$\begin{aligned}L^{-1}\frac{j!s^{a-b}}{(s^{a}+\lambda)^{j+1}}=t^{aj+b-1}E_{a,b}^{(j)}(-\lambda t^{a})\\Re(s)>|\lambda|^{1/a}\end{aligned} \quad (13)$$

At present, focus on the M-L function and the recognition of its importance have remarkably increased from an analytical standpoints to the description of fractional-order control systems and fractional viscoelastic models. Their definition and properties are now available in currently published books and surveys concerning about fractional calculus, integral and differential equations, mechanics, and etc (Agrawal, 2001 ; Elshehawey et al., 2001 ; Narahar et al., 2001 ; Xu and Tan, 2001).

If

$$k\neq0,\ \eta=\frac{k}{m}\frac{s^{-\alpha}}{s^{2-\alpha}+c/m}$$

Eq. (10) can be integrated

$$\tilde{G}(s)=\frac{1}{ms^{2}+cs^{\alpha}+k}=\frac{1}{k}\frac{\eta}{1+\eta} \quad (14)$$

Defined as

$$r=|s|=\sqrt{Re(s)^{2}+Im(s)^{2}}$$

for any $m>0$, $c\neq 0$, $k\neq 0$, when $Re(s)\geq\xi=\left(\frac{m+|c|+|k|}{m}\right)^{\frac{1}{2-\alpha}}$, results in

$$r\geq\xi=\left(\frac{m+|c|+|k|}{m}\right)^{\frac{1}{2-\alpha}}$$

The $|\eta|$ of η in Eq. (14) is given in Appendix (Eqs. (a3)).

Then, for $Re(s)\geq\xi$, the equation is obtained as

$$\begin{aligned}\tilde{G}(s)&=\frac{1}{ms^2+cs^\alpha+k}=\frac{1}{k}\frac{\eta}{1+\eta}=\frac{1}{k}\sum_{n=0}^{+\infty}(-1)^n\eta^{n+1}\\&=\frac{1}{k}\sum_{n=0}^{+\infty}(-1)^n\left(\frac{k}{m}\right)^{n+1}\frac{s^{-\alpha(n+1)}}{(s^{2-\alpha}+c/m)^{n+1}}\\&=\frac{1}{k}\sum_{n=0}^{+\infty}(-1)^n\left(\frac{k}{m}\right)^{n+1}\frac{s^{2-\alpha-(2+\alpha n)}}{(s^{2-\alpha}+c/m)^{n+1}}\end{aligned}\tag{15}$$

Considering Eq. (13), when $Re(s)\geq\max(\xi,|c/m|^{1/(2-\alpha)})=\xi$, the inversion of Eq. (15) in terms of the linear property of Laplace transform can be expressed as

$$\begin{aligned}G(t)&=\sum_{n=0}^{+\infty}(-1)^n\frac{k^n}{n!m^{n+1}}t^{(2-\alpha)n+(2+\alpha n)-1}E^{(n)}_{2-\alpha,2+\alpha n}\left(-\frac{c}{m}t^{2-\alpha}\right)\\&=\sum_{n=0}^{+\infty}(-1)^n\frac{k^n}{n!m^{n+1}}t^{2n+1}E^{(n)}_{2-\alpha,2+\alpha n}\left(-\frac{c}{m}t^{2-\alpha}\right)\end{aligned}\tag{16}$$

where

$$\begin{aligned}&E^{(n)}_{2-\alpha,2+\alpha n}\left(-\frac{c}{m}t^{2-\alpha}\right)\\&=\sum_{j=0}^{+\infty}\frac{(j+n)!}{j!}\left(-\frac{c}{m}\right)^j\frac{t^{(2-\alpha)j}}{\Gamma((2-\alpha)j+2n+2)}\end{aligned}$$

i.e.

$$G(t)=\sum_{n=0}^{+\infty}\sum_{j=0}^{+\infty}(-1)^{n+j}\frac{(n+j)!k^nc^j}{m!j!m^{n+j+1}}\frac{t^{2n+(2-\alpha)j+1}}{\Gamma((2-\alpha)j+2n+2)}\tag{17}$$

Differentiating Eq. (16) with respect to time, the derivative can be written as

$$\dot{G}(t)=\sum_{n=0}^{+\infty}\sum_{j=0}^{+\infty}(-1)^{n+j}\frac{(n+j)!k^nc^j}{n!j!m^{n+j+1}}\frac{t^{2n+(2-\alpha)j}}{\Gamma((2-\alpha)j+2n+1)}\tag{18}$$

Similarly, the equation is denoted as,

$$\ddot{G}(t)=\sum_{n=0}^{+\infty}\sum_{j=0,n\neq 0}^{+\infty}(-1)^{n+j}\frac{(n+j)!k^nc^j}{n!j!m^{n+j+1}}\frac{t^{2n+(2-\alpha)j-1}}{\Gamma((2-\alpha)j+2n)}\tag{19}$$

Substituting $t=0$ into Eqs. (17), (18) and (19) results in

$$G(0)=0,\ \dot{G}(0)=\frac{1}{m},\ \ddot{G}(0)=0\tag{20}$$

Then the responses of displacement and velocity in Eq. (8) and (9) can be represented as

$$y(t)=my_0\dot{G}(t)+my_1G(t)+\int_0^t G(t-\tau)f(\tau)d\tau\tag{21}$$

$$\dot{y}(t)=my_0\ddot{G}(t)+my_1\dot{G}(t)+\int_0^t\dot{G}(t-\tau)f(\tau)d\tau\tag{22}$$

The above equations, which represent general analysis solutions for the displacement and velocity in Eq. (1) corresponding to the fractional derivative, are similar to the Duhamel integral solution for a linear system.

4. Application Analysis of Vibration under Different Forcing Conditions

A dynamic system is often subjected to some type of external force or excitation. It may be harmonic, non-harmonic but periodic, non-periodic, or random in nature. In this part, the dynamic response of a single degree of freedom system under different kinds of excitations is represented. For simplicity, the following discussed cases will be all based on zero initial conditions. In order to obtain the analysis solution, the following integral formulae will be used as

$$\int_0^t(t-\tau)^\xi\cos(\omega\tau)d\tau=\frac{t^{1+\xi}F(1,1+1/2\xi,3/2+1/2\xi,-1/4t^2\omega^2)}{1+\xi}\tag{23}$$

$$\int_0^t(t-\tau)^\xi\sin(\omega\tau)d\tau=\frac{t^{\xi+2}\omega F(1,2+1/2\xi,3/2+1/2\xi,-1/4t^2\omega^2)}{(\xi+1)(\xi+2)}\tag{24}$$

$$\int_0^t(t-\tau)^\xi h^{\omega\tau}d\tau=\frac{t^{1+\xi}F(1,2+\xi,\omega\ln(h)t)}{1+\xi}\tag{25}$$

where, $F(u,w,z)$ is the generalized hyper-geometric function (Slater, 1966) which is defined by

$$F(u,w,z)={}_jF_l(u_1,u_2,\cdots;w_1,w_2,\cdots;z)=\sum_{\zeta=0}^{+\infty}\frac{\Pi_{i=1}^{j}\frac{\Gamma(u_i+\zeta)}{\Gamma(u_i)}z^\zeta}{\Pi_{i=1}^{l}\frac{\Gamma(w_i+\zeta)}{\Gamma(w_i)}\zeta!}\tag{26}$$

where j and l are the number of terms in vectors u and w, respectively.

Shown as below, the approach is applied to some detailed dynamic systems under different forcing conditions.

4.1 Response under exponential force

If the forcing function is given by $f(t)=h^{vt+\theta}$ the responses of the motion becomes

$$y(t)=\int_0^t G(t-\tau)h^{v\gamma+\theta}d\tau \qquad (27)$$

$$\dot{y}(t)=\int_0^t \dot{G}(t-\tau)h^{v\gamma-\theta}d\tau \qquad (28)$$

Substituting Eqs. (17) and (18) into Eqs. (27) and (28), respectively, results in Eq. (29) as below

$$\begin{aligned} y(t)&=\int_0^t G(t-\tau)h^{v\tau+\theta}d\tau \\ &=\sum_{n=0}^{+\infty}\sum_{j=0}^{+\infty}(-1)^{n+j}\frac{(n+j)!k^n+c^j}{n!j!m^{n+j+1}}\frac{\int_0^t(t-\tau)^{2n+(2-\alpha)j+1}h^{v\tau+\theta}d\tau}{\Gamma((2-\alpha)j+2n+2)} \\ &=\sum_{n=0}^{+\infty}\sum_{j=0}^{+\infty}(-1)^{n+j}\frac{(n+j)!k^nc^j}{n!j!m^{n+j+1}}\frac{t^{2n+(2-\alpha)j+2}F(\gamma_1,\gamma_2,\nu\ln(h)t)}{\Gamma((2-\alpha)j+2n+3)} \end{aligned} \qquad (29)$$

where, similarly, the equation is given by

$$\begin{aligned} \dot{y}(t)&=\int_0^t \dot{G}(t-\tau)h^{v\tau+\theta}d\tau \\ &=\sum_{n=0}^{+\infty}\sum_{j=0}^{+\infty}(-1)^{n+j}\frac{(n+j)!k^nc^j}{n!j!m^{n+j+1}}\frac{\int_0^t(t-\tau)^{2n+(2-\alpha)j}h^{v\tau+\theta}d\tau}{\Gamma((2-\alpha)j+2n+2)} \\ &=\sum_{n=0}^{+\infty}\sum_{j=0}^{+\infty}(-1)^{n+j}\frac{(n+j)!k^nc^j}{n!j!m^{n+j+1}}\frac{t^{2n+(2-\alpha)j+1}F(\gamma_1,\gamma_2,\nu\ln(h)t)}{\Gamma((2-\alpha)j+2n+2)} \end{aligned} \qquad (30)$$

where, $\gamma_1=1$, $\gamma_2=2+(2n+(2-\alpha)j)$.

Figure 1 illustrates the responses, $y(t)$ and $\dot{y}(t)$, under existing force $f(t)=e^{vt}$ for $k=400$, $c=20$, $\alpha=\pi/4$ and $v=-6$, -3, -2, -1, 0, 0.5 and 1 for different lines.

4.2 Response under general forcing conditions

Considering the response of the system under an arbitrary smooth external force, $\phi(x)$ and $\psi(x)$ are regarded as the orthogonal scaling and wavelet functions (Suarez and Shokooh, 1997; Sweldens and Piessens, 1994; Wang and Zhou, 1998; Wang, 2001; Wim, 1995). Here, $f(x)$ is a smooth L^2R function in the form of $f(x)$ and is square integrable. Defining P_n, Q_j to be project operators, then equation can obtained,

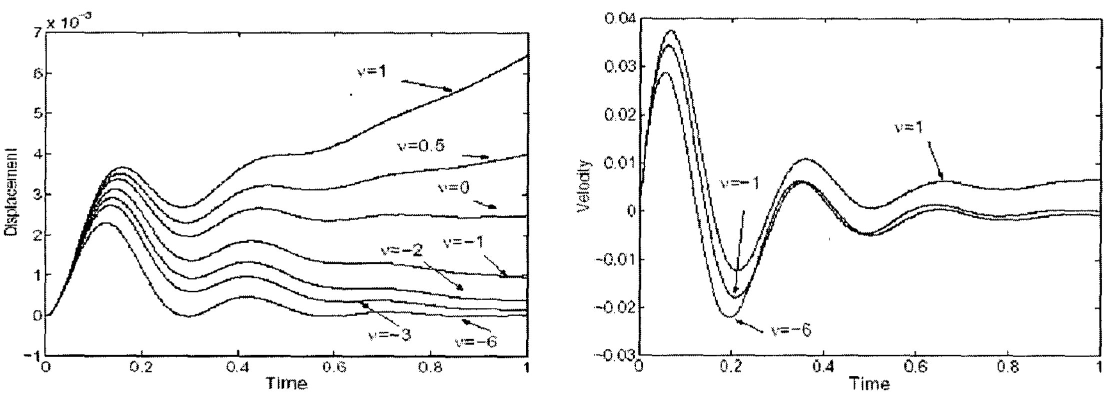
$$\begin{aligned} f(x)&=P_nf(x)+\sum_{j=0}^{+\infty}Q_jf(x) \\ &=\sum_{i=-0}^{+\infty}c_{n,i},\phi_{n,i}(x)+\sum_{j=0}^{+\infty}\sum_{i=-\infty}^{+\infty}d_{j,i}\psi_{j,i}(x) \end{aligned} \qquad (31)$$

The coefficients $c_{n,i}$, $d_{j,i}$, $\phi_{n,i}$, $\psi_{j,i}$ in Eq. (31) is given in Appendix (Eq. (a4)).

According to the theory of multi-resolution analysis (Sweldens and Piessens, 1994; Wang, 2001), the two project operators of P_n, Q_j have the following relationship

$$Q_j=P_{j+1}-P_j$$

Coiflets have been shown to be excellent for the sampling approximation of smooth functions (Sweldens and Piessens, 1994; Wang, 2001; Wim, 1995). According to the character of Coiflets (Sweldens and Piessens, 1994; Wim, 1995) and the assumption of $\phi(x)=[0, 3N-1]$, the wavelet function and scaling function can be constructed by the coefficients of Table 1 (Wang, 2001), using the two scale relation as below,



(a) Displacement (b) Velocity

Fig. 1 Dependence of single-degree-of-freedom system response on the coefficient v under force $f(t)=e^{vt}$

Table 1 Coiflets filter coefficients for $N=2$, 4 and 6

k	$N=2(M_1=4)$	$N=4(M_1=4)$	$N=6(M_1=7)$
0	5.456145913796356e−002	1.689380907695821e−003	−2.392638657280051e−003
1	−1.795614591379636e−001	−1.816639282073453e−002	−4.932601854180402e−003
2	−1.091229182759271e−001	3.507862062605389e−002	2.714039971139949e−002
3	8.591229182759271e−001	7.074394036809258e−002	3.064755594619984e−002
4	1.054561459137964e+000	−2.197082915811749e−001	−1.393102370707997e−001
5	3.204385408620364e−001	−1.013118304071172e−001	−8.060653071779983e−002
6		8.067593419102440e−001	6.459945432939942e−001
7		1.061135780078056e+000	1.116266213257999e+000
8		3.968448038803485e−001	5.381890557079980e−001
9		−1.047986487449172e−002	−9.961543386239989e−002
10		−2.066385574316280e−002	−7.992313943479994e−002
11		−1.921632058008399e−003	5.149146293240031e−002
12			1.238869565706006e−002
13			−1.583178039255944e−002
14			−2.717178600539990e−003
15			2.886948664020020e−003
16			6.304993947079994e−004
17			−3.058339735960013e−004

$$\phi(x)=\sum_{k=0}^{3N-1}a_k\phi(2x-k) \tag{32}$$

where a_k, $k=0, 1, \cdots, 3N-1$, are the filter coefficients, it yields

$$\int_{-\infty}^{+\infty}\phi(t)\,dt=1,\ \int_{-\infty}^{+\infty}(t-M_1)^i\phi(t)\,dt=0,\quad \int_{-\infty}^{+\infty}t^j\psi(t)\,dt=0 \tag{33}$$

where $1\leq i<N-1$, $0<N-1$, then the equation is given by

$$c_{n,k}=\int_{-\infty}^{+\infty}f(x)\,\phi_{n,k}(x)\,dx\approx 2^{n/2}f\left(\frac{k+M_1}{2^n}\right) \tag{34}$$

with a degree of accuracy of $N-1$. Combining Eq.s (32) and (33) they yields

$$f(x)\approx\bar{P}_nf(x)=\sum_{k=-\infty}^{+\infty}f\left(\frac{k+M_1}{2^n}\right)\phi(2^nx-k) \tag{35}$$

where

$$f(x)=\lim_{n\to\infty}\bar{P}_nf(x)$$

Such an approach of approximation is attractive due to its simplicity and the high degree of accuracy of $N-1$. Moreover, if it has $f(x)\in C^\gamma$, $\gamma\leq N-1$, the precision of approximation in Eq. (35) subsequently becomes (Sweldens and Piessens, 1994; Wim, 1995)

$$\|f(x)-\bar{P}_nf(x)\|_2\leq O(2^{-n\gamma}) \tag{36}$$

Assumed that the external force $f(t)$ is smooth enough at the initial point $t=0$, define $\chi_{0,0}(\lambda)$, $\chi_{0,1}(\lambda)$, $\chi_{1,0}(\lambda)$, $\chi_{1,1}(\lambda)$, $\chi_{0,0}(\lambda)$ and $g(\lambda, t)$, $g_\lambda(\lambda, t)$.

$\chi_{0,0}(\lambda)$, $\chi_{0,1}(\lambda)$, $\chi_{1,0}(\lambda)$, $\chi_{1,1}(\lambda)$, $\chi_{0,0}(\lambda)$ and $g(\lambda, t)$, $g_\lambda(\lambda, t)$ in Eqs. (36) ~ (42) are given in Appendix (Eqs. (a5)).

For almost all reasonable finite functions, it's defined as $g(\lambda, t)\lambda_{[0,T]}(t)\in L^2(R)$ and $g_\lambda(\lambda, t)\lambda_{[0,T]}(t)\in L^2R$ for any $\lambda\leq t$, where $\chi_{[0,T]}(t)$ is defined as

$$\chi_{[0,T]}(t)=\begin{cases}1 t=[0,\ T]\\ 0 t\in(-\infty,\ 0)\cup(T,\ \infty)\end{cases}$$

Simultaneously

$$\begin{aligned} g(\lambda, t)_{\chi[0,T]}(t) &\approx P_j[g(\lambda, t)_{\chi[0,T]}(t)]\\ &=\sum_{i=-\infty}^{\infty}g\left(\lambda, \frac{i+M_1}{2^j}\right)_{\lambda[0,T]}\left(\frac{i+M_1}{2^j}\right)\emptyset(2^jt-i)\\ &=\sum_{i=0}^{2^n}g\left(\lambda, \frac{i}{2^j}\right)\emptyset(2^jt-i+M_1)\end{aligned} \tag{37}$$

$$\begin{aligned} g_\lambda(\lambda, t)_{\chi[0,T]}(t) &\approx P_{jg\lambda}[g(\lambda, t)_{\chi[0,T]}(t)]\\ &=\sum_{i=-\infty}^{\infty}g_\lambda\left(\lambda, \frac{i+M_1}{2^j}\right)_{\lambda[0,T]}\left(\frac{i+M_1}{2^j}\right)\emptyset(2^jt-i)\\ &=\sum_{i=0}^{2^n}g_\lambda\left(\lambda, \frac{i}{2^j}\right)\emptyset(2^jt-i+M_1)\end{aligned} \tag{38}$$

In general, the multi-resolution analysis theory states that

$$g(\lambda,\ t)=\lim_{j\to+\infty} P_j g(\lambda,\ t),\ t\in[0,\ T] \tag{39}$$

$$g_\lambda(\lambda,\ t)=\lim_{j\to+\infty} P_j g(\lambda,\ t),\ t\in[0,\ T] \tag{40}$$

However, Eqs. (37) and (38) are not used for practical computing, because wavelets can effectively detect singularities and the possible presence of artificial discontinuities at the end points, 0 and T which are likely to introduce significant errors.

Thus, the following equations are used, which are based on the definition of Eqs. (35) and (36), to approximate g and g_λ as expressed

$$g(\lambda,\ t)\chi_{[0,T]}(t) \approx \sum_{i=1-3N}^{2^n-1} g\left(\lambda,\ \frac{i+M_1}{2^j}\right)\varnothing(2^j t-i)\chi_{[0,T]}(t) \tag{41}$$

$$g_\lambda(\lambda,\ t)\chi_{[0,T]}(t) \approx \sum_{i=1-3N}^{2^n-1} g_\lambda\left(\lambda,\ \frac{i+M_1}{2^j}\right)\varnothing(2^j t-i)\chi_{[0,T]}(t) \tag{42}$$

Therefore, the final solutions of $y(t)$ and $\dot{y}(t)$ can be obtained.

$$y(t)=my_0\dot{G}(t)+my_1G(t)+\int_0^t G(t-\tau)f(\tau)\,d\tau \tag{43}$$

$$\dot{y}(t)=my_0\ddot{G}(t)+my_1\dot{G}(t)+\int_0^t \dot{G}(t-\tau)f(\tau)\,d\tau \tag{44}$$

The detailed calculation and reasoning of Eqs. (43) and (44) is given in Appendix (Eqs. (a6)).

Figure 2 highlights the comparison between the analysis solution and wavelet solution obtained by $y(t)=my_0\dot{G}(t)+my_1G(t)+\sum_{i=1-3N}^{2^n-1} g\left(t,\frac{i+M_1}{2^j}\right)[\varnothing^f(2^j t-i)\varnothing^f(-i)]$, $t=[0,\ T]$ appeared in Appendix (Eqs. (a6)), for $k=400$, $c=20$, $\alpha=\pi/4$ and $f(t)=e^{-6t}$ and scale $j=4$, 6, which illustrate the efficiency of wavelet approximation. It can be observed that even small scale j can make high precision.

Applying the exciting force $f(t)=10^{-t}\ln(t^2+3)\sin(4\pi t)^6$, as shown in Fig. 3(a), and using the Coiflets with $N=18$, $M_1=7$ Fig. 3(b) shows the response for $k=100$, $c=40$ and $\alpha=\pi/4$, $\pi/6$, $\pi/15$ scale $j=10$. When the exciting force becomes $f(t)=2^{-t}\ln(t^2+7/10)\sin(10\pi t)^9$ as also shown in Fig. 4(a), and using the Coiflets with $N=18$,

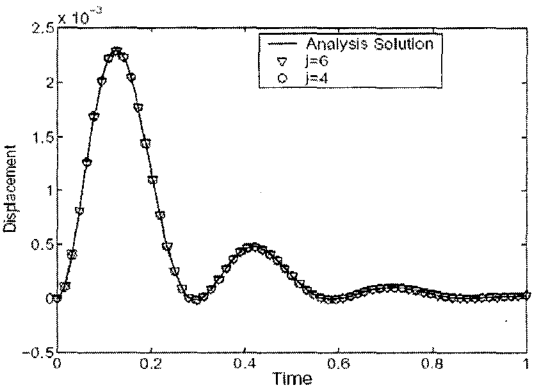
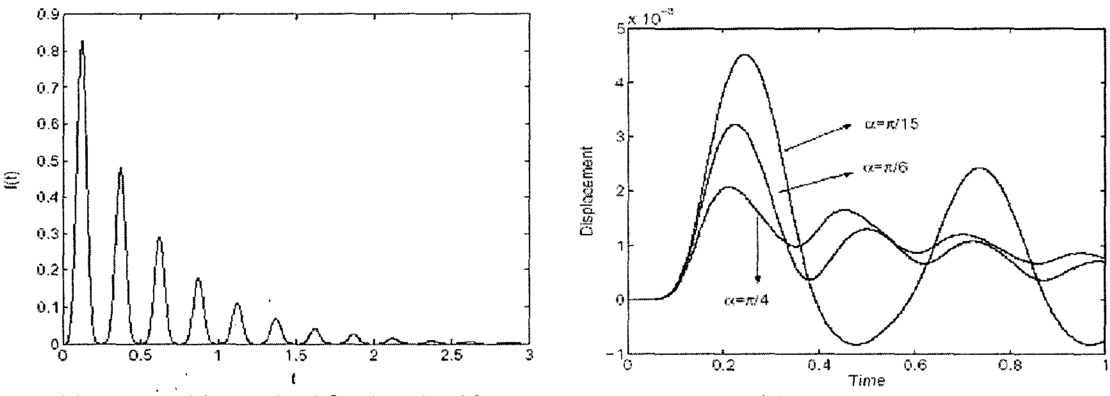
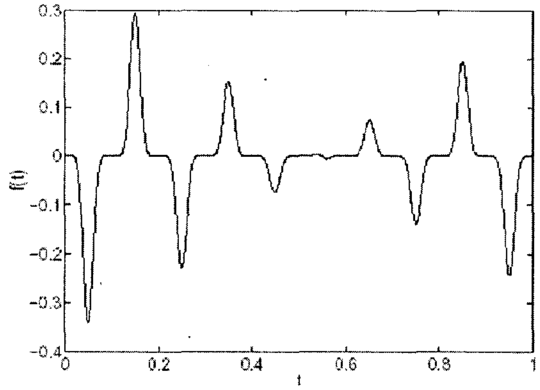


Fig. 2 Comparison between the analysis solution and the wavelet solutions

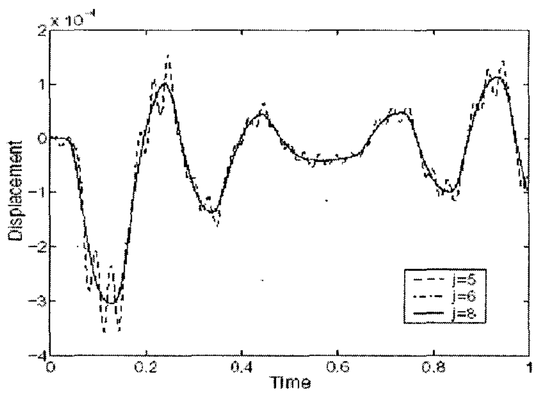


(a) Force, $f(t)=10^{-t}\ln(t^2+3)\sin(4\pi t)^6$ (b) Displacement

Fig. 3 Dependence of single-degree-of-freedom system response on the scale of the scaling function



(a) Force, $f(t) = 2^{-t}\ln(t^2+7/10)\sin(10\pi t)^9$



(b) Displacement

Fig. 4 Dependence of single-degree-of-freedom system response on the scale of the scaling function

$M_1=7$ finally Fig. 4(b) also illustrates the response for $k=100$, $c=40$, $\alpha=\pi/4$ and scale $j=$ 5, 6, 8.

5. Conclusions

This approach is based on the Laplace transform, fractional Green's function and inverse Laplace transform. The deductive principle and procedure is presented.

By means of the new approach, the analysis solutions of a dynamic system of a spring-mass-damper system of single-degree of freedom under general forcing conditions, whose damping is described by a fractional derivative of any order α, $0<\alpha<1$, with both irrational and rational fractional values are accomplished. By contrast in studies reported earlier, the previous methods can only obtain a limited number of analysis solutions when the fractional orders are rational, but simultaneously have great unreliability. The solution, which is obtained with the help of the new approach, is extremely reliable and can be universally applied.

Moreover, the dynamic models are established with confirmative solutions. The cases of analysis solutions derived from the dynamic systems under general forcing conditions accurately verify the feasibility very well. Furthermore, from these cases, changes in the order of the damping differential operators α or the damping coefficient c, corresponding to the change of the proportion between the viscous and elastic properties of the material, are revealed.

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Appendix

The differentiative coefficient of $\delta(t)$ and $\int_0^t G(t-\tau) f(\tau) d\tau$ in Eq. (8) are given by :

$$\begin{aligned} &\dot{\delta}(t) = \frac{d}{dt}\delta(t) \\ &\frac{d}{dt}\left[\int_0^t G(t-\tau) f(\tau) d\tau\right] \\ &= f(t) G(0) + \int_0^t \dot{G}(t-\tau) f(\tau) d\tau \end{aligned} \tag{a1}$$

The power series appeared in Eqs. (12) and (13) are defined as below :

$$\begin{aligned} &E_{a,b}(z) = \sum_{l=0}^{+\infty} \frac{z^l}{\Gamma(al+a)},\ a>0,\ b>0 \\ &E_{a,b}^{(j)}(z) = \frac{d^j}{dz^j} E_{a,b}(z) \\ &= \sum_{l=0}^{+\infty} \frac{(l+j)!z^l}{l!\Gamma(al+a_j+b)},\ j=0,\ 1,\ 2,\ \cdots \end{aligned} \tag{a2}$$

The $|\eta|$ of η appeared in Eq. (14) is given as below:

$$
\begin{aligned}
|\eta| &= \left|\frac{k}{m}\frac{s^{-\alpha}}{s^{2-\alpha}+c/m}\right| \\
&= \frac{1}{|s^2m/k+s^\alpha c/k|} \le \left|\frac{1}{|s^2m/k|-|s^\alpha c/k|}\right| \\
&= \left|\frac{1}{r^\alpha(r^{2-\alpha}m/|k|-|c/k|)}\right| \\
&\le \left|\frac{1}{\xi^\alpha(\xi^{2-\alpha}m/|k|-|c/k|)}\right| \\
&= \left|\frac{1}{\left(\frac{m+|c|+|k|}{m}\right)^{\frac{\alpha}{2-\alpha}}(1+m/|k|)}\right| < 1
\end{aligned} \tag{a3}
$$

The coefficients $c_{n,i}$, $d_{j,i}$, $\phi_{n,i}$, $\psi_{j,i}$ in Eq. (31) is given by:

$$
\begin{aligned}
c_{n,i} &= \int_{-\infty}^{+\infty} f(x)\,\phi_{n,i}(x)\,dx \\
d_{j,i} &= \int_{-\infty}^{+\infty} f(x)\,\psi_{ji}(x)\,dx \\
\phi_{n,i}(x) &= 2^{n/2}\phi(2^n x - i) \\
\psi_{j,i}(x) &= 2^{j/2}\psi(2^j x - i)
\end{aligned} \tag{a4}
$$

The equations of $\chi_{0,0}(\lambda)$, $\chi_{0,1}(\lambda)$, $\chi_{1,0}(\lambda)$, $\chi_{1,1}(\lambda)$, $\chi_{0,0}(\lambda)$ and $g(\lambda, t)$, $g_\lambda(\lambda, t)$ appeared in Eqs. (36) ~ (42) are given by:

$$\chi_{0,0}(\lambda) = [f(\lambda-t)\,G(t)]_{t=0} = f(\lambda)\,G(0)$$

$$
\begin{aligned}
\chi_{0,1}(\lambda) &= \left\{\frac{\partial}{\partial t}[f(\lambda-t)\,G(t)]\right\}_{t=0} \\
&= f(\lambda)\,\dot{G}(0) - \dot{f}(\lambda)\,G(0)
\end{aligned}
$$

$$
\begin{aligned}
\chi_{0,2}(\lambda) &= \left\{\frac{\partial^2}{\partial t^2}[f(\lambda-t)\,G(t)]\right\}_{t=0} \\
&= \ddot{f}(\lambda)\,G(0) - 2\dot{f}(\lambda)\,\dot{G}(0) - f(\lambda)\,\ddot{G}(0)
\end{aligned}
$$

$$\chi_{1,0}(\lambda) = [\dot{f}(\lambda-t)\,G(t)]_{t=0} = \dot{f}(\lambda)\,G(0)$$

$$
\begin{aligned}
\chi_{1,1}(\lambda) &= \left\{\frac{\partial}{\partial t}[\dot{f}(\lambda-t)\,G(t)]\right\}_{t=0} \\
&= \dot{f}(\lambda)\,\dot{G}(_0) - \ddot{f}(\lambda)\,G(0)
\end{aligned}
$$

and then defining $g(\lambda, t)$, $g_\lambda(\lambda, t)$ as shown below

$$
g(\lambda, t) = \begin{cases} f(\lambda-t)\,G(t) & t=(0, \infty) \\ \frac{1}{2}\chi_{0,2}(\lambda)\,t^2 + \chi_{0,1}(\lambda)\,t + \chi_{0,0} & t\in(\delta, 0) \\ 0 & t\in(\infty, \delta) \end{cases} \tag{a5}
$$

$$
g_\lambda(\lambda, t) = \begin{cases} \dot{f}(\lambda-t)\,G(t) & t\in(0, \mathit{infty}) \\ \chi_{1,1}(\lambda)\,t + \chi_{1,0} & t\in(\delta, 0) \\ 0 & t\in(\infty, \delta) \end{cases}
$$

where

$$\delta = (1-3N)/2^n$$

n is the scale of scaling functions which is used to approximate g and g_λ.

The final solutions of $y(t)$ and $\dot{y}(t)$ appeared in Eqs. (43) and (44) are given by:

$$
\begin{aligned}
y(t) &= my_0\dot{G}(t) + my_1G(t) + \int_0^t G(t-\tau)\,f(\tau)\,d\tau \\
&= my_0\dot{G}(t) + my_1G(t) + \int_0^t G(\tau)\,f(t-\tau)\,d\tau \\
&= my_0\dot{G}(t) + my_1G(t) + \int_0^t g(t, \tau)\,d\tau \\
&\approx my_0\dot{G}(t) + my_1G(t) \\
&\quad + \sum_{i=0}^{2n} g\left(t, \frac{i}{2^j}\right)\int_0^1 \varnothing(2^j\tau - i + M_1)\,d\tau \\
&\approx my_0\dot{G}(t) + my_1G(t) \\
&\quad + \sum_{i=0}^{2n} g\left(t, \frac{i}{2^j}\right)[\varnothing^f(2^jt-i+M_1) - \varnothing^f(M_1-i)], \\
&\qquad t\in[0, T]
\end{aligned}
$$

or

$$
\begin{aligned}
y(t) &= my_0\dot{G}(t) + my_1(t) \\
&\quad + \sum_{i=1-3N}^{2n-1} g\left(t, \frac{i+M_1}{2^j}\right)[\varnothing^f(2^jt-i) - \varnothing^f(-i)], \\
&\qquad t\in[0, T]
\end{aligned}
$$

$$
\begin{aligned}
g(\lambda, t)_{\chi[0,t]}(t) &\approx P_j[g(\lambda, t)_{\chi[0,T]}(t)] \\
&= \sum_{i=-\infty}^{\infty} g\left(\lambda, \frac{i+M_1}{2^j}\right)_{\chi[0,t]}\left(\frac{i+M_1}{2^j}\right)\varnothing(2^jt-i)
\end{aligned}
$$

$$
\begin{aligned}
\dot{y}(t) &= my_0\ddot{G}(t) + my_1\dot{G}(t) + \int_0^t \dot{G}(t-\tau)\,f(\tau)\,d\tau \\
&= my_0\ddot{G}(t) + my_1\dot{G}(t) + \int_0^t g_t(t, \tau)\,d\tau \\
&\approx my_0\ddot{G}(t) + my_1\dot{G}(t) \\
&\quad + \sum_{i=0}^{S} g_t\left(t, \frac{i}{2^j}\right)\int_0^t \varnothing(2^j\tau - i + M_1)\,d\tau \\
&\approx my_0\ddot{G}(t) + my_1\dot{G}(t) \\
&\quad + \sum_{i=0}^{S} g_t\left(t, \frac{i}{2^j}\right)\int_0^t \varnothing(2^j\tau - i + M_1)\,d\tau
\end{aligned} \tag{a6}
$$

or

$$
\begin{aligned}
\dot{y}(t) &= my_0\dot{G}(t) + my_1G(t) + \sum_{i=1-3N}^{2n-1} g_t\left(t, \frac{i+M_1}{2^j}\right) \\
&\quad [\varnothing^f(2^jt-i) - \varnothing^f(-i)], \; t\in[0, T]
\end{aligned}
$$

where

$$\varnothing^f(x) = \int_0^x \varnothing(z)\,dz$$

which can be obtained by the wavelet approximations methods (Wang, 2001).